Exercise 9.6

In each of the following differential equations given in Exercises 1 to 4, find the general solution:

1. $\dfrac{dy}{dx} + 2y = \sin x$

Sol. The given differential equation is $\dfrac{dy}{dx} + 2y = \sin x$

| Standard form of linear differential equation

Comparing with $\dfrac{dy}{dx} + Py = Q$, we have P = 2 and Q = sin x

$\int P\,dx = \int 2\,dx = 2\int 1\,dx = 2x$ I.F. $= e^{\int P\,dx} = e^{2x}$

Solution is $y(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c$

or $y\,e^{2x} = \int e^{2x}\sin x\,dx + c$

or $y\,e^{2x} = I + c$ ...(*i*)

where $I = \int \underset{\text{I}}{e^{2x}}\ \underset{\text{II}}{\sin x}\,dx$ ....(*ii*)

Applying Product Rule of Integration

$$\left[\int \text{I}\,.\,\text{II}\,dx = \text{I}\int \text{II}\,dx - \int\left(\frac{d}{dx}(\text{I})\int \text{II}\,dx\right)dx\right],$$

$$= e^{2x}(-\cos x) - \int 2e^{2x}(-\cos x)\,dx$$

or $I = -e^{2x}\cos x + 2\int \underset{\text{I}}{e^{2x}}\ \underset{\text{II}}{\cos x}\,dx$

Again applying Product Rule,

$$I = -e^{2x}\cos x + 2\left[e^{2x}\sin x - \int 2e^{2x}\sin x\,dx\right]$$

$\Rightarrow$ $I = -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx$

or $I = e^{2x}(-\cos x + 2\sin x) - 4I$

Transposing $5I = e^{2x}(2\sin x - \cos x)$

$\therefore$ $I = \dfrac{e^{2x}}{5}(2\sin x - \cos x)$

Putting this value of I in (*i*), the required solution is

$$y\,e^{2x} = \frac{e^{2x}}{5}(2\sin x - \cos x) + c$$

Dividing every term by e^{2x}, $y = \dfrac{1}{5}(2\sin x - \cos x) + \dfrac{c}{(e^{2x})}$

or $y = \dfrac{1}{5}(2\sin x - \cos x) + c\,e^{-2x}$

which is the required general solution.

2. $\dfrac{dy}{dx} + 3y = e^{-2x}$

Sol. The given differential equation is $\dfrac{dy}{dx} + 3y = e^{-2x}$

| Standard form of linear differential equation

Comparing with $\dfrac{dy}{dx}$ + Py = Q, we have P = 3 and Q = e^{-2x}

$\int P\ dx = \int 3\ dx = 3\int 1\ dx = 3x$ I.F. = $e^{\int P\,dx} = e^{3x}$

Solution is $y(\text{I.F.}) = \int Q\,(\text{I.F.})\ dx + c$

or $y\ e^{3x} = \int e^{-2x}\ e^{3x}\ dx + c$ or $= \int e^{-2x+3x}\ dx + c = \int e^{x}\ dx + c$

or $y\ e^{3x} = e^{x} + c$

Dividing every term by e^{3x},

$y = \dfrac{e^{x}}{e^{3x}} + \dfrac{c}{e^{3x}}$ or $y = e^{-2x} + ce^{-3x}$

which is the required general solution.

3. $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$

Sol. The given differential equation is $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$

It is of the form $\dfrac{dy}{dx}$ + Py = Q Comparing P = $\dfrac{1}{x}$, Q = x^2

$\int P\ dx = \int \dfrac{1}{x}\ dx = \log x$ $\therefore$ I.F. = $e^{\int P\,dx} = e^{\log x} = x$

The general solution is $y(\text{I.F.}) = \int Q\,(\text{I.F.})\ dx + c$

or $yx = \int x^2 . x\ dx + c = \int x^3\,dx + c$ or $xy = \dfrac{x^4}{4} + c.$

4. $\dfrac{dy}{dx} + (\sec x)\ y = \tan x \left(0 \le x < \dfrac{\pi}{2}\right)$

Sol. The given differential equation is $\dfrac{dy}{dx} + (\sec x)\ y = \tan x$

It is of the form $\dfrac{dy}{dx}$ + Py = Q.

Comparing P = sec x, Q = tan x

$\int P\ dx = \int \sec x\ dx = \log(\sec x + \tan x)$

I.F. = $e^{\int P\,dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$

The general solution is $y(\text{I.F.}) = \int Q\,(\text{I.F.})\ dx + c$

or $y(\sec x + \tan x) = \int \tan x\ (\sec x + \tan x)\ dx + c$

$= \int (\sec x \tan x + \tan^2 x)\ dx + c = \int (\sec x \tan x + \sec^2 x - 1)\ dx + c$

$= \sec x + \tan x - x + c$

or $\quad y(\sec x + \tan x) = \sec x + \tan x - x + c.$

For each of the following differential equations given in Exercises 5 to 8, find the general solution:

5. $\cos^2 x \dfrac{dy}{dx} + y = \tan x \left(0 \le x < \dfrac{\pi}{2}\right)$

Sol. The given differential equation is $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

Dividing throughout by $\cos^2 x$ to make the coefficient of $\dfrac{dy}{dx}$ unity,

$$\frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x} \Rightarrow \frac{dy}{dx} + (\sec^2 x)\, y = \sec^2 x \tan x$$

It is of the form $\dfrac{dy}{dx} + \text{P}y = \text{Q}$.

Comparing $\text{P} = \sec^2 x$, $\text{Q} = \sec^2 x \tan x$

$\int \text{P}\, dx = \int \sec^2 x\, dx = \tan x \qquad \text{I.F.} = e^{\int \text{P}\, dx} = e^{\tan x}$

The general solution is $y(\text{I.F.}) = \int \text{Q}\,(\text{I.F.})\, dx + c$

or $\quad y e^{\tan x} = \int \sec^2 x \tan x \,.\, e^{\tan x} dx + c \qquad ...(i)$

Put $\quad \tan x = t$. Differentiating $\sec^2 x\, dx = dt$

$\therefore \int \sec^2 x \tan x\, e^{\tan x} dx = \int \underset{\text{I}}{t}\ \underset{\text{II}}{e^t}\, dt$

Applying integration by Product Rule,

$= t \,.\, e^t - \int 1 \,.\, e^t dt = t \,.\, e^t - e^t = (t - 1)\, e^t = (\tan x - 1)\, e^{\tan x}$

Putting this value in eqn. (i), $y e^{\tan x} = (\tan x - 1)\, e^{\tan x} + c$

Dividing every term by $e^{\tan x}$,

$y = (\tan x - 1) + c e^{-\tan x}$ which is the required general solution.

6. $x \dfrac{dy}{dx} + 2y = x^2 \log x$

Sol. The given differential equation is $x \dfrac{dy}{dx} + 2y = x^2 \log x$

Dividing every term by x $\left(\text{To make coeff. of } \dfrac{dy}{dx} \text{ unity}\right)$

$$\frac{dy}{dx} + \frac{2}{x} y = x \log x$$

It is of the form $\dfrac{dy}{dx} + \text{P}y = \text{Q}$.

Comparing $\text{P} = \dfrac{2}{x}$, $\text{Q} = x \log x \qquad \int \text{P}\, dx = 2 \int \dfrac{1}{x} dx = 2 \log x$

$$\text{I.F.} = e^{\int \text{P}\, dx} = e^{2 \log x} = e^{\log x^2} = x^2 \quad | \because e^{\log f(x)} = f(x)$$

The general solution is $y(\text{I.F.}) = \int \text{Q}\,(\text{I.F.})\, dx + c$

or $\quad y x^2 = \int (x \log x) \,.\, x^2 dx + c = \int (\log x) \,.\, x^3 dx + c$

$$= \log x \cdot \frac{x^4}{4} - \int \frac{1}{x} \cdot \frac{x^4}{4}\, dx + c \qquad = \frac{x^4}{4} \log x - \frac{1}{4} \int x^3\, dx + c$$

or $\qquad yx^2 = \frac{x^4}{4} \log x - \frac{x^4}{16} + c.$

Dividing by x^2, $y = \frac{x^2}{4} \log x - \frac{x^2}{16} + \frac{c}{x^2}$

$$y = \frac{x^2}{16} (4 \log x - 1) + \frac{c}{x^2}.$$

7. $\boldsymbol{x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x}$

Sol. The given differential equation is $x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x$

Dividing every term by $x \log x$ to make the coefficient of $\frac{dy}{dx}$ unity, $\frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2}{x^2}$

Comparing with $\frac{dy}{dx} + \text{P}y = \text{Q}$, we have

$$\text{P} = \frac{1}{x \log x} \text{ and } \text{Q} = \frac{2}{x^2}$$

$$\int \text{P}\, dx = \int \frac{1}{x \log x}\, dx = \int \frac{1/x}{\log x}\, dx = \log (\log x)$$

$$\left[\because \int \frac{f'(x)}{f(x)} dx = \log f(x)\right]$$

$$\text{I.F.} = e^{\int \text{P}\, dx} = e^{\log (\log x)} = \log x$$

The general solution is $y(\text{I.F.}) = \int \text{Q(I.F.)}\, dx + c$

or $\quad y \log x = \int \frac{2}{x^2} \log x\, dx = 2 \int \underset{\text{I}}{(\log x)}\, \underset{\text{II}}{x^{-2}}\, dx + c$

Applying Product Rule of integration,

$$= 2\left[(\log x) \frac{x^{-1}}{-1} - \int \frac{1}{x} \cdot \frac{x^{-1}}{-1} dx\right] + c \quad = 2\left[-\frac{\log x}{x} + \int x^{-2}\, dx\right] + c$$

$$= 2\left[-\frac{\log x}{x} + \frac{x^{-1}}{-1}\right] + c \quad \text{or} \quad y \log x = \frac{-2}{x}(1 + \log x) + c.$$

8. $\boldsymbol{(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx \ (x \neq 0)}$

Sol. The given differential equation is $(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx$

Dividing every term by dx, $(1 + x^2) \frac{dy}{dx} + 2xy = \cot x$

Dividing every term by $(1 + x^2)$ to make coefficient of $\frac{dy}{dx}$ unity,

$$\frac{dy}{dx} + \frac{2x}{1+x^2}\, y = \frac{\cot x}{1+x^2}$$

Comparing with $\frac{dy}{dx}$ + Py = Q, we have

$P = \frac{2x}{1+x^2}$ and $Q = \frac{\cot x}{1+x^2}$

$\int P\, dx = \int \frac{2x}{1+x^2}\, dx = \log |1 + x^2|$ $\left[\because \int \frac{f'(x)}{f(x)} dx = \log|f(x)|\right]$

$= \log (1 + x^2)$ $[\because 1 + x^2 > 0 \Rightarrow |1 + x^2| = 1 + x^2]$

$$\text{I.F.} = e^{\int P\, dx} = e^{\log(1+x^2)} = 1 + x^2$$

Solution is $y(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + c$

$\Rightarrow y(1 + x^2) = \int \frac{\cot x}{1+x^2}(1 + x^2)\, dx + c$

$\Rightarrow y(1 + x^2) = \int \cot x + c \Rightarrow y(1 + x^2) = \log|\sin x| + c$

Dividing by $1 + x^2$, $y = \frac{\log|\sin x|}{1+x^2} + \frac{c}{1+x^2}$

or $y = (1 + x^2)^{-1} \log|\sin x| + c\,(1 + x^2)^{-1}$

which is the required general solution.

For each of the differential equations in Exercises 9 to 12, find the general solution:

9. $x\frac{dy}{dx} + y - x + xy\cot x = 0, (x \neq 0)$

Sol. The given differential equation is

$$x\frac{dy}{dx} + y - x + xy \cot x = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} + y + xy\cot x = x$$

$$\Rightarrow \quad x\frac{dy}{dx} + (1 + x\cot x)\, y = x$$

Dividing every term by x to make coefficient of $\frac{dy}{dx}$ unity,

$$\frac{dy}{dx} + \frac{(1 + x\cot x)}{x}\, y = 1$$

Comparing with $\frac{dy}{dx}$ + Py = Q, we have

$$P = \frac{1 + x\cot x}{x} \quad \text{and} \quad Q = 1$$

$$\int P\, dx = \int \frac{(1 + x\cot x)}{x}\, dx = \int \left(\frac{1}{x} + \frac{x\cot x}{x}\right) dx = \int \left(\frac{1}{x} + \cot x\right) dx$$

$$\Rightarrow \quad \int P\, dx = \log x + \log \sin x = \log (x \sin x)$$

$$\text{I.F.} = e^{\int P\,dx} = e^{\log (x \sin x)} = x \sin x$$

Solution is $y(\text{I.F.}) = \int Q(\text{I.F.})\, dx + c$

or $y(x \sin x) = \int \underset{\text{I}}{x}\ \underset{\text{II}}{\sin x}\ dx + c$

$$\left(\text{Applying Product Rule, } \int \text{I . II}\, dx = \text{I} \int \text{II}\, dx - \int \left(\frac{d}{dx}(\text{I}) \int \text{II}\, dx\right) dx\right)$$

$\Rightarrow$ $y(x \sin x) = x(-\cos x) - \int 1\,(-\cos x)\ dx + c$

$$= -x \cos x + \int \cos x\, dx + c$$

or $y(x \sin x) = -x \cos x + \sin x + c$

Dividing by $x \sin x$, $y = \dfrac{-x\cos x}{x \sin x} + \dfrac{\sin x}{x \sin x} + \dfrac{c}{x \sin x}$

or $$y = -\cot x + \frac{1}{x} + \frac{c}{x \sin x}$$

which is the required general solution.

10. $(x + y) \dfrac{dy}{dx} = 1$

Sol. The given differential equation is

$(x + y) \dfrac{dy}{dx} = 1$ $\Rightarrow$ $dx = (x + y)\, dy$

$\Rightarrow$ $\dfrac{dx}{dy} = x + y$ $\Rightarrow$ $\dfrac{dx}{dy} - x = y$

| Standard form of linear differential equation

Comparing with $\dfrac{dx}{dy} + Px = Q$, we have, $P = -1$ and $Q = y$

$\int P\, dy = \int -1\, dy = -\int 1\, dy = -y$ $\qquad$ $\text{I.F.} = e^{\int P\,dy} = e^{-y}$

$\therefore$ Solution is $x(\text{I.F.}) = \int Q(\text{I.F.})\, dy + c$

or $xe^{-y} = \int \underset{\text{I}}{y}\ \underset{\text{II}}{e^{-y}}\ dy + c$

$$\left(\text{Applying Product Rule, } \int \text{I . II}\, dy = \text{I} \int \text{II}\, dy - \int \left(\frac{d}{dy}(\text{I}) \int \text{II}\, dy\right) dy\right)$$

$\Rightarrow xe^{-y} = y \dfrac{e^{-y}}{-1} - \int 1 . \dfrac{e^{-y}}{-1}\ dy + c$ $\quad = -ye^{-y} + \int e^{-y}\ dy + c$

$$= -ye^{-y} + \frac{e^{-y}}{-1} + c$$

$\Rightarrow xe^{-y} = -ye^{-y} - e^{-y} + c$

Dividing every term by e^{-y}, $x = -y - 1 + \frac{c}{(e^{-y})}$

or $\quad x + y + 1 = ce^y$

which is the required general solution.

11. $\boldsymbol{y\,dx + (x - y^2)\,dy = 0}$

Sol. The given differential equation is $y\,dx + (x - y^2)\,dy = 0$

Dividing by dy, $y\frac{dx}{dy} + x - y^2 = 0$ or $y\frac{dx}{dy} + x = y^2$

Dividing every term by y (to make coefficient of $\frac{dx}{dy}$ unity),

$$\frac{dx}{dy} + \frac{1}{y}x = y \quad | \text{ Standard form of linear differential equation}$$

Comparing with $\frac{dx}{dy} + Px = Q$, we have

$$P = \frac{1}{y} \text{ and } Q = y$$

$$\int P\,dy = \int \frac{1}{y}\,dy = \log y$$

$$\text{I.F.} = e^{\int P\,dy} = e^{\log y} = y$$

Solution is $x(\text{I.F.}) = \int Q(\text{I.F.})\,dy + c$

$$\Rightarrow x\,.\,y = \int yy\,dy + c \Rightarrow xy = \int y^2\,dy + c \Rightarrow xy = \frac{y^3}{3} + c$$

Dividing by y, $x = \frac{y^2}{3} + \frac{c}{y}$

which is the required general solution.

12. $\boldsymbol{(x + 3y^2)\frac{dy}{dx} = y\ (y > 0)}$

Sol. The given differential equation is $(x + 3y^2)\frac{dy}{dx} = y$

$$\Rightarrow y\,dx = (x + 3y^2)\,dy \Rightarrow y\frac{dx}{dy} = x + 3y^2 \Rightarrow y\frac{dx}{dy} - x = 3y^2$$

Dividing every term by y (to make coefficient of $\frac{dx}{dy}$ unity),

$$\frac{dx}{dy} - \frac{1}{y}x = 3y \quad | \text{ Standard form of linear differential equation}$$

Comparing with $\frac{dx}{dy} + Px = Q$, we have $P = \frac{-1}{y}$ and $Q = 3y$

$$\int P\,dy = -\int \frac{1}{y}\,dy = -\log y = (-1)\log y = \log y^{-1}$$

$$\text{I.F.} = e^{\int P\,dy} = e^{\log y^{-1}} = y^{-1} = \frac{1}{y}$$

Solution is $x(\text{I.F.}) = \int Q(\text{I.F.})\, dy + c$

$\Rightarrow x \cdot \frac{1}{y} = \int 3y \cdot \frac{1}{y}\, dy + c \Rightarrow \frac{x}{y} = 3\int 1\, dy + c = 3y + c$

Cross – Multiplying, $x = 3y^2 + cy$

which is the required general solution.

For each of the differential equations given in Exercises 13 to 15, find a particular solution satisfying the given condition:

13. $\frac{dy}{dx} + 2y \tan x = \sin x;\ y = 0$ **when** $x = \frac{\pi}{3}$

Sol. The given differential equation is

$$\frac{dy}{dx} + 2y \tan x = \sin x;\ y = 0 \text{ when } x = \frac{\pi}{3}.$$

(It is standard form of linear differential equation)

Comparing with $\frac{dy}{dx} + Py = Q$, we have

$$P = 2 \tan x \quad \text{and} \quad Q = \sin x$$

$$\int P\, dx = 2\int \tan x\, dx = 2 \log \sec x = \log (\sec x)^2$$

$(\because n \log m = \log m^n)$

$$\text{I.F.} = e^{\int P\, dx} = e^{\log (\sec x)^2} = (\sec x)^2 = \sec^2 x$$

$\therefore$ Solution is $y(\text{I.F.}) = \int Q(\text{I.F.})\, dx + c$

$\Rightarrow \quad y \sec^2 x = \int \sin x \sec^2 x\, dx + c$

$$= \int \frac{\sin x}{\cos^2 x}\, dx + c = \int \frac{\sin x}{\cos x \cdot \cos x}\, dx + c$$

or $\quad y \sec^2 x = \int \tan x \sec x\, dx + c = \sec x + c$

$\Rightarrow \quad \frac{y}{\cos^2 x} = \frac{1}{\cos x} + c$

Multiplying by L.C.M. $= \cos^2 x$,

$$y = \cos x + c \cos^2 x \quad ...(i)$$

To find c: $y = 0$ when $x = \frac{\pi}{3}$ (given)

$\therefore$ From (i), $0 = \cos \frac{\pi}{3} + c \cos^2 \frac{\pi}{3}$

or $\quad 0 = \frac{1}{2} + c\left(\frac{1}{2}\right)^2$ or $0 = \frac{1}{2} + \frac{c}{4}$

$$\Rightarrow \quad \frac{c}{4} = \frac{-1}{2} \qquad\qquad \Rightarrow \quad c = -2$$

Putting $c = -2$ in (i), the required particular solution is

$$y = \cos x - 2\cos^2 x.$$

14. $(1 + x^2)\dfrac{dy}{dx} + 2xy = \dfrac{1}{1+x^2}$; $y = 0$ **when** $x = 1$

Sol. The given differential equation is

$$(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2} \; ; \; y = 0 \text{ when } x = 1$$

Dividing every term by $(1 + x^2)$ to make coefficient of $\dfrac{dy}{dx}$ unity,

$$\frac{dy}{dx} + \frac{2x}{1+x^2}\, y = \frac{1}{(1+x^2)^2}$$

Comparing with $\dfrac{dy}{dx} + \text{P}y = \text{Q}$, we have

$$\text{P} = \frac{2x}{1+x^2} \quad \text{and} \quad \text{Q} = \frac{1}{(1+x^2)^2}$$

$$\int \text{P}\, dx = \int \frac{2x}{1+x^2}\, dx = \int \frac{f'(x)}{f(x)}\, dx \quad = \log f(x) = \log(1 + x^2)$$

$$\text{I.F.} = e^{\int \text{P}\, dx} = e^{\log(1 + x^2)} = 1 + x^2$$

Solution is $\quad y(\text{I.F.}) = \int \text{Q(I.F.)}\, dx + c$

or $\quad y(1 + x^2) = \int \dfrac{1}{(1+x^2)^2}(1 + x^2)\, dx + c$

or $\quad y(1 + x^2) = \int \dfrac{1}{x^2+1}\, dx + c = \tan^{-1} x + c$

or $\quad y(1 + x^2) = \tan^{-1} x + c \qquad \ldots(i)$

To find c**:** $y = 0$ when $x = 1$

Putting $y = 0$ and $x = 1$ in (i), $\quad 0 = \tan^{-1} 1 + c$

or $\quad 0 = \dfrac{\pi}{4} + c \quad \left[\because \tan\dfrac{\pi}{4} = 1\right] \quad \Rightarrow \quad c = -\dfrac{\pi}{4}$

Putting $c = -\dfrac{\pi}{4}$ in (i), required particular solution is

$$y(1 + x^2) = \tan^{-1} x - \frac{\pi}{4}.$$

15. $\frac{dy}{dx} - 3y \cot x = \sin 2x$ **;** $y = 2$ **when** $x = \frac{\pi}{2}$

Sol. The given differential equation is $\frac{dy}{dx} - 3y \cot x = \sin 2x$

Comparing with $\frac{dy}{dx} + Py = Q$, we have

$$P = -3 \cot x \text{ and } Q = \sin 2x$$

$$\int P\, dx = -3 \int \cot x\, dx = -3 \log \sin x = \log (\sin x)^{-3}$$

$$\text{I.F.} = e^{\int P\, dx} = e^{\log (\sin x)^{-3}} = (\sin x)^{-3} = \frac{1}{\sin^3 x}$$

The general solution is $y(\text{I.F.}) = \int Q(\text{I.F.})\, dx + c$

or $y \frac{1}{\sin^3 x} = \int \sin 2x \,.\, \frac{1}{\sin^3 x}\, dx + c$

or $\frac{y}{\sin^3 x} = \int \frac{2 \sin x \cos x}{\sin^3 x}\, dx + c \quad = 2 \int \frac{\cos x}{\sin^2 x}\, dx + c$

$= 2\int \frac{\cos x}{\sin x \,.\, \sin x}\, dx + c \quad = 2 \int \operatorname{cosec} x \cot x\, dx = -2 \operatorname{cosec} x + c$

or $\frac{y}{\sin^3 x} = -\frac{2}{\sin x} + c$

Multiplying every term by L.C.M. $= \sin^3 x$

$$y = -2 \sin^2 x + c \sin^3 x \qquad \text{...}(i)$$

To find c**:** Putting $y = 2$ and $x = \frac{\pi}{2}$ (given) in (i),

$2 = -2 \sin^2 \frac{\pi}{2} + c \sin^3 \frac{\pi}{2}$ or $2 = -2 + c$ or $c = 4$

Putting $c = 4$ in (i), the required particular solution is

$$y = -2 \sin^2 x + 4 \sin^3 x.$$

16. Find the equation of the curve passing through the origin, given that the slope of the tangent to the curve at any point (*x*, *y*) is equal to the sum of coordinates of that point.

Sol. Given: Slope of the tangent to the curve at any point (x, y) = Sum of coordinates of the point (x, y).

$\Rightarrow \frac{dy}{dx} = x + y \qquad \Rightarrow \frac{dy}{dx} - y = x$

Comparing with $\frac{dy}{dx} + Py = Q$, we have $\qquad P = -1$ and $Q = x$

$\int P\, dx = \int -1\, dx = -\int 1\, dx = -x \qquad \text{I.F.} = e^{\int P\, dx} = e^{-x}$

Solution is $y(\text{I.F.}) = \int Q(\text{I.F.})\, dx + c$

i.e., $\qquad ye^{-x} = \int \underset{\text{I}}{x}\, \underset{\text{II}}{e^{-x}}\, dx + c$

$\left[\text{Applying Product Rule: } \int \text{I . II } dx = \text{I} \int \text{II } dx - \int \frac{d}{dx}(\text{I}) \left(\int \text{II } dx\right) dx\right]$

$$\Rightarrow \quad ye^{-x} = x\,\frac{e^{-x}}{-1} - \int 1 . \frac{e^{-x}}{-1}\, dx + c$$

or $\quad ye^{-x} = -\,xe^{-x} + \int e^{-x}\, dx + c \quad$ or $\quad ye^{-x} = -\,xe^{-x} + \dfrac{e^{-x}}{-1} + c$

or $ye^{-x} = -\,xe^{-x} - e^{-x} + c \quad$ or $\quad \dfrac{y}{e^x} = -\,\dfrac{x}{e^x} - \dfrac{1}{e^x} + c$

Multiplying by L.C.M. $= e^x$, $y = -\,x - 1 + ce^x$ ...(*i*)

To find *c*: Given: Curve (*i*) passes through the origin (0, 0).

Putting $x = 0$ and $y = 0$ in (*i*), $\quad 0 = 0 - 1 + c$

or $\quad -\,c = -\,1 \quad$ or $\quad c = 1$

Putting $c = 1$ in (*i*), equation of required curve is

$$y = -\,x - 1 + e^x \quad \text{or} \quad x + y + 1 = e^x.$$

17. Find the equation of the curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

Sol. According to question,

Sum of the coordinates of any point say (*x*, *y*) on the curve.

= Magnitude of the slope of the tangent to the curve + 5
↓
(because of exceeds)

i.e., $\quad x + y = \dfrac{dy}{dx} + 5$

$$\Rightarrow \quad \frac{dy}{dx} + 5 = x + y \quad \Rightarrow \quad \frac{dy}{dx} - y = x - 5$$

Comparing with $\dfrac{dy}{dx} + \text{P}y = \text{Q}$, we have

$$\text{P} = -\,1 \quad \text{and} \quad \text{Q} = x - 5$$

$\int \text{P}\, dx = \int -1\, dx = -\int 1\, dx = -\,x \qquad$ I.F. $= e^{\int \text{P}\,dx} = e^{-x}$

Solution is $\quad y(\text{I.F.}) = \int \text{Q}\,(\text{I.F.})\, dx + c$

or $\quad ye^{-x} = \int \underset{\text{I}}{(x-5)}\, \underset{\text{II}}{e^{-x}}\, dx + c$

$\left[\text{Applying Product Rule: } \int \text{I . II } dx = \text{I} \int \text{II } dx - \int \frac{d}{dx}(\text{I}) \left(\int \text{II } dx\right) dx\right]$

or $\quad ye^{-x} = (x - 5)\,\dfrac{e^{-x}}{-1} - \int 1 . \dfrac{e^{-x}}{-1}\, dx + c$

or $\quad ye^{-x} = -\,(x - 5)e^{-x} + \int e^{-x}\, dx + c$

or $\quad ye^{-x} = -(x-5)\,e^{-x} + \dfrac{e^{-x}}{-1} + c$

or $\quad \dfrac{y}{(e^x)} = -\dfrac{(x-5)}{(e^x)} - \dfrac{1}{(e^x)} + c$

Multiplying both sides by L.C.M. $= e^x$

$$y = -(x-5) - 1 + ce^x$$

or $\quad y = -x + 5 - 1 + ce^x \quad$ or $\quad x + y = 4 + ce^x \qquad ...(i)$

To find c: Curve (i) passes through the point (0, 2).

Putting $x = 0$ and $y = 2$ in (i),

$$2 = 4 + ce^0 \quad \text{or} \quad -2 = c$$

Putting $c = -2$ in (i), required equation of the curve is

$$x + y = 4 - 2e^x \quad \text{or} \quad y = 4 - x - 2e^x.$$

18. Choose the correct answer:

The integrating factor of the differential equation

$$x\frac{dy}{dx} - y = 2x^2 \text{ is}$$

(A) e^{-x} **(B)** e^{-y} **(C)** $\dfrac{1}{x}$ **(D)** x

Sol. The given differential equation is $x\dfrac{dy}{dx} - y = 2x^2$

Dividing every term by x to make coefficient of $\dfrac{dy}{dx}$ unity,

$\dfrac{dy}{dx} - \dfrac{1}{x}y = 2x \quad$ | Standard form of linear differential equation

Comparing with $\dfrac{dy}{dx} + \text{P}y = \text{Q}$, we have $\text{P} = \dfrac{-1}{x}$ and $\text{Q} = 2x$

$\therefore \int \text{P}\,dx = \int \dfrac{-1}{x}\,dx = -\log x = \log x^{-1} \quad [\because n\log m = \log m^n]$

$\text{I.F.} = e^{\int \text{P}\,dx} = e^{\log x^{-1}} = x^{-1} = \dfrac{1}{x} \qquad [\because e^{\log f(x)} = f(x)]$

$\therefore$ Option (C) is the correct answer.

19. Choose the correct answer:

The integrating factor of the differential equation

$$(1 - y^2)\frac{dx}{dy} + yx = ay \ (-1 < y < 1)$$

(A) $\dfrac{1}{y^2 - 1}$ **(B)** $\dfrac{1}{\sqrt{y^2 - 1}}$ **(C)** $\dfrac{1}{1 - y^2}$ **(D)** $\dfrac{1}{\sqrt{1 - y^2}}$

Sol. The given differential equation is

$$(1 - y^2)\frac{dx}{dy} + yx = ay \ (-1 < y < 1)$$

Dividing every term by $(1 - y^2)$ to make coefficient of $\dfrac{dx}{dy}$ unity,

$$\frac{dx}{dy} + \frac{y}{1-y^2}\, x = \frac{ay}{1-y^2}$$

| Standard form of linear differential equation

Comparing with $\frac{dx}{dy}$ + Px = Q, we have

$$P = \frac{y}{1-y^2} \quad \text{and} \quad Q = \frac{ay}{1-y^2}$$

$$\therefore \quad \int P\, dy = \int \frac{y}{1-y^2}\, dy = \frac{-1}{2} \int \frac{-2y}{1-y^2}\, dy$$

$$= \frac{-1}{2} \log (1 - y^2) \qquad \left[\because \int \frac{f'(y)}{f(y)} = \log f(y)\right]$$

$$= \log (1 - y^2)^{-1/2}$$

$$\text{I.F.} = e^{\int P\, dy} = e^{\log (1-y^2)^{-1/2}}$$

$$= (1 - y^2)^{-1/2} \qquad [\because \; e^{\log f(x)} = f(x)]$$

$$= \frac{1}{\sqrt{1-y^2}}$$

$\therefore$ Option (D) is the correct answer.