Exercise 8.2

Question 1:

Find the coefficient of x^5 in $(x+3)^8$

Solution 1:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$

Assuming that x^5 occurs in the $(r+1)^{th}$ term of the expansion $(x+3)^8$, we obtain

$T_{r+1} = {}^8C_r (x)^{8-r} (3)^r$

Comparing the indices of x in x^5 in T_{r+1},

We obtain r = 3

Thus, the coefficient of x^5 is ${}^8C_3 (3)^3 = \frac{8!}{3!5!} \times 3^3 = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3 \cdot 2.5!} \cdot 3^3 = 1512$.

Question 2:

Find the coefficient of a^5b^7 in $(a-2b)^{12}$

Solution 2:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$

Assuming that a^5b^7 occurs in the $(r+1)^{th}$ term of the expansion $(a-2b)^{12}$, we obtain

$T_{r+1} = {}^{12}C_r (a)^{12-r} (-2b)^r = {}^{12}C_r (-2)^r (a)^{12-r} (b)^r$

Comparing the indices of a and b in a^5b^7 in T_{r+1},

We obtain r = 7

Thus, the coefficient of a^5b^7 is

${}^{12}C_7 (-2)^7 = \frac{12!}{7!5!} \cdot 2^7 = \frac{12 \cdot 11.10 \cdot 9 \cdot 8 \cdot 7!}{5 \cdot 4.3 \cdot 2 \cdot 7!} \cdot (-2)^7 = -(792)(128) = -101376$.

Question 3:

Write the general term in the expansion of $(x^2 - y)^6$

Solution 3:

It is known that the general term T_{r+1} {which is the $(r+1)^{th}$ term} in the binomial expansion of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r\, a^{n-r} b^r$.

Thus, the general term in the expansion of $(x^2 - y^6)$ is

$$T_{r+1} = {}^6C_r \left(x^2\right)^{6-r} (-y)^r = (-1)^r\, {}^6C_r . x^{12-2r} . y^r$$

Question 4:

Write the general term in the expansion of $(x^2 - yx)^{12}$, $x \neq 0$

Solution 4:

It is known that the general term T_{r+1} {which is the $(r+1)^{th}$ term} in the binomial expansion of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r\, a^{n-r} b^r$.

Thus, the general term in the expansion of $(x^2 - yx)^{12}$ is

$$T_{r+1} = {}^{12}C_r \left(x^2\right)^{12-r} (-yx)^r = (-1)^r\, {}^{12}C_r . x^{24-2r} . y^r = (-1)^r\, {}^{12}C_r . x^{24-r} . y^r$$

Question 5:

Find the 4th term in the expansion of $(x-2y)^{12}$.

Solution 5:

It is known $(r+1)^{th}$ term, T_{r+1}, in the binomial expansion of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r\, a^{n-r} b^r$.

Thus, the 4th term in the expansion of $(x^2 - 2y)^{12}$ is

$$T_4 = T_{3+1} = {}^{12}C_3 (x)^{12-3} (-2y)^3 = (-1)^3 \cdot \frac{12!}{3!9!} \cdot x^9 \cdot (2)^3 \cdot y^3 = -\frac{12 \cdot 11 \cdot 10}{3 \cdot 2} \cdot (2)^3 x^9 y^3 = -1760 x^9 y^3$$

Question 6:

Find the 13th term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}, x \neq 0$

Solution 6:

It is known $(r+1)^{th}$ term, T_{r+1}, in the binomial expansion of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r\, a^{n-r} b^r$.

Thus, the 13^{th} term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$ is

$$T_{13} = T_{12+1} = {}^{18}C_{12}(9x)^{18-12}\left(-\frac{1}{3\sqrt{x}}\right)^{12}$$

$$= (-1)^{12}\frac{18!}{12!6!}(9)^6(x)^6\left(\frac{1}{3}\right)^{12}\left(\frac{1}{\sqrt{x}}\right)^{12}$$

$$= \frac{18\cdot17\cdot16\cdot15\cdot14\cdot13.12!}{12!.6\cdot5\cdot4\cdot3\cdot2}\cdot x^6\left(\frac{1}{x^6}\right).3^{12}\left(\frac{1}{3^{12}}\right) \qquad \left[9^6 = \left(3^2\right)^6 = 3^{12}\right]$$

$$= 18564$$

Question 7:

Find the middle terms in the expansions of $\left(3 - \frac{x^3}{6}\right)^7$

Solution 7:

It is known that in the expansion of $(a+b)^n$, in n is odd, then there are two middle terms,

Namely $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term.

Therefore, the middle terms in the expansion $\left(3 - \frac{x^3}{6}\right)^7$ are $\left(\frac{7+1}{2}\right)^{th} = 4^{th}$ and $\left(\frac{7+1}{2}+1\right)^{th} = 5^{th}$ term

$$T_4 = T_{3+1} = {}^7C_3(3)^{7-3}\left(-\frac{x^3}{6}\right)^3 = (-1)^3\frac{7!}{3!4!}\cdot3^4\cdot\frac{x^9}{6^3}$$

$$= -\frac{7\cdot6\cdot5.4!}{3\cdot2.4!}\cdot3^4\cdot\frac{1}{2^3\cdot3^3}\cdot x^9 = -\frac{105}{8}x^9$$

$$T_5 = T_{4+1} = {}^7C_4(3)^{7-4}\left(-\frac{x^3}{6}\right)^4 = (-1)^4\frac{7!}{4!3!}\cdot3^3\cdot\frac{x^{12}}{6^4}$$

$$= \frac{7\cdot6\cdot5.4!}{4!.3\cdot2}\cdot\frac{3^3}{2^4\cdot3^4}\cdot x^{12} = \frac{35}{48}x^{12}$$

Thus, the middle terms in the expansion of $\left(3 - \frac{x^3}{6}\right)^7$ are $-\frac{105}{8}x^9$ and $\frac{35}{48}x^{12}$.

Question 8:

Find the middle terms in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$

Solution 8:

It is known that in the expansion of $(a+b)^n$, in n is even, then the middle term is $\left(\frac{n}{2}+1\right)^{th}$ term.

Therefore, the middle term in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ is $\left(\frac{10}{2}+1\right)^{th}=6^{th}$

$$T_4=T_{5+1}={}^{10}C_5\left(\frac{x}{3}\right)^{10-5}(9y)^5=\frac{10!}{5!5!}\cdot\frac{x^5}{3^5}\cdot 9^5\cdot y^5$$

$$=\frac{10\cdot 9\cdot 8\cdot 7\cdot 6.5!}{5\cdot 4\cdot 3\cdot 2.5!}\cdot\frac{1}{3^5}\cdot 3^{10}\cdot x^5y^5 \qquad \left[9^5=\left(3^2\right)^5=3^{10}\right]$$

$$=252\times 3^5\cdot x^5\cdot y^5=6123x^5y^5$$

Thus, the middle term in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ is $61236x^5y^5$.

Question 9:

In the expansion of $(1+a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

Solution 9:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by

$T_{r+1}={}^nC_r a^{n-r}b^r$

Assuming that a^m occurs in the $(r+1)^{th}$ term of the expansion $(1+a)^{m+n}$, we obtain

$T_{r+1}={}^{m+n}C_r(1)^{m+n-r}(a)^r={}^{m+n}C_r a^r$

Comparing the indices of a in a^m in T_{r+1},

We obtain r = m

Therefore, the coefficient of a^m is

$${}^{m+n}C_m=\frac{(m+n)!}{m!(m+n-m)!}=\frac{(m+n)!}{m!n!}\ldots\ldots(1)$$

Assuming that a^n occurs in the $(k+1)^{th}$ term of the expansion $(1+a)^{m+n}$, we obtain

$T_{k+1}={}^{m+n}C_k(1)^{m+n-k}(a)^k={}^{m+n}C_k(a)^k$

Comparing the indices of a in a^n and in T_{k+1},

We obtain

$k=n$

Therefore, the coefficient of a^n is

$$^{m+n}C_n = \frac{(m+n)!}{n!(m+n-n)!} = \frac{(m+n)!}{n!m!} ....(2)$$

Thus, from (1) and (2), it can be observed that the coefficients of a^m and a^n in the expansion of $(1+a)^{m+n}$ are equal.

Question 10:

The coefficients of the $(r-1)^{th}$, r^{th} and $(r+1)^{th}$ terms in the expansion of $(x+1)^n$ are in the ratio $1:3:5$. Find n and r.

Solution 10:

It is known that $(k+1)^{th}$ term, (T_{k+1}), in the binomial expansion of $(a+b)^n$ is given by

$T_{k+1} = {}^nC_k a^{n-k} b^k$.

Therefore, $(r-1)^{th}$ term in the expansion of $(x+1)^n$ is

$$T_{r-1} = {}^nC_{r-2}(x)^{n-(r-2)}(1)^{(r-2)} = {}^nC_{r-2}x^{n-r+2}$$

$(r+1)$ term in the expansion of $(x+1)^n$ is

$$T_{r+1} = {}^nC_r (x)^{n-r}(1)^r = {}^nC_r x^{n-r}$$

r^{th} term in the expansion of $(x+1)^n$ is

$$T_r = {}^nC_{r-1}(x)^{n-(r-1)}(1)^{(r-1)} = {}^nC_{r-1}x^{n-r+1}$$

Therefore, the coefficients of the $(r-1)^{th}$, r^{th} and $(r+1)^{th}$ terms in the expansion of $(x+1)^n$ ${}^nC_{r-2}$, ${}^nC_{r-1}$, and ${}^nC_r$ are respectively. Since these coefficients are in the ratio $1:3:5$, we obtain

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3} \text{ and } \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{5}$$

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{n!}{(r-2)!(n-r+2)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{(r-1)(r-2)!(n-r+1)!}{(r-2)!(n-r+2)!(n-r+1)!}$$

$$= \frac{r-1}{n-r+2}$$

$$\therefore \frac{r-1}{n-r+2} = \frac{1}{3}$$

$$\Rightarrow 3r-3 = n-r+2$$

$$\Rightarrow n-4r+5 = 0 \ ......(1)$$

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{n!}{(r-1)!(n-r+1)} \times \frac{r!(n-r)!}{n!} = \frac{r(r-1)!(n-r)!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{r}{n-r+1}$$

$\therefore \dfrac{r}{n-r+1}=\dfrac{3}{5}$

$\Rightarrow 5r=3n-3r+3$

$\Rightarrow 3n-8r+3=0 \qquad ....(2)$

Multiplying (1) by 3 and subtracting it from (2), we obtain

$4r-12=0$

$\Rightarrow r=3$

Putting the value of r in (1), we obtain n

$-12+5=0$

$\Rightarrow n=7$

Thus, $n=7$ and $r=3$

Question 11:

Prove that the coefficient of x^n in the expansion of $(1+x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1+x)^{2n-1}$.

Solution 11:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by

$T_{r+1}={}^nC_r a^{n-r} b^r$.

Assuming that x^n occurs in the $(r+1)^{th}$ term of the expansion of $(1+x)^{2n}$, we obtain

$T_{r+1}={}^{2n}C_r (1)^{2n-r}(x)^r={}^{2n}C_r (x)^r$

Comparing the indices of x in x^n and in T_{r+1}, we obtain $r=n$

Therefore, the coefficient of x^n in the expansion of $(1+x)^{2n}$ is

$${}^{2n}C_n=\frac{(2n)!}{n!(2n-n)!}=\frac{(2n)!}{n!n!}=\frac{(2n)!}{(n!)^2} \quad ....(1)$$

Assuming that x^n occurs in the $(k+1)^{th}$ term of the expansion of $(1+x)^{2n-1}$, we obtain

$T_{k+1}={}^{2n}C_k (1)^{2n-1-k}(x)^k={}^{2n}C_k (x)^k$

Comparing the indices of x in x^n and in T_{k+1}, we obtain $k=n$

Therefore, the coefficient of x^n in the expansion of $(1+x)^{2n-1}$ is

$${}^{2n-1}C_n=\frac{(2n-1)!}{n!(2n-1-n)!}=\frac{(2n-1)!}{n!(n-1)!}$$

$$=\frac{2n.(2n-1)!}{2n.n!(n-1)!}=\frac{(2n)!}{2.n!n!}=\frac{1}{2}\left[\frac{(2n)!}{(n!)^2}\right] \quad ....(2)$$

From (1) and (2), it is observed that

$$\frac{1}{2}\left({}^{2n}C_n\right)={}^{2n-1}C_n$$

$\Rightarrow {}^{2n}C_n = 2\left({}^{2n-1}C_n\right)$

Therefore, the coefficient of x^n expansion of $(1+x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1+x)^{2n-1}$.

Hence proved.

Question 12:

Find a positive value of m for which the coefficient of x^2 in the expansion $(1+x)^m$ is 6.

Solution 12:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Assuming that x^2 occurs in the $(r+1)^{th}$ term of the expansion of $(1+x)^m$, we obtain

$T_{r+1} = {}^mC_r (1)^{m-r} (x)^r = {}^mC_r (x)^r$

Comparing the indices of x in x^2 and in T_{r+1}, we obtain $r = 2$

Therefore, the coefficient of x^2 is ${}^mC_2$

It is given that the coefficient of x^2 in the expansion $(1+x)^m$ is 6.

$\therefore {}^mC_2 = 6$

$\Rightarrow \frac{m!}{2!(m-2)!} = 6$

$\Rightarrow \frac{m(m-1)(m-2)!}{2 \times (m-2)!} = 6$

$\Rightarrow m(m-1) = 12$

$\Rightarrow m^2 - m - 12 = 0$

$\Rightarrow m^2 - 4m + 3m - 12 = 0$

$\Rightarrow m(m-4) + 3(m-4) = 0$

$\Rightarrow (m-4)(m+3) = 0$

$\Rightarrow (m-4) = 0 \; or \; (m+3) = 0$

$\Rightarrow m = 4$ or $m = -3$

Thus, the positive value of m, for which the coefficient of x^2 in the expansion $(1+x)^m$ is 6, is 4.