Exercise 4.11

Q1

Prove that $\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)=\tan^{-1}\left(\frac{2}{9}\right)$

Solution

$$\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)=\tan^{-1}\left(\frac{2}{9}\right)$$

$$\text{LHS} = \tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right) \quad \left\{\text{Since } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right\}$$

$$= \tan^{-1}\left(\frac{\frac{13+7}{91}}{\frac{91-1}{91}}\right)$$

$$= \tan^{-1}\left(\frac{20}{91}\times\frac{91}{90}\right)$$

$$= \tan^{-1}\left(\frac{2}{9}\right)$$

$$= \text{RHS}$$

Hence proved.

Q2

Prove that $\sin^{-1}\left(\frac{12}{13}\right)+\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{63}{16}\right)=\pi$

Solution

$$\sin^{-1}\left(\frac{12}{13}\right)+\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{63}{16}\right)=\pi$$

LHS $=\sin^{-1}\left(\frac{12}{13}\right)+\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{63}{16}\right)$

$$=\tan^{-1}\left(\frac{\frac{12}{13}}{\sqrt{1-\left(\frac{12}{13}\right)^2}}\right)+\tan^{-1}\left(\frac{\sqrt{1-\left(\frac{4}{5}\right)^2}}{\frac{4}{5}}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$\left\{\text{Since } \sin^{-1}x=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\cos^{-1}x=\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right\}$$

$$=\tan^{-1}\left(\frac{\frac{12}{13}}{\frac{5}{13}}\right)+\tan^{-1}\left(\frac{\frac{3}{5}}{\frac{4}{5}}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\tan^{-1}\left(\frac{12}{5}\right)+\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\pi+\tan^{-1}\left(\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$\left\{\text{Since } \tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)\text{ if } x>0,\ y>0 \text{ and } xy>0\right\}$$

$$=\pi+\tan^{-1}\left(\frac{\frac{63}{20}}{-\frac{16}{20}}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\pi+\tan^{-1}\left(-\frac{63}{16}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\pi-\tan^{-1}\left(\frac{63}{16}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$\left\{\text{Since } \tan^{-1}(-x)=-\tan^{-1}x\right\}$$

$=\pi$

Hence,

$$\sin^{-1}\left(\frac{12}{13}\right)+\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{63}{16}\right)=\pi$$

Q3

Prove the following result

$$\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\sin^{-1}\frac{1}{\sqrt{5}}$$

Solution

$\tan^{-1}1/4 + \tan^{-1}2/9 = \dfrac{\tan^{-1}(1/4+2/9)}{(1-1/4\times 2/9)}$

$= \tan^{-1}\dfrac{(17/36)}{(34/36)}$

$= \tan^{-1}(1/2)$

Let $\tan^{-1}(1/2) = \theta$

$\tan\theta = 1/2$

we know that $\tan\theta = \dfrac{\text{opposite side}}{\text{adjacent side}}$

so if opposite side = 1 unit

adjacent side = 2 unit, then hypotenuse = $\sqrt{5}$ unit

so $\sin\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}}$

so $\sin\theta = 1/\sqrt{5}$

so $\theta = \sin^{-1}(1/\sqrt{5}) = \tan^{-1}(1/2)$

Q4

Find the value of $\tan^{-1}\left(\dfrac{x}{y}\right) - \tan^{-1}\left(\dfrac{x-y}{x+y}\right)$

Solution

We know that, $\tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\dfrac{A-B}{1+AB}\right)$, *if* $AB > -1$

Consider the given expression $\tan^{-1}\left(\dfrac{x}{y}\right) - \tan^{-1}\left(\dfrac{x-y}{x+y}\right)$:

$$\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right) = \tan^{-1}\left(\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1+\left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right)$$

$$= \tan^{-1}\left(\frac{x(x+y) - y(x-y)}{y(x+y) + x(x-y)}\right)$$

$$= \tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

Q5

Solve the equation for x :

$\tan^{-1}2x + \tan^{-1}3x = n\pi + \dfrac{3\pi}{4}$

Solution

Given

$$\tan^{-1}2x+\tan^{-1}3x=n\pi+\frac{3\pi}{4} \qquad \text{---(i)}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x+3x}{1-2x\times 3x}\right)=n\pi+\frac{3\pi}{4} \qquad \left\{\text{Since } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } xy<1\right\}$$

$$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right)=n\pi+\frac{3\pi}{4},\ 6x^2<1$$

$$\Rightarrow \frac{5x}{1-6x^2}=\tan\left(n\pi+\frac{3\pi}{4}\right),\ 6x^2<1$$

$$\Rightarrow \frac{5x}{1-6x^2}=-1,\ 6x^2<1$$

$$\Rightarrow 5x=-1+6x^2,\ 6x^2<1$$

$$\Rightarrow 6x^2-5x-1=0,\ x^2<\frac{1}{6}$$

$$\Rightarrow 6x^2-6x+x-1=0,\ -\frac{1}{\sqrt{6}}<x<\frac{1}{\sqrt{6}}$$

$$\Rightarrow 6x(x-1)+1(x-1)=0,\ -\frac{1}{\sqrt{6}}<x<\frac{1}{\sqrt{6}}$$

$$\Rightarrow (6x+1)(x-1)=0$$

$$\Rightarrow 6x+1=0 \quad \text{or} \quad x-1=0$$

$$\Rightarrow x=-\frac{1}{6} \text{ or } \quad x=1$$

Since $x=1\notin\left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$

So, $x=1$ is not root of the given equation (i).

Since,

$$x=-\frac{1}{6}\in\left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

So,

$x=-\frac{1}{6}$ is the root of the gien equation (i).

Hence,

$$x=-\frac{1}{6}$$

Q6

Solve the equation for x :

$$\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$$

Solution

Given,

$$\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31} \quad ---(i)$$

$$\Rightarrow \tan^{-1}\left[\frac{(x+1)+(x-1)}{1-(x+1)(x-1)}\right]=\tan^{-1}\frac{8}{31},\ (x+1)(x-1)<1$$

$$\left\{\text{Since } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy},\ \text{if } xy<1\right\}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x}{1-\left(x^2-1\right)}\right]=\tan^{-1}\frac{8}{31}, \quad \left(x^2-1\right)<1$$

$$\Rightarrow \tan^{-1}\left[\frac{2x}{1-x^2+1}\right]=\tan^{-1}\frac{8}{31},\ x^2<2$$

$$\Rightarrow \frac{2x}{2-x^2}=\frac{8}{31}, \quad -\sqrt{2}<x<\sqrt{2}$$

$$\Rightarrow 8x^2+62x-16=0, \quad -\sqrt{2}<x<\sqrt{2}$$

$$\Rightarrow 4x^2+31x-8=0, \quad -\sqrt{2}<x<\sqrt{2}$$

$$\Rightarrow 4x(x+8)-1(x+8)=0, \quad -\sqrt{2}<x<\sqrt{2}$$

$$\Rightarrow (4x-1)(x+8)=0, \quad -\sqrt{2}<x<\sqrt{2}$$

$$\Rightarrow x=\frac{1}{4} \quad \text{or} \quad x=-8$$

But, $x=-8\notin\left(-\sqrt{2},\ \sqrt{2}\right)$

$\Rightarrow$ $x=8$ is not root of the given equation (i)

For $x=\frac{1}{4}\in\left(-\sqrt{2},\ \sqrt{2}\right)$

$\Rightarrow$ $x=\frac{1}{4}$ is a root of the equation (i)

Hence,

$$x=\frac{1}{4}$$

Q7

Solve the equation for x:

$$\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$$

Solution

Given,

$$\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$$

$$\Rightarrow \tan^{-1}(x-1)+\tan^{-1}(x+1)+\tan^{-1}x=\tan^{-1}3x$$

$$\Rightarrow \tan^{-1}\left[\frac{(x-1)+(x+1)}{1-(x-1)(x+1)}\right]+\tan^{-1}x=\tan^{-1}3x$$

$$\left\{\text{Since, } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right),\ \text{if } xy<1\right\}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{1-x^2+1}\right)+\tan^{-1}x=\tan^{-1}3x, \quad x^2-1<1$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{2-x^2}\right)+\tan^{-1}x=\tan^{-1}3x, \quad x^2<2$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2x}{2-x^2}+x}{1-\left(\frac{2x}{2-x^2}\right)x}\right]=\tan^{-1}3x, \quad \frac{2x^2}{2-x^2}<1$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2x+2x-x^3}{2-x^2}}{\frac{2-x^2-2x^2}{2-x^2}}\right]=\tan^{-1}3x$$

$$\Rightarrow \tan^{-1}\left[\frac{4x-x^3}{2-3x^2}\right]=\tan^{-1}3x, \quad 2x^2<2-x^2$$

$$\Rightarrow \frac{4x-x^3}{2-3x^2}=3x, \quad 3x^2<2$$

$$\Rightarrow 4x-x^3=6x-9x^3, \quad x^2<\frac{2}{3}$$

$$\Rightarrow 9x^3-x^3+4x-6x=0$$

$$\Rightarrow 8x^3-2x=0$$

$$\Rightarrow 2x\left(4x^2-1\right)=0$$

$$\Rightarrow x=0,\ x=\frac{1}{2},\ x=-\frac{1}{2} \text{ all satisfies } x^2<\frac{2}{3} \text{ ed}$$

$$x=0,\ \pm\frac{1}{2}$$

Q8

Solve the following equations for x :

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)-\frac{1}{2}\tan^{-1}x=0, \text{ where } x>0$$

Solution

$$\tan^{-1}\frac{1-x}{1+x}=\frac{1}{2}\tan^{-1}x$$

$$\Rightarrow \tan^{-1}1-\tan^{-1}x=\frac{1}{2}\tan^{-1}x \qquad \left[\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}\right]$$

$$\Rightarrow \frac{\pi}{4}=\frac{3}{2}\tan^{-1}x$$

$$\Rightarrow \tan^{-1}x=\frac{\pi}{6}$$

$$\Rightarrow x=\tan\frac{\pi}{6}$$

$$\therefore x=\frac{1}{\sqrt{3}}$$

Q9

Solve the equation for x :

$\cot^{-1}x-\cot^{-1}(x+2)=\frac{\pi}{12}$, where $x>0$

Solution

Given,

$$\cot^{-1}x - \cot^{-1}(x+2) = \frac{\pi}{12}, \text{ where } x > 0$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{x}\right) - \tan^{-1}\left(\frac{1}{x+2}\right) = \frac{\pi}{12} \qquad \left\{\text{Since } \cot^{-1}x = \tan^{-1}\frac{1}{x}\right\}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{1}{x} - \frac{1}{x+2}}{1 + \frac{1}{x}\times\frac{1}{x+2}}\right) = \frac{\pi}{12} \qquad \left\{\text{Since, } \tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}\right\}$$

$$\Rightarrow \left(\frac{\frac{x+2-x}{x(x+2)}}{\frac{x(x+2)+1}{x(x+2)}}\right) = \tan\frac{\pi}{12}$$

$$\Rightarrow \frac{2}{x^2+2x+1} = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$\Rightarrow \frac{2}{(x+1)^2} = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{4}\times\tan\frac{\pi}{3}} \qquad \left\{\text{Since, } \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}\right\}$$

$$\Rightarrow \frac{2}{(x+1)^2} = \frac{\sqrt{3}-1}{1+\sqrt{3}}$$

$$\Rightarrow \frac{2}{(x+1)^2} = \frac{\sqrt{3}-1}{\sqrt{3}+1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$\Rightarrow \frac{2}{(x+1)^2} = \frac{3-1}{\left(\sqrt{3}+1\right)^2}$$

$$\Rightarrow (x+1)^2 = \left(\sqrt{3}+1\right)^2$$

$$\Rightarrow x+1 = \pm\left(\sqrt{3}+1\right)$$

$$\Rightarrow x+1 = \sqrt{3}+1 \text{ or } x+1 = -\sqrt{3}-1$$

$$\Rightarrow x = \sqrt{3}+1-1 \text{ or } x = -\sqrt{3}-2$$

$$\Rightarrow x = \sqrt{3} \text{ or } x = -\left(\sqrt{3}+2\right)$$

Given, $x > 0$, so

$x = \sqrt{3}$

Q10

Solve the equation for x :

$$\tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1}\left(\frac{8}{79}\right), \; x > 0$$

Solution

Given,

$$\tan^{-1}(x+2)+\tan^{-1}(x-2)=\tan^{-1}\left(\frac{8}{79}\right),\ x>0$$

$$\Rightarrow \tan^{-1}\left[\frac{(x+2)+(x-2)}{1-(x+2)(x-2)}\right]=\tan^{-1}\frac{8}{79} \qquad \left\{\text{Since, } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right\}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x}{1-x^2+4}\right]=\tan^{-1}\frac{8}{79}$$

$$\Rightarrow \frac{2x}{5-x^2}=\frac{8}{79}$$

$$\Rightarrow 40-8x^2=158x$$

$$\Rightarrow 8x^2+158x-40=0$$

$$\Rightarrow 4x^2+79x-20=0$$

$$\Rightarrow 4x^2+80x-x-20=0$$

$$\Rightarrow 4x(x+20)-1(x+20)=0$$

$$\Rightarrow (4x-1)(x+20)=0$$

$$\Rightarrow (4x-1)=0 \quad \text{or} \quad x+20=0$$

$$\Rightarrow x=\frac{1}{4} \quad \text{or} \quad x=-20$$

Since, $x>0$, so

$$x=\frac{1}{4}$$

Q11

Solve the equation for x :

$$\tan^{-1}\frac{x}{2}+\tan^{-1}\frac{x}{3}=\frac{\pi}{4},\ 0<x<\sqrt{6}$$

Solution

Given,

$$\tan^{-1}\frac{x}{2}+\tan^{-1}\frac{x}{3}=\frac{\pi}{4},\quad 0<x<\sqrt{6}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x}{2}+\frac{x}{3}}{1-\frac{x}{2}\times\frac{x}{3}}\right]=\frac{\pi}{4} \qquad \left\{\text{Since, } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right\}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{5x}{6}}{\frac{\left(6-x^2\right)}{16}}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{5x}{6-x^2}\right]=\frac{\pi}{4}$$

$$\Rightarrow \frac{5x}{6-x^2}=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{5x}{6-x^2}=1$$

$$\Rightarrow 5x=6-x^2$$

$$\Rightarrow x^2+5x-6=0$$

$$\Rightarrow x^2+6x-x-6=0$$

$$\Rightarrow x(x+6)-1(x+6)=0$$

$$\Rightarrow (x+6)(x-1)=0$$

$$\Rightarrow x=-6 \text{ or } \quad x=1$$

But, $0<x<\sqrt{6}$, so

$x=1$

Q12

Solve the following equations for x:

$$\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$$

Solution

$$\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{x-2}{x-4}+\frac{x+2}{x+4}}{1-\left(\frac{x-2}{x-4}\right)\left(\frac{x+2}{x+4}\right)}\right)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{(x-2)(x+4)+(x+2)(x-4)}{(x-4)(x+4)}}{\frac{(x-4)(x+4)-(x-2)(x+2)}{(x-4)(x+4)}}\right)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{(x-2)(x+4)+(x+2)(x-4)}{(x-4)(x+4)-(x-2)(x+2)}\right)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{x^2+2x-8+x^2-2x-8}{(x^2-16)-(x^2-4)}\right)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x^2-16}{-12}\right)=\frac{\pi}{4}$$

$$\Rightarrow \left(\frac{x^2-8}{-6}\right)=\tan\frac{\pi}{4}$$

$$\Rightarrow \left(\frac{x^2-8}{-6}\right)=1$$

$$\Rightarrow x^2-8=-6$$

$$\Rightarrow x^2=2$$

$$\therefore x=\pm\sqrt{2}$$

Q13

Solve the following equations for x:

$$\tan^{-1}(2+x)+\tan^{-1}(2-x)=\tan^{-1}\frac{2}{3}, \text{where } x<-\sqrt{3} \text{ or}, x>\sqrt{3}$$

Solution

$$\tan^{-1}\frac{(2+x+2-x)}{[1-(2+x)(2-x)]}$$

$$\tan^{-1}\frac{4}{(x^2-3)}=\tan^{-1}2/3$$

$$4/(x^2-3)=2/3$$

$$x^2-3=6$$

$$x=3,-3$$

Q14

Sum the following series:

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{2}{9}+\tan^{-1}\frac{4}{33}+\ldots+\tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}}$$

Solution

Let $T_n = \frac{\tan^{-1} 2^{n-1}}{1+2^{2n-1}}$

$T_n = \tan^{-1}\frac{\left(2^n - 2^{n-1}\right)}{1+2^n 2^{n-1}}$

$= \tan^{-1}2^n - \tan^{-1}2^{n-1}$

So, $T_1 = \tan^{-1}2 - \tan^{-1}1$

$T_2 = \tan^{-1}4 - \tan^{-1}2$

$T_n = \tan^{-1}2^n - \tan^{-1}2^{n-1}$

Adding all the terms we get

$\tan^{-1}2^n - \tan^{-1}1$

$\tan^{-1}2^n - \pi/4$