Exercise 4.3

Q1

Find the principal value of $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Solution

We know that, for any $x \in R$, $\tan^{-1} x$ represents an angle in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x.

So,

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \text{An angle in } \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \text{ whose tangest is } \frac{1}{\sqrt{3}}$$

$$= \frac{\pi}{6}$$

$\therefore \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$.

Q2

Find the principal value of each of the following:

$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution

We know that for any $x \in R$, $\tan^{-1} x$ represents an angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x.

$\therefore \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

$\therefore$ Principle value of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is $-\frac{\pi}{6}$.

Q3

Find the principal value of each of the following:

$\tan^{-1}\left(\cos\frac{\pi}{2}\right)$

Solution

$\tan^{-1}\left(\cos\frac{\pi}{2}\right) = \tan^{-1}(0)$

We know that for any $x \in R$, $\tan^{-1}x$ represents an angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x.

$\therefore \tan^{-1}(0) = 0$

$\therefore$ Principle value of $\tan^{-1}\left(\cos\frac{\pi}{2}\right)$ is 0.

Q4

Find the principal value of each of the following:

$\tan^{-1}\left(2\cos\frac{2\pi}{3}\right)$

Solution

$\tan^{-1}\left(2\cos\frac{2\pi}{3}\right) = \tan^{-1}\left(2\times\frac{-1}{2}\right) = \tan^{-1}(-1)$

We know that for any $x \in R$, $\tan^{-1}x$ represents an angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x.

$\therefore \tan^{-1}(-1) = -\frac{\pi}{4}$

$\therefore$ Principle value of $\tan^{-1}\left(2\cos\frac{2\pi}{3}\right)$ is $-\frac{\pi}{4}$

Q5

For the principal values, evaluate $\tan^{-1}(-1) + \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution

Let $\tan^{-1}(-1) = x$. Then, $\tan x = -1 = -\tan\left(\frac{\pi}{4}\right) = \tan\left(\pi - \frac{\pi}{4}\right)$

$\therefore \tan^{-1}(-1) = \frac{3\pi}{4}$

Let $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = y$. Then, $\cos y = \frac{-1}{\sqrt{2}} = -\cos\left(\frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right)$

$\therefore \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

$\therefore \tan^{-1}(-1) + \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4} + \frac{3\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$

Q6

For the principal values, evaluate each of the following :

$\tan^{-1}\left(2\sin\left(4\cos^{-1}\frac{\sqrt{3}}{2}\right)\right)$

Solution

$\tan^{-1}\left(2\sin\left(4\cos^{-1}\frac{\sqrt{3}}{2}\right)\right)=\tan^{-1}\left(2\sin\left(4\times\frac{\pi}{6}\right)\right)=\tan^{-1}\left(2\times\frac{\sqrt{3}}{2}\right)=\tan^{-1}\left(\sqrt{3}\right)$

We know that for any $x\in R$, $\tan^{-1}x$ represents an angle in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangent is x.

$\therefore \tan^{-1}\left(\sqrt{3}\right)=\frac{\pi}{6}$

$\therefore$ Principle value of $\tan^{-1}\left(2\sin\left(4\cos^{-1}\frac{\sqrt{3}}{2}\right)\right)$ is $\frac{\pi}{3}$.

Q7

Find the value of $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

Solution

Let $\tan^{-1}(1)=x$. Then, $\tan x=1=\tan\frac{\pi}{4}$.

$\therefore \tan^{-1}(1)=\frac{\pi}{4}$

Let $\cos^{-1}\left(-\frac{1}{2}\right)=y$. Then, $\cos y=-\frac{1}{2}=-\cos\left(\frac{\pi}{3}\right)=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\left(\frac{2\pi}{3}\right)$.

$\therefore \cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$

Let $\sin^{-1}\left(-\frac{1}{2}\right)=z$. Then, $\sin z=-\frac{1}{2}=-\sin\left(\frac{\pi}{6}\right)=\sin\left(-\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

$\therefore \tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

$=\frac{\pi}{4}+\frac{2\pi}{3}-\frac{\pi}{6}$

$=\frac{3\pi+8\pi-2\pi}{12}=\frac{9\pi}{12}=\frac{3\pi}{4}$

Q8

Evaluate each of the following:

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left(-\sqrt{3}\right)+\tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right)$$

Solution

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left(-\sqrt{3}\right)+\tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right)$$

$$=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left(-\sqrt{3}\right)+\tan^{-1}(-1)$$

We know that for any $x \in R$, $\tan^{-1}x$ represents an angle in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangent is x.

$\therefore \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$

$\tan^{-1}\left(-\sqrt{3}\right)=-\frac{\pi}{3}$

$\tan^{-1}(-1)=-\frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left(-\sqrt{3}\right)+\tan^{-1}(-1)=-\frac{\pi}{6}-\frac{\pi}{3}-\frac{\pi}{4}=-\frac{3\pi}{4}$$

$\therefore$ Principle value of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left(-\sqrt{3}\right)+\tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right)$ is $-\frac{3\pi}{4}$.

Q9

Evaluate each of the following:

$$\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left\{\cos\left(\frac{13\pi}{6}\right)\right\}$$

Solution

$$\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left\{\cos\left(\frac{13\pi}{6}\right)\right\}$$

$$=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

We know that for any $x \in R$, $\tan^{-1}x$ represents an angle in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangent is x.

$\therefore \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$

We know that for any $x \in [-1, 1]$, $\cos^{-1}x$ represents an angle in $[0, \pi]$ whose cosine is x.

$\therefore \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$

$\therefore \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{6}+\frac{\pi}{6}=0$

$\therefore$ Principle value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left\{\cos\left(\frac{13\pi}{6}\right)\right\}$ is 0.