Miscellaneous Exercise

Question 1:

The relation f is defined by $f(x)=\begin{cases} x^2, & 0\le x\le 3 \\ 3x, & 3\le x\le 10 \end{cases}$

The relation g is defined by $g(x)=\begin{cases} x^2, & 0\le x\le 2 \\ 3x, & 2\le x\le 10 \end{cases}$

Show that f is a function and g is not a function.

Solution 1:

The relation f is defined as

$$f(x)=\begin{cases} x^2, & 0\le x\le 3 \\ 3x, & 3\le x\le 10 \end{cases}$$

It is observed that for

$0\le x<3,\qquad f(x)=x^2$

$3<x\le 10,\qquad f(x)=3x$

Also, at $x=3, f(x)=3^2=9$ or $f(x)=3\times 3=9$ i.e., at $x=3, f(x)=9$

Therefore, for $0\le x\le 10$, the images of $f(x)$ are unique. Thus, the given relation is a function.

The relation g is defined as

$$g(x)=\begin{cases} x^2, & 0\le x\le 2 \\ 3x, & 2\le x\le 10 \end{cases}$$

It can be observed that for $x=2, g(x)=2^2=4$ and $g(x)=3\times 2=6$

Hence, element 2 of the domain of the relation g corresponds to two different images i.e., 4 and 6.

Hence, this relation is not a function.

Question 2:

If $f(x)=x^2$, find $\dfrac{f(1.1)-f(1)}{(1.1-1)}$

Solution 2:

$f(x)=x^2$

$$\therefore \frac{f(1.1)-f(1)}{(1.1-1)}=\frac{(1.1)^2-(1)^2}{(1.1-1)}=\frac{1.21-1}{0.1}=\frac{0.21}{01}=2.1$$

Question 3:

Find the domain of the function $f(x)=\frac{x^2+2x+1}{x^2-8x+12}$

Solution 3:

The given function is $f(x)=\frac{x^2+2x+1}{x^2-8x+12}$

$$f(x)=\frac{x^2+2x+1}{x^2-8x+12}=\frac{x^2+2x+1}{(x-6)(x-2)}$$

It can be seen that function f is defined for all real numbers except at x = 6 and x = 2. Hence, the domain of f is $\mathbf{R}-\{2,6\}$.

Question 4:

Find the domain and the range of the real function f defined by $f(x)=\sqrt{(x-1)}$

Solution 4:

The given real function is $f(x)=\sqrt{(x-1)}$

It can be seen that $\sqrt{(x-1)}$ is defined for $f(x)=x\geq 1$.

Therefore, the domain of f is the set of all real numbers greater than or equal to 1 i.e., the domain of $f=[1,\infty)$.

As $x\geq 1\Rightarrow (x-1)\geq 0\Rightarrow\sqrt{(x-1)}\geq 0$

Therefore, the range of f is the set of all real numbers greater than or equal to 0 i.e., the range of $f=[0,\infty)$.

Question 5:

Find the domain and the range of the real function f defined by $f(x)=|x-1|$.

Solution 5:

The given real function is $f(x)=|x-1|$.

It is clear that $|x-1|$ is defined for all real numbers.

$\therefore$ Domain of $f=\mathrm{R}$

Also, for $x\in\mathrm{R}=|x-1|$ assumes all real numbers.

Hence, the range of *f* is the set of all non-negative real numbers.

Question 6:

Let $f=\left\{\left(x,\frac{x^2}{1+x^2}\right):x\in\mathrm{R}\right\}$

be a function from **R** into **R**. Determine the range of *f*.

Solution 6:

$$f=\left\{\left(x,\frac{x^2}{1+x^2}\right): x\in\mathbf{R}\right\}$$

$$=\left\{(0,0),\left(\pm 0.5,\frac{1}{5}\right),\left(\pm 1,\frac{1}{2}\right),\left(\pm 1.5,\frac{9}{13}\right),\left(\pm 2,\frac{4}{5}\right),\left(3,\frac{9}{10}\right),\left(4,\frac{16}{17}\right),....\right\}$$

The range of f is the set of all second elements. It can be observed that all these elements are greater than or equal to 0 but less than 1.

[Denominator is greater numerator]. Thus, range of f = [0, 1)

Question 7:

Let $f,g:\mathrm{R}\to\mathrm{R}$ be defined, respectively by $f(x)=x+1, g(x)=2x-3$. Find $f+g, f-g$ and $\frac{f}{g}$.

Solution 7:

$f,g:\mathrm{R}\to\mathrm{R}$ is defined as $f(x)=x+1,\ g(x)=2x-3$

$(f+g)(x)=f(x)+g(x)=(x+1)+(2x-3)=3x-2$

$\therefore (f+g)(x)=3x-2$

$(f-g)(x)=f(x)-g(x)=(x+1)-(2x-3)=x+1-2x+3=-x+4$

$\therefore (f-g)(x)=-x+4$

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}, g(x)\neq 0, x\in\mathrm{R}$$

$$\therefore\left(\frac{f}{g}\right)(x)=\frac{x+1}{2x-3}, 2x-3\neq 0 \text{ or } 2x\neq 3$$

$$\therefore\left(\frac{f}{g}\right)(x)=\frac{x+1}{2x-3}, x\neq\frac{3}{2}$$

Question 8:

Let $f=\{(1,1),(2,3),(0,-1),(-1,-3)\}$ be a function from **Z** to **Z** defined by $f(x)=ax+b$, for some integers a,b. Determine a,b.

Solution 8:

$f=\{(1,1),(2,3),(0,-1),(-1,-3)\}$ and $f(x)=ax+b$

$(1,1)\in f \Rightarrow f(1)=1 \Rightarrow a\times 1+b=1$

$\Rightarrow a+b=1$

$(0,-1)\in f \Rightarrow f(0)=-1 \Rightarrow a\times 0+b=-1$

On substituting $b=-1$ in $a+b=1$

We obtain $a+(-1)=1 \Rightarrow a=1+1=2$. Thus, the respective values of a and b are 2 and -1.

Question 9:

Let R be a relation from **N** to **N** defined by $R=\{(a,b): a,b \in \mathbf{N} \text{ and } a=b^2\}$. Are the following true?

(i) $(a,a) \in R$, for all $a \in \mathbf{N}$

(ii) $(a,b) \in R$, implies $(b,a) \in R$

(iii) $(a,b) \in R, (b,c) \in R$ implies $(a,c) \in R$.

Justify your answer in each case.

Solution 9:

$R=\{(a,b): a,b \in \mathbf{N} \text{ and } a=b^2\}$

(i) It can be seen that $2 \in \mathbf{N}$; however, $2 \neq 2^2 = 4$.

Therefore, the statement "$(a,a) \in R$, for all $a \in \mathbf{N}$" is not true.

(ii) It can be seen that $(9,3) \in \mathbf{N}$ because $9,3 \in \mathbf{N}$ and $9=3^2$. Now, $3 \neq 9^2 = 81$; therefore, $(3,9) \notin \mathbf{N}$

Therefore, the statement "$(a,b) \in R$, implies "$(b,a) \in R$" is not true.

(iii) It can be seen that $(9,3) \in R, (16,4) \in R$ because $9,3,16,4 \in \mathbf{N}$ and $9=3^2$ and $16=4^2$.

Now, $9 \neq 4^2 = 16$; therefore, $(9,4) \notin \mathbf{N}$

Therefore, the statement "$(a,b) \in R, (b,c) \in R$ implies $(a,c) \in R$" is not true.

Question 10:

Let $A=\{1,2,3,4\}, B=\{1,5,9,11,15,16\}$ and $f=\{(1,5),(2,9),(3,1),(4,5),(2,11)\}$. Are the following true?

(i) f is a relation from A to B

(ii) f is a function from A to B

Justify your answer in each case.

Solution 10:

$A=\{1,2,3,4\}$ and $B=\{1,5,9,11,15,16\}$

$$\therefore\ A \times B = \{(1,1),(1,5),(1,9),(1,11),(1,15),(1,16),(2,1),(2,5),(2,9),(2,11),(2,15),(2,16)$$
$$(3,1),(3,5),(3,9),(3,11),(3,15),(3,16),(4,1),(4,5),(4,9),(4,11),(4,15),(4,16)\}$$

It is given that $f=\{(1,5),(2,9),(3,1),(4,5),(2,11)\}$

(i) A relation from a non-empty set A to a non-empty set B is a subset of the Cartesian product $A \times B$.

Thus, f is a relation from A to B.

(ii) Since the same first element i.e, 2 corresponds to two different images i.e., 9 and 11, relation f is not a function.

Question 11:

Let f be the subset of $\mathbf{Z}\times\mathbf{Z}$ defined by $f=\{(ab,a+b):a,b\in\mathbf{Z}\}$. If f a function from **Z** to **Z**: Justify your answer.

Solution 11:

The relation *f* is defined as $f=\{(ab,a+b):a,b\in\mathbf{Z}\}$

We know that a relation *f* from a set A to a set B is said to be a function if every element of set A has unique images in set B.

Since $(2,6,-2,-6\in Z,(2\times6,2+6),(-2\times-6,-2+-6))\in f$ i.e., $(12,8),(12,-8)\in f$

It can be seen that the same first element i.e., 12 corresponds to two different images i.e., 8 and -8. Thus, relation *f* is not a function.

Question 12:

Let $A=\{9,10,11,12,13\}$ and let $f:A\to\mathbf{N}$ be defined by $f(n)=$ the highest prime factor of *n*. Find the range of *f*.

Solution 12:

$A=\{9,10,11,12,13\}$ and let $f:A\to\mathbf{N}$ is defined as $f(n)=$ The highest prime factor of *n*

Prime factor of 9 = 3

Prime factors of 10 = 2, 5

Prime factor of 11 = 11

Prime factor of 12 = 2, 3

Prime factor of 13 = 13

$\therefore\ f(9)=$ The highest prime factor of 9 = 3

$f(10)=$ The highest prime factor of 10 = 5

$f(11)=$ The highest prime factor of 11 = 11

$f(12)=$ The highest prime factor 12 = 3

$f(13)=$ The highest prime factor of 13 = 13

The range of *f* is the set of all $f(n)$, where $n\in A$.

$\therefore$ Range of $f=\{3,5,11,13\}$
