Miscellaneous Exercise

Question 1: Find the derivative of the following functions from first principle:

(i) $-x$

(ii) $(-x)^{-1}$

(iii) $\sin(x + 1)$

(iv) $\cos\left(x - \frac{\pi}{8}\right)$

Solution 1: (i) Let $f(x) = -x$. Accordingly, $f(x + h) = -(x + h)$

By first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{-x - h + x}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h}$$

$$= \lim_{h \to 0}(-1) = -1$$

(ii) Let $f(x) = (-x)^{-1} = \frac{1}{-x} = \frac{-1}{x}$. Accordingly, f(x + h) = $\frac{-1}{(x+h)}$

By first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-1}{(x+h)} - \left(\frac{-1}{x}\right)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-x+(x+h)}{x(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{h}{x(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{x(x+h)}$$

$$= \frac{1}{x.x} = \frac{1}{x^2}$$

(iii) Let f(x) = sin(x + 1). Accordingly, f(x + h) = sin(x + h + 1)

By first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}[\sin(x+h+1) - \sin(x+1)]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{2x+h+2}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h \to 0}\left[\cos\left(\frac{2x+h+2}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\cos\left(\frac{2x+h+2}{2}\right)\cdot\lim_{\frac{h}{2}\to 0}\frac{1}{h}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \qquad \left[\text{As } h\to 0\Rightarrow\frac{h}{2}\to 0\right]$$

$$=\cos\left(\frac{2x+0+2}{2}\right)\cdot 1 \qquad \left[\lim_{h\to 0}\frac{\sin x}{x}=1\right]$$

$$=\cos(x+1)$$

(iv) Let f(x) = $\cos\left(x-\frac{\pi}{8}\right)$. Accordingly, f(x + h) = $\cos\left(x+h-\frac{\pi}{8}\right)$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\cos\left(x+h-\frac{\pi}{8}\right)-\cos\left(x-\frac{\pi}{8}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[-2\sin\frac{\left(x+h-\frac{\pi}{8}+x-\frac{\pi}{8}\right)}{2}\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[-2\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$=\lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$=\lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\right]\cdot\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \qquad \left[\text{As } h\to 0\Rightarrow\frac{h}{2}\to 0\right]$$

$$=-\sin\left(\frac{2x+0-\frac{\pi}{4}}{2}\right)\cdot 1$$

$$= -\sin\left(x - \frac{\pi}{8}\right)$$

Question 2: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): (x + a)

Solution 2: Let f(x) = x + a. Accordingly, f(x + h) = x + h + a

By first principle,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{x+h+a-x-a}{h}$$

$$= \lim_{h\to 0}\left(\frac{h}{h}\right)$$

$$= \lim_{h\to 0}(1)$$

= 1

Question 3: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non- zero constants and m and n are integers): (px + q)$\left(\frac{r}{x} + s\right)$

Solution 3: Let f(x) = (px + q)$\left(\frac{r}{x} + s\right)$

By Leibnitz product rule,

$$f'(x) = (px+q)\left(\frac{r}{x}+s\right)' + \left(\frac{r}{x}+s\right)(px+q)'$$

$$= (px+q)\left(rx^{-1}+s\right)' + \left(\frac{r}{x}+s\right)(p)$$

$$= (px+q)\left(-rx^{-2}\right) + \left(\frac{r}{x}+s\right)p$$

$= (px+q)\left(\frac{-r}{x^2}\right)+\left(\frac{r}{x}+s\right)p$

$= \frac{-px}{x}-\frac{qr}{x^2}+\frac{pr}{x}+ps$

$= ps-\frac{qr}{x^2}$

Question 4: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): (ax + b) $(cx + d)^2$

Solution 4: Let $f'(x)=(ax+b)(cx+d)^2$

By Leibnitz product rule,

$f'(x)=(ax+b)\frac{d}{dx}(cx+d)^2\frac{d}{dx}(ax+b)$

$= (ax+b)\frac{d}{dx}(c^2x^2+2cdx^2)+(cx+d)^2\frac{d}{dx}(ax+b)$

$= (ax+b)\left[\frac{d}{dx}(c^2x^2)+\frac{d}{dx}(2cdx)+\frac{d}{dx}d^2\right]+(cx+d)^2\left[\frac{d}{dx}ax+\frac{d}{dx}b\right]$

$= (ax+b)(2c^2x+2cd)+(cx+d)^2a$

$= 2c(ax+b)(cx+d)+a(cx+d)^2$

Question 5: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non zero constants and m and n are integers): $\frac{ax+b}{cx+d}$

Solution 5: Let $f(x)=\frac{ax+b}{cx+d}$

By quotient rule,

$f'(x)=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$

$= \frac{(cx+d)(a)-(ax+d)(c)}{(cx+d)^2}$

$= \dfrac{acx+ad-acx-bc}{(cx+d)^2}$

$= \dfrac{ad-bc}{(cx+d)^2}$

Question 6: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}$

Solution 6: Let $f(x)=\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}=\dfrac{\frac{x+1}{x}}{\frac{x-1}{x}}=\dfrac{x+1}{x-1}$, where x ≠ 0

By quotient rule,

$$f^{'}(x)=\frac{(x-1)\frac{d}{dx}(x-1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2}, x\neq 0,1$$

$= \dfrac{(x-1)(1)-(x+1)(1)}{(x-1)^2}, x\neq 0,1$

$= \dfrac{x-1-x-1}{(x-1)^2}, x\neq 0,1$

$= \dfrac{-2}{(x-1)^2}, x\neq 0,1$

Question 7: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\dfrac{1}{ax^2+bx+c}$

Solution 7: Let $f(x)=\dfrac{1}{ax^2+bx+c}$

By quotient rule,

$$f^{'}(x)=\frac{(ax^2+bx+c)\frac{d}{dx}(1)-\frac{d}{dx}(ax^2+bx+c)}{(ax^2+bx+c)^2}$$

$$=\frac{(ax^2+bx+c)(0)-(2ax+b)}{(ax^2+bx+c)^2}$$

$$=\frac{-(2ax+b)}{(ax^2+bx+c)^2}$$

Question 8: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{ax+b}{px^2+qx+r}$

Solution 8: Let $f(x)=\frac{ax+b}{px^2+qx+r}$

By quotient rule,

$$f^{'}(x)=\frac{(px^2+qx+r)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(px^2+qx+r)}{(px^2+qx+r)^2}$$

$$=\frac{(px^2+qx+r)(a)-(ax+b)(2px+q)}{(px^2+qx+r)^2}$$

$$=\frac{apx^2+aqx+ar-aqx+2npx+bq}{(px^2+qx+r)^2}$$

$$=\frac{-apx^2+2bpx+ar-bq}{(px^2+qx+r)^2}$$

Question 9: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{px^2+qx+r}{ax+b}$

Solution 9: Let $f(x)=\frac{px^2+qx+r}{ax+b}$

By quotient rule,

$$f'(x)=\frac{(ax+b)\frac{d}{dx}(px^2+qx+r)-(px^2+qx+r)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

$$=\frac{(ax+b)(2px+q)-(px^2+qx+r)(a)}{(ax+b)^2}$$

$$=\frac{2apx^2+aqx+2bpx+bq-aqx^2-aqx-ar}{(ax+b)^2}$$

$$=\frac{apx^2+2bpx+bq-ar}{(ax+b)^2}$$

Question 10: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{a}{x^4}-\frac{b}{x^2}+\cos x$

Solution 10: Let $f(x)=\frac{a}{x^4}-\frac{b}{x^2}+\cos x$

$$f'(x)=\frac{d}{dx}\left(\frac{a}{x^4}\right)-\frac{d}{dx}\left(\frac{a}{x^2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\frac{d}{dx}\left(x^{-4}\right)-b\frac{d}{dx}\left(x^{-2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\left(-4x^{-5}\right)-b\left(-2x^{-3}\right)+(-\sin x)\quad\left[\frac{d}{dx}\left(x^n\right)=nx^{n-1}\text{ and }\frac{d}{dx}(\cos x)=-\sin x\right]$$

$$=\frac{-4a}{x^5}+\frac{2b}{x^3}-\sin x$$

Question 11: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non zero constants and m and n are integers): $4\sqrt{x}-2$

Solution 11: Let f(x) = $4\sqrt{x}-2$

$$f'(x)=\frac{d}{dx}(4\sqrt{x}-2)=\frac{d}{dx}(4\sqrt{x})-\frac{d}{dx}(2)$$

$$=4\frac{d}{dx}(x^{\frac{1}{2}})-0=4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$

$$=\left(2x^{-\frac{1}{2}}\right)=\frac{2}{\sqrt{x}}$$

Question 12: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax + b)^n$

Solution 12: Let $f(x) = (ax + b)^n$. Accordingly, $f(x + h) = \{a(x + h) + b\}^n = (ax + ah + b)^n$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(ax+ah+b)-(ax+b)^n}{h}$$

$$=\lim_{h\to 0}\frac{(ax+b)^n\left(1+\frac{ah}{ax+b}\right)^n-(ax+b)^n}{h}$$

$$=(ax+b)^n\lim_{h\to 0}\frac{1}{h}\left[\left\{1+n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)}{2}\left(\frac{ah}{ax+b}\right)^2+\cdots\right\}-1\right]\quad\text{(using binomial theorem)}$$

$$=(ax+b)^n\lim_{h\to 0}\frac{1}{h}\left[n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)a^2h^2}{2(ax+b)^2}+\cdots(\text{Terms containing higher degrees of h})\right]$$

$$=(ax+b)^n\lim_{h\to 0}\left[\frac{na}{(ax+b)}+\frac{n(n-1)a^2h^2}{2(ax+b)^2}+\cdots\right]$$

$$=(ax+b)^n\left[\frac{na}{(ax+b)}+0\right]$$

$$=na\frac{(ax+b)^n}{ax+b}$$

$$=na(ax+b)^{n-1}$$

Question 13: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax + b)^n (cx + d)^m$

Solution 13: Let $f(x) = (ax + b)^n (cx + d)^m$

By Leibnitz product rule,

$$f'(x) = (ax+b)^n \frac{d}{dx}(cx+d)^m + (cx+d)^m \frac{d}{dx}(ax+b)^n \qquad \ldots (1)$$

Now let $f_1(x) = (cx + d)^m$

$f_1(x + h) = (cx + ch + d)^m$

$$f_1'(x) = \lim_{h\to 0}\frac{f_1(x+h) - f_1(x)}{h}$$

$$= \lim_{h\to 0}\frac{(cx+ch+d)^m - (cx+d)^m}{h}$$

$$= (cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{ch}{cx+d}\right)^m - 1\right]$$

$$= (cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{mch}{(cx+d)} + \frac{m(m-1)}{2}\frac{c^2h^2}{(cx+d)^2} + \cdots\right)^m - 1\right]$$

$$= (cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\frac{mch}{(cx+d)} + \frac{m(m-1)c^2h^2}{2(cx+d)^2} + \cdots \text{(Terms containing higher degree oh h)}\right]$$

$$= (cx+d)^m \lim_{h\to 0}\left[\frac{mc}{(cx+d)} + \frac{m(m-1)c^2h^2}{2(cx+d)^2} + \cdots\right]$$

$$= (cx+d)^m \left[\frac{mch}{(cx+d)} + 0\right]$$

$$= \frac{mc(cx+d)^m}{(cx+d)}$$

$$= mc(cx+d)^{m-1}$$

$$\frac{d}{dx}(cx+d)^m = mc(cx+d)^{m-1} \qquad \ldots (2)$$

Similarly, $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$ ... (3)

Therefore, from (1), (2), and (3), we obtain

$$f'(x) = (ax+b)^n\{mc(cx+d)^{m-1}\} + (cx+d)^m\{na(ax+b)^{n-1}\}$$

$$= (ax+b)^{n-1}(cx+d)^{m-1}[mc(ax+b)+na(cx+d)]$$

Question 14: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): sin (x + a)

Solution 14: Let, f(x) = sin(x + a)

f(x + h) = sin(x + h + a)

By first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+h+a)-\sin(x+a)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left[\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]\right]$$

$$= \lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right)\cdot\lim_{\frac{h}{2}\to 0}\left[\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right] \qquad \left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right]$$

$$= \cos\left(\frac{2x+2a}{2}\right)\times 1 \qquad \left[\lim_{h\to 0}\frac{\sin x}{x}=1\right]$$

$$= \cos(x+a)$$

Question 15: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): cosec x cot x

Solution 15: Let f(x) = cosec x cot x

By Leibnitz product rule,

f'(x) = cosec x(cot x)'+cot x(cosec x)' ….(1)

Let $f_1(x)$ = cot x. Accordingly, $f_1(x + h)$ = cot (x + h)

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cot(x+h)-\cot(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos(x)}{\sin x}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\sin x\cos(x+h)-\cos x\sin(x+h)}{\sin x\sin(x+h)}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\sin(x-x+h)}{\sin x\sin(x+h)}\right)$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right]$$

$$=\frac{-1}{\sin x}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\sin(x+h)}\right)$$

$$=\frac{-1}{\sin x}\cdot 1\cdot\left(\lim_{h\to 0}\frac{1}{\sin(x+0)}\right)$$

$$=\frac{-1}{\sin^2 x}$$

$$=-\text{cosec}^2 x$$

$$\therefore (\cot x)'=-\text{cosec}^2 x \qquad \ldots (2)$$

Now, let $f_2(x)$ = cosec x. Accordingly, $f_2(x + h)$ = cosec(x + h)

By first principle,

$$f_2'(x) = \lim_{h\to 0} \frac{f_2(x+h) - f_2(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}[\operatorname{cosec}(x+h) - \operatorname{cosec}(x)]$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sin(x+h)} - \frac{1}{\sin x}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)}\right)$$

$$= \frac{1}{\sin x} \cdot \lim_{h\to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x} \cdot \lim_{h\to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x} \cdot \lim_{h\to 0}\left[\frac{-\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{-1}{\sin x} \cdot \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h\to 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$= \frac{-1}{\sin x} \cdot 1 \cdot \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+0)}$$

$$= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$= -\operatorname{cosec} x \cdot \cot x$$

$$\therefore (\operatorname{cosec} x)' = -\operatorname{cosec} x \cdot \cot x$$

From (1), (2), and (3), we obtain

$$f'(x) = \operatorname{cosec} x(-\operatorname{cosec}^2 x) + \cot x(-\operatorname{cosec} x \cot x)$$

$$= -\text{cosec}^3 x - \cot^2 x \text{cosec}\, x$$

Question 16: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{\cos x}{1+\sin x}$

Solution 16: Let f(x) = $\frac{\cos x}{1+\sin x}$

By quotient rule,

$$f'(x) = \frac{(1+\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - 1}{(1+\sin x)^2}$$

$$= \frac{-(1-\sin x)}{(1+\sin x)^2}$$

$$= \frac{-1}{(1+\sin x)^2}$$

Question 17: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non zero constants and m and n are integers): $\frac{\sin x + \cos x}{\sin x - \cos x}$

Solution 17: Let f (x) = $\frac{\sin x + \cos x}{\sin x - \cos x}$

By quotient rule,

$$f'(x)=\frac{(\sin x-\cos x)\frac{d}{dx}(\sin x+\cos x)-(\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)}{(\sin x+\cos x)^2}$$

$$=\frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x+\cos x)^2}$$

$$=\frac{-(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x+\cos x)^2}$$

$$=\frac{-[\sin^2 x+\cos^2 x-2\sin x\cos x+\sin^2 x+\cos^2 x+2\sin x\cos x]}{(\sin x+\cos x)^2}$$

$$=\frac{-[1+1]}{(\sin x-\cos x)^2}$$

$$=\frac{-2}{(\sin x-\cos x)^2}$$

Question 18: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{\sec x-1}{\sec x+1}$

Solution 18: Let f(x) = $\frac{\sec x-1}{\sec x+1}$

$$f(x)=\frac{\frac{1}{\cos x}-1}{\frac{1}{\cos x}+1}=\frac{1-\cos x}{1+\cos x}$$

By quotient rule,

$$f'(x)=\frac{(1+\cos x)\frac{d}{dx}(1-\cos x)-(1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)(\sin x)-(1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

$$=\frac{\sin x+\cos x\sin x+\sin x-\sin x\cos x}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{\left(1+\frac{1}{\sec x}\right)^2} = \frac{2\sin x}{\frac{(\sec x+1)^2}{\sec^2 x}}$$

$$= \frac{2\sin x \sec^2 x}{(\sec x+1)^2}$$

$$= \frac{\frac{2\sin x}{\cos x}\sec x}{(\sec x+1)^2}$$

$$= \frac{2\sec x \tan x}{(\sec x+1)^2}$$

Question 19: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\sin^n x$

Solution 19: Let $y = \sin^n x$.

Accordingly, for n = 1, y = sin x.

$\therefore \frac{dy}{dx} = \cos x$, i.e., $\frac{d}{dx}\sin x = \cos x$

For n = 2, $y = \sin^2 x$.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin x \sin x)$

= (sin x)'(sin x + sin x(sin x)' [By Leibnitz product rule]

= cos x sin x + sin x cos x

= 2sin x cos x …. (1)

For n = 3, $y = \sin^3 x$.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin x \sin^2 x)$

= (sin x)'$\sin^2 x$ + sin x(sin x)' [By Leibnitz product rule]

= cos x $\sin^2 x$+sin x(2sin x cos x) [Using (1)]

= cos x $\sin^2 x$ + $\sin^2 x$ cos x

$= 3\sin^2 x \cos x$

We assert that $\frac{d}{dx}(\sin^n x) = n\sin^{(n-1)} x \cos x$

Let our assertion be true for n = k.

i.e., $\frac{d}{dx}(\sin^k x) = k\sin^{(k-1)} x \cos x$ …. (2)

Consider

$\frac{d}{dx}(\sin^{k+1} x) = \frac{d}{dx}(\sin x \sin^{(k)} x)$

$= (\sin x)' \sin^k x + \sin x (\sin^k x)'$ [By Leibnitz product rule]

$= \cos x \sin^k x + \sin x (k\sin^{k-1} \cos x)$ [Using (2)]

$= \cos x \sin^k x + 2 \sin^k x \cos x$

$= (k + 1) \sin^k x \cos x$

Thus, our assertion is true for n = k + 1.

Hence, by mathematical induction, $\frac{d}{dx}(\sin^n x) = n\sin^{(n-1)} x \cos x$

Question 20: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{a + b\sin x}{c + d\cos x}$

Solution 20: Let f(x) = $\frac{a + b\sin x}{c + d\cos x}$

By quotient rule,

$$f'(x) = \frac{(c + d\cos x)\frac{d}{dx}(a + b\sin x) - (a + b\sin x)\frac{d}{dx}(c + d\cos x)}{(c + d\cos x)^2}$$

$$= \frac{(c + d\cos x)(b\cos x) - (a + b\sin x)(-d\sin x)}{(c + d\cos x)^2}$$

$$= \frac{cb\cos x + bd\cos^2 x + ad\sin x + bd\sin^2 x}{(c + d\cos x)^2}$$

$$= \frac{bc\cos x + ad\sin x + bd(\cos^2 x + \sin^2 x)}{(c + d\cos x)^2}$$

$$= \frac{bc\cos x + ad\sin x + bd}{(c + d\cos x)^2}$$

Question 21: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{\sin(x+a)}{\cos x}$

Solution 21: Let f(x) = $\frac{\sin(x+a)}{\cos x}$

By quotient rule,

$$f'(x) = \frac{\cos x \frac{d}{dx}[\sin(x+a)] - \sin(x+a)\frac{d}{dx}\cos x}{\cos^2 x}$$

$$f'(x) = \frac{\cos x \frac{d}{dx}[\sin(x+a)] - \sin(x+a)\frac{d}{dx}(-\sin x)}{\cos^2 x} \quad \ldots \text{(i)}$$

Let g(x) = sin(x + a). Accordingly, g(x + h) = sin (x + h + a)

By first principle,

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}[\sin(x+h+a) - \sin(x+a)]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h \to 0}\left[\cos\left(\frac{2x+2a+h}{h}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$

$$= \lim_{h \to 0} \cos\left(\frac{2x+2a+h}{h}\right) \cdot \lim_{h \to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\} \qquad \left[\text{As } h \to 0 \Rightarrow \frac{h}{2} \to 0\right]$$

$$= \left(\cos\frac{2x+2a}{2}\right) \times 1 \qquad \left[\lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

$$= \cos(x+a) \qquad \ldots \text{(ii)}$$

From (i) and (ii), we obtain

$$f'(x) = \frac{\cos x \cos(x+a) + \sin x \sin(x+a)}{\cos^2 x}$$

$$= \frac{\cos(x+a-x)}{\cos^2 x}$$

$$= \frac{\cos a}{\cos^2 x}$$

Question 22: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): x^4 (5 sin x – 3 cos x)

Solution 22: Let f (x) = x^4 (5 sin x – 3 cos x)

By product rule,

$$f'(x) = x^4 \frac{d}{dx}(5\sin x - 3\cos x) + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$

$$= x^4\left[5\frac{d}{dx}(\sin x) - 3\frac{d}{dx}(\cos x)\right] + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$

$$= x^4\left[5\cos x - 3(-\sin x)\right] + (5\sin x - 3\cos x)(4x^3)$$

$$= x^3[5x\cos x + 3x\sin x + 20\sin x - 12\cos x]$$

Question 23: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): (x^2 + 1) cos x

Solution 23: Let f(x) = $(x^2 + 1)$ cos x

By product rule,

$$f'(x) = (x^2+1)\frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2+1)$$

$$= (x^2+1)(-\sin x) + \cos x(2x)$$

$$= -x^2 \sin x - \sin x + 2x \cos x$$

Question 24: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax^2 + \sin x)(p + q\cos x)$

Solution 24: Let f(x) = $(ax^2 + \sin x)(p + q \cos x)$

By product rule,

$$f'(x) = (ax^2+\sin x)\frac{d}{dx}(p+q\cos x) + (p+q\cos x)\frac{d}{dx}(ax^2+\sin x)$$

$$= (ax^2+\sin x)(-q\sin x) + (p+q\cos x)(2ax+\cos x)$$

$$= -q\sin x(ax^2+\sin x) + (p+q\cos x)(2ax+\cos x)$$

Question 25: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): (x + cos x) (x – tan x)

Solution 25: Let f(x) = (x + cos x) (x – tan x)

By product rule,

$$f'(x) = (x+\cos x)\frac{d}{dx}(x-\tan x) + (x-\tan x)\frac{d}{dx}(x+\cos x)$$

$$= (x+\cos x)\left[\frac{d}{dx}(x) - \frac{d}{dx}(\tan x)\right] + (x-\tan x)(1-\sin x)$$

$$= (x+\cos x)\left[1 - \frac{d}{dx}(\tan x)\right] + (x-\tan x)(1-\sin x) \quad \ldots \text{(i)}$$

Let g(x) = tan x. Accordingly, g(x + h) = tan(x + h)

By first principle,

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos(x+h)}\right)$$

$$=\frac{1}{\cos x}\cdot 1\cdot\left(\frac{1}{\cos(x+0)}\right)$$

$$=\frac{1}{\cos^2 x}$$

$$=\sec^2 x \qquad \ldots\text{(ii)}$$

Therefore, from (i) and (ii) ,We obtain

$$f'(x)=(x+\cos x)(1-\sec^2 x)+(x-\tan x)(1-\sin x)$$

$$=(x+\cos x)(-\tan^2 x)+(x-\tan x)(1-\sin x)$$

$$=-\tan^2 x(x+\cos x)+(x-\tan x)(1-\sin x)$$

Question 26: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{4x+5\sin x}{3x+7\cos x}$

Solution 26: Let f(x) = $\frac{4x+5\sin x}{3x+7\cos x}$

By quotient rule,

$$f'(x)=\frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x)-(4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)\left[4\frac{d}{dx}(x)+5\frac{d}{dx}(\sin x)\right]-(4x+5\sin x)\left[3\frac{d}{dx}(x)+7\frac{d}{dx}(\cos x)\right]}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)[4x+5\cos x]-(4x+5\sin x)[3-7\sin x]}{(3x+7\cos x)^2}$$

$$=\frac{12x+15x\cos x+28x\cos x+35\cos^2 x-12x+28x\sin x-15\sin x+35(\cos^2 x+\sin^2 x)}{(3x+7\cos x)^2}$$

$$=\frac{15x\cos x+28\cos x+28x\sin x-15\sin x+35(\cos^2 x+\sin^2 x)}{(3x+7\cos x)^2}$$

$$=\frac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$$

Question 27: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$

Solution 27: Let f(x) = $\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$

By quotient rule,

$$f'(x)=\cos\left(\frac{\pi}{4}\right)\left[\frac{\sin x\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(\sin x)}{\sin^2 x}\right]$$

$$= \cos\left(\frac{\pi}{4}\right)\left[\frac{\sin x(2x) - x^2(\cos x)}{\sin^2 x}\right]$$

$$= \frac{x\cos\frac{\pi}{4}[2\sin x - x\cos x]}{\sin^2 x}$$

Question 28: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x}{1+\tan x}$

Solution 28: Let f(x) = $\frac{x}{1+\tan x}$

$$f'(x) = \frac{(1+\tan x)\frac{d}{dx}(x) - (x)\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= f'(x) = \frac{(1+\tan x) - x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \quad \ldots(i)$$

Let g(x) = 1 + tan x. .Accordingly, g(x + h) = 1 + tan(x+h).

By first principle,

$$g'(x) = \lim_{h\to 0}\frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{1+\tan(x+h) - 1 - \tan(x)}{h}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x - \sin x\cos x(x+h)}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sinh}{\cos(x+h)\cos x}\right]$$

$$= \left(\lim_{h\to 0}\frac{\sinh}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$= 1\times\frac{1}{\cos^2} = \sec^2 x$$

$$\Rightarrow \frac{d}{dx}(1+\tan^{2} x) = \sec^2 x \qquad \ldots(ii)$$

From (i) and (ii), we obtain

$$f^{'}(x) = \frac{1+\tan x - x\sec^2 x}{(1+\tan x)^2}$$

Question 29: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): (x + sec x) (x – tan x)

Solution 29: Let f(x) = (x + sec x) (x – tan x)

By product rule,

$$f(x) = (x+\sec x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\sec x)$$

$$= (x+\sec x)\left[\frac{d}{dx}(x)-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[\frac{d}{dx}(x)-\frac{d}{dx}\sec x\right]$$

$$= f(x+\sec x)\left[1-\frac{d}{dx}\tan x)\right]+(x-\tan x)\left[1+\frac{d}{dx}\sec x\right] \qquad ...(i)$$

Let f_1 (x) = tan x, f_2 (x) = sec x

Accordingly, f_1 (x + h)-tan(x + h) and f_2 (x + h) = sec (x + h)

$$f_1^{'}(x) = \lim_{h\to 0}\left(\frac{f_1(x+h)-f_1(x)}{h}\right)$$

$$= \lim_{h\to 0}\left[\frac{\tan(x+h)-\tan(x)}{h}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos x(x+h)}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sinh}{\cos(x+h)\cos x}\right]$$

$$= \left(\lim_{h\to 0}\frac{\sinh}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$= 1\times\frac{1}{\cos^2}=\sec^2 x$$

$$\Rightarrow \frac{d}{dx}(1+\tan^2 x)=\sec^2 x \qquad \ldots \text{(ii)}$$

$$f'_2(x)=\lim_{h\to 0}\left(\frac{f_2+(x+h)-f_2(x)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\sec(x+h)-\sec(x)}{h}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\cos x-\cos(x+h)}{\cos(x+h)\cos x}\right)$$

$$=\frac{1}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right]$$

$$= \frac{1}{\cos x} \lim_{h \to 0} \frac{1}{h} \left[\frac{\sin\left(\frac{2x+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}}{\cos(x+h)} \right]$$

$$= \sec x \frac{\left\{\lim_{h \to 0} \sin\left(\frac{2x+h}{2}\right)\right\}\left\{\lim_{\frac{h}{2} \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}}{\lim_{h \to 0} \cos(x+h)}$$

$$= \sec x \cdot \frac{\sin x \cdot 1}{\cos x}$$

$$\Rightarrow \frac{d}{dx} \sec x = \sec x \tan x$$

From (i), (ii), and (iii), we obtain

$$f'(x) = (x + \sec x)(1 - \sec^2 x) + (x - \tan x)(1 + \sec x \tan x)$$

Question 30: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x}{\sin^n x}$

Solution 30: Let f(x) = $\frac{x}{\sin^n x}$

By quotient rule,

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

It can be easily shown that $\frac{d}{dx} \sin^n x = n \sin^{n-1} x \cos x$

Therefore,

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

$$= \frac{\sin^n x.1 - x(n \sin^{n-1} x \cos x)}{\sin^{2n} x}$$

$$= \frac{\sin^{n-1} x(\sin x - nx \cos x)}{\sin^{2n} x}$$

$$= \frac{\sin x - nx \cos x}{\sin^{n+1} x}$$