Exercise 4.12

Q1

Evaluate $\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right)$

Solution

$\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right)$

$$= \cos\left[\sin^{-1}\left(\frac{3}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}+\frac{5}{13}\sqrt{1-\left(\frac{3}{5}\right)^2}\right)\right] \quad \left\{\text{Since } \sin^{-1}x+\sin^{-1}y=\sin^{-1}\left[x\sqrt{1-y^2}+y\sqrt{1-x^2}\right]\right\}$$

$$= \cos\left[\sin^{-1}\left(\frac{3}{5}\times\frac{12}{13}+\frac{5}{13}\times\frac{4}{5}\right)\right]$$

$$= \cos\left[\sin^{-1}\left(\frac{56}{65}\right)\right]$$

$$= \cos\left[\cos^{-1}\left(\sqrt{1-\left(\frac{56}{65}\right)^2}\right)\right] \quad \left\{\text{Since } \sin^{-1}x=\cos^{-1}\left(\sqrt{1-x^2}\right)\right\}$$

$$= \cos\left[\cos^{-1}\left(\frac{33}{65}\right)\right]$$

$$= \frac{33}{65} \quad \left\{\text{Since } \cos\left(\cos^{-1}x\right)=x \text{ as } x\in[0,\ 1]\right\}$$

Hence,

$$\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right)=\frac{33}{65}$$

Q2

Prove that $\sin^{-1}\frac{63}{65}=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$

Solution

$$\text{RHS} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$$

$$= \sin^{-1}\frac{5}{13} + \sin^{-1}\sqrt{1-\frac{9}{25}} \qquad \left[\text{Since } \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}\right]$$

$$= \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{4}{5}$$

$$= \tan^{-1}\left(\frac{\frac{5}{13}}{\sqrt{1-\frac{25}{169}}}\right) + \tan^{-1}\left(\frac{\frac{4}{5}}{\sqrt{1-\frac{16}{25}}}\right)$$

$$= \tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left(\frac{4}{3}\right)$$

$$= \tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\cdot\frac{4}{3}}\right)$$

$$= \tan^{-1}\left(\frac{63}{16}\right) \qquad \ldots\ldots(1)$$

Again, $\sin^{-1}\frac{63}{65} = \tan^{-1}\frac{\frac{63}{65}}{\sqrt{1-\left(\frac{63}{65}\right)^2}}$

$$= \tan^{-1}\frac{63}{16} \qquad \ldots\ldots(2)$$

Hence from (1) and (2), $\sin^{-1}\frac{63}{65} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

Q3

Prove $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

Solution

Let $\sin^{-1}\frac{5}{13}=x$. Then, $\sin x=\frac{5}{13}\Rightarrow\cos x=\frac{12}{13}$.

$\therefore \tan x=\frac{5}{12}\Rightarrow x=\tan^{-1}\frac{5}{12}$

$\therefore \sin^{-1}\frac{5}{13}=\tan^{-1}\frac{5}{12}$ ...(1)

Let $\cos^{-1}\frac{3}{5}=y$. Then, $\cos y=\frac{3}{5}\Rightarrow\sin y=\frac{4}{5}$.

$\therefore \tan y=\frac{4}{3}\Rightarrow y=\tan^{-1}\frac{4}{3}$

$\therefore \cos^{-1}\frac{3}{5}=\tan^{-1}\frac{4}{3}$ ...(2)

Using (1) and (2), we have

$$\begin{aligned}\text{R.H.S.}&=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}\\&=\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}\\&=\tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right)\qquad\left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]\\&=\tan^{-1}\left(\frac{15+48}{36-20}\right)\\&=\tan^{-1}\frac{63}{16}\\&=\text{L.H.S.}\end{aligned}$$

Concept Insight:
As L.H.S is $\tan^{-1}$ express the terms in R.H.S in the form of $\tan^{-1}$

Q4

Prove $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Solution

$$\begin{aligned}\text{L.H.S.}&=\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}\\&=\frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)\\&=\frac{9}{4}\left(\cos^{-1}\frac{1}{3}\right)\qquad ....(1)\quad\left[\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]\end{aligned}$$

Now, let $\cos^{-1}\frac{1}{3}=x$. Then, $\cos x=\frac{1}{3}\Rightarrow\sin x=\sqrt{1-\left(\frac{1}{3}\right)^2}=\frac{2\sqrt{2}}{3}$.

$\therefore x=\sin^{-1}\frac{2\sqrt{2}}{3}\Rightarrow\cos^{-1}\frac{1}{3}=\sin^{-1}\frac{2\sqrt{2}}{3}$

$\therefore \text{L.H.S.}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}=\text{R.H.S.}$

Q5

Solve the following:

$$\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$$

Solution

$$\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$$
$$\Rightarrow \sin^{-1} x + \sin^{-1} 2x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$
$$\Rightarrow \sin^{-1} 2x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1} x$$
$$\Rightarrow \sin^{-1} 2x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\sqrt{1-x^2} - x\sqrt{1-\frac{3}{4}}\right)$$
$$\Rightarrow 2x = \left(\frac{\sqrt{3}}{2}\sqrt{1-x^2} - \frac{x}{2}\right)$$
$$\Rightarrow \frac{5}{2}x = \frac{\sqrt{3}}{2}\sqrt{1-x^2}$$
$$\Rightarrow 25x^2 = 3\left(1-x^2\right)$$
$$\Rightarrow 28x^2 = 3$$
$$\Rightarrow x^2 = \frac{3}{28}$$
$$\Rightarrow x = \frac{1}{2}\sqrt{\frac{1}{7}}$$

Q6

Solve the following :

$$\cos^{-1} x + \sin^{-1} \frac{x}{2} = \frac{\pi}{6}$$

Solution

$$\cos^{-1}x + \sin^{-1}\frac{x}{2} = \frac{\pi}{6}$$

$$\Rightarrow \sin^{-1}\frac{x}{2} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \sin^{-1}\frac{x}{2} = \sin^{-1}\left[\frac{1}{2}\sqrt{1-1+x^2} - \sqrt{1-x^2}\sqrt{1-\frac{1}{4}}\right]$$

$$\Rightarrow \frac{x}{2} = \frac{x}{2} - \frac{\sqrt{3}\sqrt{1-x^2}}{2}$$

$$\Rightarrow \frac{\sqrt{3}\sqrt{1-x^2}}{2} = 0$$

$$\Rightarrow \sqrt{1-x^2} = 0$$

$$\Rightarrow x = \pm 1$$