Exercise 9.4

Question 1:

Find the sum to n terms of the series $1\times2+2\times3+3\times4+4\times5+....$

Solution 1:

The given series is $1\times2+2\times3+3\times4+4\times$
$5+.....n^{th}$ term, $a_n=n(n+1)$

$$\therefore S_n=\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}k(k+1)$$

$$=\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k$$

$$=\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{2}\left(\frac{2n+1}{3}+1\right)$$

$$=\frac{n(n+1)}{2}\left(\frac{2n+4}{3}\right)$$

$$=\frac{n(n+1)(n+2)}{3}$$

Question 2:

Find the sum to n terms of the series $1\times2\times3+2\times3\times4+3\times4\times5+....$

Solution 2:

The given series is $1\times2\times3+2\times3\times4+3\times4\times5+.....n^{th}$ term,

$$a_n=n(n+1)(n+2)$$

$$=(n^2+n)(n+2)$$

$$=n^3+3n^2+2n$$

$$\therefore S_n=\sum_{k=1}^{n}a_k$$

$$=\sum_{k=1}^{n}k^3+3\sum_{k=1}^{n}k^2+2\sum_{k=1}^{n}k$$

$$=\left[\frac{n(n+1)}{2}\right]^2+\frac{3n(n+1)(2n+1)}{6}+\frac{2n(n+1)}{2}$$

$$=\left[\frac{n(n+1)}{2}\right]^2+\frac{n(n+1)(2n+1)}{2}+n(n+1)$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + 2n + 1 + 2\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{n^2 + n + 4n + 6}{2}\right]$$

$$= \frac{n(n+1)}{4}\left(n^2 + 5n + 6\right)$$

$$= \frac{n(n+1)}{4}\left(n^2 + 2n + 3n + 6\right)$$

$$= \frac{n(n+1)\left[n(n+2) + 3(n+2)\right]}{4}$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

Question 3:

Find the sum to n terms of the series $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots$

Solution 3:

The given series is $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots\ldots n^{th}$ term,

$$a_n = (2n+1)n^2 = 2n^3 + n^2$$

$$\therefore S_n = \sum_{k=1}^{n} a_k$$

$$= \sum_{k=1}^{n} = \left(2k^3 + k^2\right) = 2\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2$$

$$= 2\left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n^2(n+1)^2}{2} + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2}\left[n(n+1) + \frac{2n+1}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2 + 3n + 2n + 1}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2 + 5n + 1}{3}\right]$$

$$= \frac{n(n+1)\left(3n^2 + 5n + 1\right)}{6}$$

Question 4:

Find the sum to n terms of the series $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+.....$

Solution 4:

The given series is $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+.....$

n^{th} term, $a_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$ [By partial fractions]

$a_1=\frac{1}{1}-\frac{1}{2}$

$a_2=\frac{1}{2}-\frac{1}{3}$

$a_3=\frac{1}{3}-\frac{1}{4}....$

$a_n=\frac{1}{n}-\frac{1}{n+1}$

Adding the above terms column wise, we obtain

$$a_1+a_2+....+a_n=\left[\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+....\frac{1}{n}\right]-\left[\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.....\frac{1}{n+1}\right]$$

$$\therefore S_n=1-\frac{1}{n+1}=\frac{n+1-1}{n+1}=\frac{n}{n+1}$$

Question 5:

Find the sum to n terms of the series $5^2+6^2+7^2+.....+20^2$

Solution 5:

The given series is $5^2+6^2+7^2+.....+20^2$ n^{th} term,

$$a_n=(n+4)^2=n^2+8n+16$$

$$\therefore\ S_n=\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}(k^2+8k+16)$$

$$=\sum_{k=1}^{n}k^2+8\sum_{k=1}^{n}k+\sum_{k=1}^{n}16$$

$$=\frac{n(n+1)(2n+1)}{6}+\frac{8n(n+1)}{2}+16n$$

16^{th} term is $(16+4)^2=20^2$

$$\therefore S_{16}=\frac{16(16+1)(2\times 16+1)}{6}+\frac{8\times 16\times(16+1)}{2}+16\times 16$$

$$=\frac{(16)(17)(33)}{6}+\frac{(8)\times 16\times(16+1)}{2}+16\times 16$$

$$= \frac{(16)(17)(33)}{6} + \frac{(8)(16)(17)}{2} + 256$$
$$= 1496 + 1088 + 256$$
$$= 2840$$
$$\therefore 5^2 + 6^2 + 7^2 + \dots\dots + 20^2 = 2840.$$

Question 6:

Find the sum to n terms of the series $3\times8+6\times11+9\times14+\dots$

Solution 6:

The given series is $3\times8+6\times11+9\times14+\dots\dots a_n$

$$= \left(n^{th} \text{ term of } 3,6,9\dots.\right)\times\left(n^{th} \text{ term of } 8,11,14\dots.\right)$$
$$= (3n)(3n+5)$$
$$= 9n^2 + 15n$$
$$\therefore S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}\left(9k^2+15k\right)$$
$$= \sum_{k=1}^{n} k^2 = 15\sum_{k=1}^{n} k$$
$$= 9\times\frac{n(n+1)(2n+1)}{6} + 15\times\frac{n(n+1)}{2}$$
$$= \frac{3n(n+1)(2n+1)}{2} + \frac{15n(n+1)}{2}$$
$$= \frac{3n(n+1)}{2}(2n+1+5)$$
$$= \frac{3n(n+1)}{2}(2n+6)$$
$$= 3n(n+1)(n+3)$$

Question 7:

Find the sum to n terms of series $1^2+\left(1^2+2^2\right)+\left(1^2+2^2+3^2\right)+\dots.$

Solution 7:

The given series is $1^2+\left(1^2+2^2\right)+\left(1^2+2^2+3^2\right)+\dots.a_n$

$$= \left(1^2+2^2+3^3+\dots..+n^2\right)$$
$$= \frac{n(n+1)(2n+1)}{6}$$

$$=\frac{n\left(2n^2+3n+1\right)}{6}=\frac{2n^3+3n^2+n}{6}$$
$$=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$$
$$\therefore S_n=\sum_{k=1}^{n}a_k$$
$$=\sum_{k=1}^{n}\left(\frac{1}{3}k^3+\frac{1}{2}k^2+\frac{1}{6}k\right)$$
$$=\frac{1}{3}\sum_{k=1}^{n}k^3+\frac{1}{2}\sum_{k=1}^{n}k^2+\frac{1}{6}\sum_{k=1}^{n}k$$
$$=\frac{1}{3}\frac{n^2(n+1)^2}{(2)^2}+\frac{1}{2}\times\frac{n(n+1)(2n+1)}{6}+\frac{1}{6}\times\frac{n(n+1)}{2}$$
$$=\frac{n(n+1)}{6}\left[\frac{n(n+1)}{2}+\frac{(2n+1)}{2}+\frac{1}{2}\right]$$
$$=\frac{n(n+1)}{6}\left[\frac{n^2+n+2n+1+1}{2}\right]$$
$$=\frac{n(n+1)}{6}\left[\frac{n^2+n+2n+2}{2}\right]$$
$$=\frac{n(n+1)}{6}\left[\frac{n(n+1)+2(n+1)}{2}\right]$$
$$=\frac{n(n+1)}{6}\left[\frac{(n+1)(n+2)}{2}\right]$$
$$=\frac{n(n+1)^2(n+2)}{12}$$

Question 8:

Find the sum to n terms of the series whose n^{th} term is given by $n(n+1)(n+4)$.

Solution 8:

$$a_n=n(n+1)(n+4)=n\left(n^2+5n+4\right)=n^3+5n^2+4n$$
$$\therefore\ S_n=\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}k^3+5\sum_{k=1}^{n}k^2+4\sum_{k=1}^{n}k$$
$$=\frac{n^2(n+1)^2}{4}+\frac{5n(n+1)(2n+1)}{6}+\frac{4n(n+1)}{2}$$
$$=\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{5(2n+1)}{3}+4\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{3n^2+3n+20n+10+24}{6}\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{3n^2+23n+34}{6}\right]$$

$$=\frac{n(n+1)(3n^2+23n+34)}{12}$$

Question 9:

Find the sum to n terms of these series whose n^{th} terms is given by n^2+2^n

Solution 9:

$a_n=n^2+2^n$

$$\therefore S_n=\sum_{k=1}^{n}k^2+2^k=\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}2^k \qquad ......(1)$$

Consider $\sum_{k=1}^{n}2^k=2^1+2^2+2^3+....$

The above series $2^2+2^3....$ is a G.P. with both the first term and common ratio equal to 2.

$$\therefore \sum_{k=1}^{n}2^k=\frac{(2)\left[(2)^n-1\right]}{2-1}=2(2^n-1) \; ....(2)$$

Therefore, from (1) and (2), we obtain

$$S_n=\sum_{k=1}^{n}k^2+2(2^n-1)=\frac{n(n+1)(2n+1)}{6}+2(2^n-1)$$

Question 10:

Find the sum to n terms of the series whose n^{th} terms is given by $(2n-1)^2$

Solution 10:

$a_n=(2n-1)^2=4n^2-4n+1$

$$\therefore S_n=\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}(4k^2-4k+1)$$

$$=4\sum_{k=1}^{n}k^2-4\sum_{k=1}^{n}k+\sum_{k=1}^{n}1$$

$$=\frac{4n(n+1)(2n+1)}{6}-\frac{4n(n+1)}{2}+n$$

$$=\frac{2n(n+1)(2n+1)}{3}-2n(n+1)+n$$

$$= n\left[\frac{2\left(2n^2+3n+1\right)}{3} - 2(n+1) + 1\right]$$

$$= n\left[\frac{4n^2+6n+2-6n-6+3}{3}\right]$$

$$= n\left[\frac{4n^2-1}{3}\right]$$

$$= \frac{n(2n+1)(2n-1)}{3}$$

Miscellaneous Exercise

Question 1:

Show that the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an A.P. is equal to twice the m^{th} term.

Solution 1:

Let a and d be the first term and the common difference of the A.P. respectively. It is known that the k^{th} term of an A.P. is given by

$$a_k = a + (k-1)d$$

$$\therefore a_{m+n} = a + (m+n-1)d$$

$$a_{m-n} = a + (m-n-1)d$$

$$a_m = a + (m-1)d$$

$$\therefore a_{m+n} + a_{m-n} = a + (m+n-1)d + a + (m-n-1)d$$

$$= 2a + (m+n-1+m-n-1)d$$

$$= 2a + (2m-2)d$$

$$= 2a + 2(m-1)d$$

$$= 2\left[a + (m-1)d\right]$$

$$= 2a_m$$

Thus, the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an A.P. is equal to twice the m^{th} term.

Question 2:

Let the sum of three numbers in A.P., is 24 and their product is 440, find the numbers.

Solution 2:

Let the three numbers in A.P. be $a-d, a,$ and $a+d$.

According to the given information,