NCERT Class 12 Maths Solutions

Exercise 9.5

In each of the Exercises 1 to 5, show that the given differential equation is homogeneous and solve each of them:

1. $(x^2 + xy)\, dy = (x^2 + y^2)\, dx$

Sol. The given D.E. is

$$(x^2 + xy)\, dy = (x^2 + y^2)\, dx \qquad \ldots(i)$$

This D.E. looks to be homogeneous as degree of each coefficient of dx and dy is same throughout (here 2).

From (i), $$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy} = \frac{x^2\left(1 + \frac{y^2}{x^2}\right)}{x^2\left(1 + \frac{y}{x}\right)}$$

or $$\frac{dy}{dx} = \frac{1 + \left(\frac{y}{x}\right)^2}{1 + \left(\frac{y}{x}\right)} = F\left(\frac{y}{x}\right) \qquad \ldots(ii)$$

$\therefore$ The given D.E. is homogeneous.

Put $\dfrac{y}{x} = v$**.** Therefore $y = vx$.

$\therefore \quad \dfrac{dy}{dx} = v \,.\, 1 + x\dfrac{dv}{dx} = v + x\dfrac{dv}{dx}$

Putting these values of $\dfrac{y}{x}$ and $\dfrac{dy}{dx}$ in (*ii*), we have

$$v + x\frac{dv}{dx} = \frac{1+v^2}{1+v}$$

Transposing v to R.H.S., $x\dfrac{dv}{dx} = \dfrac{1+v^2}{1+v} - v$

$\Rightarrow \quad x\dfrac{dv}{dx} = \dfrac{1+v^2-v-v^2}{1+v} = \dfrac{1-v}{1+v}$

Cross-multiplying $\quad x(1 + v)\, dv = (1 - v)\, dx$

Separating variables $\quad \dfrac{1+v}{1-v}\, dv = \dfrac{dx}{x}$

Integrating both sides $\displaystyle\int \frac{1+v}{1-v}\, dv = \int \frac{1}{x}\, dx$

$\Rightarrow \displaystyle\int \frac{1+1-1+v}{1-v}\, dv = \log x + c \Rightarrow \int \frac{2-(1-v)}{1-v}\, dv = \log x + c$

$\Rightarrow \displaystyle\int \left(\frac{2}{1-v} - 1\right) dv = \log x + c \Rightarrow \frac{2\log(1-v)}{-1} - v = \log x + c$

$\Rightarrow \quad -2\log(1-v) - v = \log x + c$

Put $v = \dfrac{y}{x}$, $\quad -2\log\left(1-\dfrac{y}{x}\right) - \dfrac{y}{x} = \log x + c$

Dividing by -1, $\quad 2\log\left(\dfrac{x-y}{x}\right) + \dfrac{y}{x} = -\log x - c$

$\Rightarrow \log\left(\dfrac{x-y}{x}\right)^2 + \log x = -\dfrac{y}{x} - c \Rightarrow \log\left(\dfrac{(x-y)^2\, x}{x^2}\right) = -\dfrac{y}{x} - c$

$\Rightarrow \dfrac{(x-y)^2}{x} = e^{-\frac{y}{x} - c} = e^{-\frac{y}{x}}\, e^{-c} \Rightarrow (x-y)^2 = Cx\, e^{-\frac{y}{x}}$ where $C = e^{-c}$

which is the required solution.

2. $y' = \dfrac{x+y}{x}$

Sol. The given differential equation is $y' = \dfrac{x+y}{x}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{x}{x} + \dfrac{y}{x} \qquad \Rightarrow \dfrac{dy}{dx} = 1 + \dfrac{y}{x} = f\left(\dfrac{y}{x}\right) \qquad \ldots(i)$

$\therefore$ Differential equation (*i*) is homogeneous.

Put $\frac{y}{x} = v$ $\quad\therefore\quad y = vx$

$\therefore \quad \frac{dy}{dx} = v\,.\,1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$

Putting these values of $\frac{dy}{dx}$ and y in (*i*),

$v + x\frac{dv}{dx} = 1 + v \quad \Rightarrow \quad x\frac{dv}{dx} = 1 \quad \Rightarrow \quad x\,dv = dx$

Separating variables, $dv = \frac{dx}{x}$

Integrating both sides, $\int 1\,dv = \int \frac{dx}{x}$ $\quad v = \log|x| + c$

Putting $v = \frac{y}{x}$, $\frac{y}{x} = \log|x| + c \quad \therefore \quad y = x\log|x| + cx$

which is the required solution.

3. $(x - y)\,dy - (x + y)\,dx = 0$

Sol. The given differential equation is

$(x - y)\,dy - (x + y)\,dx = 0$ ...(*i*)

Differential equation (*i*) looks to be homogeneous because each coefficient of dx and dy is of degree 1.

From (*i*), $(x - y)\,dy = (x + y)\,dx$

$\therefore \quad \frac{dy}{dx} = \frac{x+y}{x-y} = \frac{x\left(1+\frac{y}{x}\right)}{x\left(1-\frac{y}{x}\right)}$ or $\frac{dy}{dx} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}} = f\left(\frac{y}{x}\right)$ ...(*ii*)

$\therefore$ Differential equation (*i*) is homogeneous.

Put $\frac{y}{x} = v$ $\quad\therefore\quad y = vx$

$\therefore \quad \frac{dy}{dx} = v\,.\,1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$

Putting these values in (*ii*), $v + x\frac{dv}{dx} = \frac{1+v}{1-v}$

Shifting v to R.H.S., $x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v+v^2}{1-v}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+v^2}{1-v}$

Cross-multiplying, $x\,(1 - v)\,dv = (1 + v^2)\,dx$

Separating variables, $\frac{(1-v)}{1+v^2}\,dv = \frac{dx}{x}$

Integrating both sides, $\int \frac{1-v}{1+v^2}\,dv = \int \frac{1}{x}\,dx + c$

$$\Rightarrow \int \frac{1}{1+v^2}\, dv - \int \frac{v}{1+v^2}\, dv = \int \frac{1}{x}\, dx + c$$

$$\Rightarrow \tan^{-1} v - \frac{1}{2} \int \frac{2v}{1+v^2}\, dv = \log x + c$$

$$\Rightarrow \tan^{-1} v - \frac{1}{2} \log (1 + v^2) = \log x + c \qquad \left[\because \int \frac{f'(v)}{f(v)} dv = \log f(v)\right]$$

Putting $v = \frac{y}{x}$, $\tan^{-1} \frac{y}{x} - \frac{1}{2} \log \left(1 + \frac{y^2}{x^2}\right) = \log x + c$

$$\Rightarrow \tan^{-1} \frac{y}{x} - \frac{1}{2} \log \left(\frac{x^2 + y^2}{x^2}\right) = \log x + c$$

$$\Rightarrow \tan^{-1} \frac{y}{x} - \frac{1}{2} [\log (x^2 + y^2) - \log x^2] = \log x + c$$

$$\Rightarrow \tan^{-1} \frac{y}{x} - \frac{1}{2} \log (x^2 + y^2) + \frac{1}{2}\, 2 \log x = \log x + c$$

$$\Rightarrow \tan^{-1} \frac{y}{x} - \frac{1}{2} \log (x^2 + y^2) = c \Rightarrow \tan^{-1} \frac{y}{x} = \frac{1}{2} \log (x^2 + y^2) + c$$

which is the required solution.

4. $(x^2 - y^2)\, dx + 2\, xy\, dy = 0$

Sol. The given differential equation is

$$(x^2 - y^2)\, dx + 2xy\, dy = 0 \qquad ...(i)$$

This differential equation looks to be homogeneous because degree of each coefficient of dx and dy is same (here 2).

From (i), $2xy\, dy = -(x^2 - y^2)\, dx$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x^2 - y^2)}{2xy} = \frac{y^2 - x^2}{2xy}$$

Dividing every term in the numerator and denominator of R.H.S. by x^2,

$$\frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)^2 - 1}{2\frac{y}{x}} = f\left(\frac{y}{x}\right) \qquad ...(ii)$$

$\therefore$ The given differential equation is homogeneous.

Put $\frac{y}{x} = v$. Therefore $y = vx$ $\therefore$ $\frac{dy}{dx} = v \,.\, 1 + x \frac{dv}{dx} = v + x \frac{dv}{dx}$

Putting these values of $\frac{y}{x}$ and $\frac{dy}{dx}$ in differential equation (ii), we have

$$v + x \frac{dv}{dx} = \frac{v^2 - 1}{2v} \quad \Rightarrow \quad x \frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{-v^2 - 1}{2v} = -\frac{(v^2 + 1)}{2v} \quad \therefore x\, 2v\, dv = -(v^2 + 1)\, dx$$

$$\Rightarrow \qquad \frac{2v\,dv}{v^2+1} = -\frac{dx}{x}$$

Integrating both sides, $\int \frac{2v}{v^2+1}\,dv = -\int \frac{1}{x}\,dx$

$$\Rightarrow \qquad \log(v^2+1) = -\log x + \log c$$

$$\Rightarrow \qquad \log(v^2+1) + \log x = \log c$$

$$\Rightarrow \qquad \log(v^2+1)x = \log c$$

$$\Rightarrow \qquad (v^2+1)x = c$$

Put $v = \frac{y}{x}$, $\left(\frac{y^2}{x^2}+1\right)x = c$ or $\left(\frac{y^2+x^2}{x^2}\right)x = c$

or $\frac{y^2+x^2}{x} = c$ or $x^2 + y^2 = cx$

which is the required solution.

5. $\boldsymbol{x^2\left(\frac{dy}{dx}\right) = x^2 - 2y^2 + xy}$

Sol. The given differential equation is $x^2\frac{dy}{dx} = x^2 - 2y^2 + xy$

The given differential equation looks to be Homogeneous as all terms in x and y are of same degree (here 2).

Dividing by x^2, $\frac{dy}{dx} = \frac{x^2}{x^2} - \frac{2y^2}{x^2} + \frac{xy}{x^2}$

or $$\frac{dy}{dx} = 1 - 2\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) \qquad ...(i)$$

$$= F\left(\frac{y}{x}\right)$$

$\therefore$ Differential equation (i) is homogeneous.

So put $\frac{y}{x} = v \qquad \therefore \quad y = vx$

$$\therefore \qquad \frac{dy}{dx} = v\,.\,1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$$

Putting these values of $\frac{y}{x}$ and $\frac{dy}{dx}$ in (i),

$$v + x\frac{dv}{dx} = 1 - 2v^2 + v \text{ or } x\frac{dv}{dx} = 1 - 2v^2 \Rightarrow x\,dv = (1-2v^2)\,dx$$

Separating variables, $\frac{dv}{1-2v^2} = \frac{dx}{x}$

Integrating both sides, $\int \frac{1}{1^2 - (\sqrt{2}v)^2}\, dv = \int \frac{1}{x}\, dx$

$$\Rightarrow \quad \frac{1}{2.1} \frac{\log\left|\frac{1+\sqrt{2}v}{1-\sqrt{2}v}\right|}{\sqrt{2} \to \text{Coefficient of } v} = \log|x| + c$$

$$\left[\because \int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|\right]$$

Putting $v = \frac{y}{x}$, $\quad \frac{1}{2\sqrt{2}} \log\left|\frac{1+\sqrt{2}\,\frac{y}{x}}{1-\sqrt{2}\,\frac{y}{x}}\right| = \log|x| + c$

Multiplying within logs by x in L.H.S.,

$$\frac{1}{2\sqrt{2}} \log\left|\frac{x+\sqrt{2}y}{x-\sqrt{2}y}\right| = \log|x| + c.$$

In each of the Exercises 6 to 10, show that the given D.E. is homogeneous and solve each of them:

6. $x\, dy - y\, dx = \sqrt{x^2+y^2}\, dx$

Sol. The given differential equation is

$x\, dy - y\, dx = \sqrt{x^2+y^2}\, dx$ or $x\, dy = y\, dx + \sqrt{x^2+y^2}\, .\, dx$

Dividing by dx

$$x\frac{dy}{dx} = y + \sqrt{x^2+y^2} \quad \text{or} \quad x\frac{dy}{dx} = y + x\sqrt{1+\left(\frac{y}{x}\right)^2}$$

Dividing by x, $\frac{dy}{dx} = \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2} = \text{F}\left(\frac{y}{x}\right)$ ...(i)

$\therefore$ Given differential equation is homogeneous.

Put $\frac{y}{x} = v$ *i.e.,* $y = vx$.

Differentiating w.r.t. x, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Putting these values of $\frac{y}{x}$ and $\frac{dy}{dx}$ in (i), it becomes

$$v + x\frac{dv}{dx} = v + \sqrt{1+v^2} \quad \text{or} \quad x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\therefore \quad x\, dv = \sqrt{1+v^2}\, dx \quad \text{or} \quad \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating both sides, $\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$

$\therefore \quad \log\left(v + \sqrt{1+v^2}\right) = \log x + \log c$

Replacing v by $\frac{y}{x}$, we have

$$\log\left(\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right) = \log cx \quad \text{or} \quad \frac{y+\sqrt{x^2+y^2}}{x} = cx$$

or $\quad y + \sqrt{x^2+y^2} = cx^2$

which is the required solution.

7. $\left\{x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right)\right\} y\, dx = \left\{y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right)\right\} x\, dy$

Sol. The given D.E. is

$$\left\{x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right)\right\} y\, dx = \left\{y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right)\right\} x\, dy$$

$$\therefore \quad \frac{dy}{dx} = \frac{\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y}{\left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x} = \frac{xy\cos\frac{y}{x} + y^2\sin\frac{y}{x}}{xy\sin\frac{y}{x} - x^2\cos\frac{y}{x}}$$

Dividing every term in R.H.S. by x^2,

$$\frac{dy}{dx} = \frac{\frac{y}{x}\cos\frac{y}{x} + \left(\frac{y}{x}\right)^2\sin\frac{y}{x}}{\frac{y}{x}\sin\frac{y}{x} - \cos\frac{y}{x}} = F\left(\frac{y}{x}\right) \quad ...(i)$$

$\therefore$ The given differential equation is homogeneous.

So let us put $\frac{y}{x} = v$. Therefore $y = vx$.

$$\therefore \quad \frac{dy}{dx} = v\,.\,1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$$

Putting these values in differential equation (i), we have

$$v + x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} \Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$$

$$= \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v} \Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

Cross-multiplying, $\quad x(v\sin v - \cos v)\, dv = 2v\cos v\, dx$

Separating variables, $\quad \frac{v\sin v - \cos v}{v\cos v}\, dv = 2\frac{dx}{x}$

Integrating both sides, $\quad \int \frac{v\sin v - \cos v}{v\cos v}\, dv = 2\int \frac{1}{x}\, dx$

Using $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$, $\Rightarrow \int \left(\frac{v\sin v}{v\cos v} - \frac{\cos v}{v\cos v}\right) dv = 2\int \frac{1}{x}\, dx$

$$\Rightarrow \quad \int \left(\tan v - \frac{1}{v}\right) dv = 2 \int \frac{1}{x}\, dx$$

$$\Rightarrow \quad \log|\sec v| - \log|v| = 2\log|x| + \log|c|$$

$$\Rightarrow \quad \log\left|\frac{\sec v}{v}\right| = \log|x|^2 + \log|c| \quad = \log(|c|\, x^2)$$

$$\Rightarrow \quad \left|\frac{\sec v}{v}\right| = |c|\, x^2 \qquad \Rightarrow \quad \frac{\sec v}{v} = \pm|c|\, x^2$$

$$\Rightarrow \quad \sec v = \pm|c|\, x^2 v$$

Putting $v = \frac{y}{x}$, $\sec \frac{y}{x} = Cx^2 \frac{y}{x}$ where $C = \pm|c|$

or $\quad \sec \frac{y}{x} = Cxy \qquad \Rightarrow \quad \frac{1}{\cos \frac{y}{x}} = Cxy$

$$\Rightarrow \quad C\, xy \cos \frac{y}{x} = 1 \qquad \Rightarrow \quad xy \cos \frac{y}{x} = \frac{1}{C} = C_1 \text{ (say)}$$

which is the required solution.

8. $\boldsymbol{x \frac{dy}{dx} - y + x \sin\left(\frac{y}{x}\right) = 0}$

Sol. The given D.E. is $x \frac{dy}{dx} - y + x \sin\left(\frac{y}{x}\right) = 0$

or $\quad x \frac{dy}{dx} = y - x \sin\left(\frac{y}{x}\right)$

Dividing every term by x, $\frac{dy}{dx} = \frac{y}{x} - \sin\left(\frac{y}{x}\right) = F\left(\frac{y}{x}\right)$ ...(*i*)

Since $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$, the given differential equation is homogeneous.

Putting $\frac{y}{x} = v$ *i.e.*, $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Putting these values of $\frac{y}{x}$ and $\frac{dy}{dx}$ in (*i*), we have

$$v + x\frac{dv}{dx} = v - \sin v$$

or $\quad x\frac{dv}{dx} = -\sin v \qquad \therefore \qquad x\, dv = -\sin v\, dx$

or $\quad \frac{dv}{\sin v} = \frac{-dx}{x} \qquad$ or $\quad \text{cosec } v\, dv = -\frac{dx}{x}$

Integrating, $\log|\text{cosec } v - \cot v| = -\log|x| + \log|c|$

or $\quad \log|\text{cosec } v - \cot v| = \log\left|\frac{c}{x}\right|$

or $\qquad \operatorname{cosec} v - \cot v = \pm \dfrac{c}{x}$

Replacing v by $\dfrac{y}{x}$, $\operatorname{cosec} \dfrac{y}{x} - \cot \dfrac{y}{x} = \dfrac{C}{x}$ where $C = \pm c$

$$\Rightarrow \quad \frac{1}{\sin \frac{y}{x}} - \frac{\cos \frac{y}{x}}{\sin \frac{y}{x}} = \frac{C}{x} \quad \Rightarrow \quad \frac{1 - \cos \frac{y}{x}}{\sin \frac{y}{x}} = \frac{C}{x}$$

Cross-multiplying, $x\left(1 - \cos \dfrac{y}{x}\right) = C \sin \dfrac{y}{x}$ which is the required solution.

9. $y\, dx + x \log \left(\dfrac{y}{x}\right) dy - 2x\, dy = 0$

Sol. The given differential equation is $y\, dx + x \left(\log \dfrac{y}{x}\right) dy - 2x\, dy = 0$

$$\therefore \quad y\, dx = 2x\, dy - x\left(\log \frac{y}{x}\right) dy \quad \text{or} \quad y\, dx = x\left(2 - \log \frac{y}{x}\right) dy$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{y}{x}}{2 - \log \frac{y}{x}} = F\left(\frac{y}{x}\right) \qquad \ldots(i)$$

Since $\dfrac{dy}{dx} = F\left(\dfrac{y}{x}\right)$, the given differential equation is homogeneous.

Putting $\dfrac{y}{x} = v$ *i.e.*, $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Putting these values of $\dfrac{y}{x}$ and $\dfrac{dy}{dx}$ in (i), we have

$$v + x\frac{dv}{dx} = \frac{v}{2 - \log v}$$

or $\quad x\dfrac{dv}{dx} = \dfrac{v}{2 - \log v} - v = \dfrac{v - 2v + v \log v}{2 - \log v} = \dfrac{-v + v \log v}{2 - \log v}$

or $\qquad x\dfrac{dv}{dx} = \dfrac{v(\log v - 1)}{2 - \log v}$

$\therefore \qquad x(2 - \log v)\, dv = v(\log v - 1)\, dx$

or $\quad \dfrac{2 - \log v}{v(\log v - 1)}\, dv = \dfrac{dx}{x} \quad$ or $\quad \dfrac{1 - (\log v - 1)}{v(\log v - 1)}\, dv = \dfrac{dx}{x}$

or $\quad \left[\dfrac{1}{v(\log v - 1)} - \dfrac{1}{v}\right] dv = \dfrac{dx}{x}$

Integrating $\int \left[\dfrac{1/v}{\log v - 1} - \dfrac{1}{v}\right] dv = \log|x| + \log|c|$

or $\log|\log v - 1| - \log|v| = \log|x| + \log|c|$

$$\left[\because \int \frac{f'(v)}{f(v)}\,dv = \log|f(v)|\right]$$

or $\log\left|\dfrac{\log v - 1}{v}\right| = \log|cx|$ or $\left|\dfrac{\log v - 1}{v}\right| = |cx|$

or $\dfrac{\log v - 1}{v} = \pm cx = Cx$ where $C = \pm c$

or $\log v - 1 = Cx\,v$

Replacing v by $\dfrac{y}{x}$, we have

$\log\dfrac{y}{x} - 1 = Cx\left(\dfrac{y}{x}\right)$ or $\log\dfrac{y}{x} - 1 = Cy$

which is a primitive (solution) of the given differential equation.

Second solution

The given D.E. is $y\,dx + x\log\left(\dfrac{y}{x}\right)dy - 2x\,dy = 0$

Dividing every term by dy,

$$y\frac{dx}{dy} - x\log\frac{x}{y} - 2x = 0 \quad \left[\because \log\frac{y}{x} = \log y - \log x = -(\log x - \log y) = -\log\frac{x}{y}\right]$$

Dividing every term by y,

$$\frac{dx}{dy} - \frac{x}{y}\log\frac{x}{y} - 2\frac{x}{y} = 0$$

$$\Rightarrow \frac{dx}{dy} = \frac{x}{y}\log\frac{x}{y} + 2\frac{x}{y} \quad \ldots (i) \quad \left(= F\left(\frac{x}{y}\right)\right)$$

$\therefore$ The given differential is homogeneous.

Put $\dfrac{x}{y} = v$ *i.e.* $x = vy$

so that $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$

Putting these values in D. E. (i), we have

$$v + y\frac{dv}{dy} = v\log v + 2v$$

$$\Rightarrow y\frac{dv}{dy} = v\log v + v = v(\log v + 1)$$

Cross-multiplying $y\,dv = v(\log v + 1)\,dy$

Separating variables $\frac{dv}{v(\log v+1)} = \frac{dy}{y}$

Integrating both sides $\int \frac{\frac{1}{v}}{\log v+1}\, dv = \int \frac{1}{y}\, dy$

$\therefore$ $\log|\log v+1| = \log|y| + \log|c| = \log|cy|$ $\left[\because \int \frac{f'(v)}{f(v)}\, dv = \log|f(v)|\right]$

$\therefore$ $\log v + 1 = \pm cy = Cy$ where $C = \pm c$

Replacing v by $\frac{x}{y}$, we have

$\log \frac{x}{y} + 1 = Cy$

or $-\log \frac{y}{x} + 1 = Cy$ $\left[\because \log \frac{x}{y} = -\log \frac{y}{x} \text{ see page 632}\right]$

Dividing by -1, $\log \frac{y}{x} - 1 = -Cy$ or $= C_1 y$ which is a primitive (solution) of the given D.E.

10. $\mathbf{(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0}$

Sol. The given differential equation is $(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$

Dividing by dy, $(1 + e^{x/y}) \frac{dx}{dy} + e^{x/y}\left(1 - \frac{x}{y}\right) = 0$

or $(1 + e^{x/y}) \frac{dx}{dy} = -\, e^{x/y}\left(1 - \frac{x}{y}\right)$ or $\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y} - 1\right)}{1 + e^{x/y}}$ ...(*i*)

which is a differential equation of the form $\frac{dx}{dy} = f\left(\frac{x}{y}\right)$.

$\therefore$ The given differential equation is homogeneous.

Hence put $\frac{x}{y} = v$ *i.e.*, $x = vy$

Differentiating w.r.t. y, $\frac{dx}{dy} = v + y\frac{dv}{dy}$

Putting these values of $\frac{x}{y}$ and $\frac{dx}{dy}$ in (*i*), we have

$$v + y\frac{dv}{dy} = \frac{e^v(v-1)}{1+e^v}.$$

Now transposing v to R.H.S.

$$y\frac{dv}{dy} = \frac{ve^v - e^v}{1+e^v} - v = \frac{ve^v - e^v - v - ve^v}{1+e^v} = \frac{-e^v - v}{1+e^v}$$

$\therefore \quad y(1 + e^v)\, dv = -(e^v + v)\, dy \quad$ or $\quad \dfrac{1+e^v}{v+e^v}\, dv = -\dfrac{dy}{y}$

Integrating, $\log |(v + e^v)| = -\log |y| + \log |c|$

Replacing v by $\dfrac{x}{y}$, we have

$$\log \left|\left(\frac{x}{y} + e^{x/y}\right)\right| = \log\left|\frac{c}{y}\right| \quad \text{or} \quad \left|\frac{x}{y} + e^{x/y}\right| = \left|\frac{c}{y}\right|$$

$$\therefore \quad \frac{x}{y} + e^{x/y} = \pm\frac{C}{y}$$

Multiplying every term by y,

$$x + y\, e^{x/y} = C \text{ where } C = \pm c$$

which is the required general solution.

For each of the differential equations in Exercises from 11 to 15, find the particular solution satisfying the given condition:

11. $(x + y)\, dy + (x - y)\, dx = 0;\ y = 1$ when $x = 1$

Sol. The given differential equation is

$$(x + y)\, dy + (x - y)\, dx = 0,\ y = 1 \text{ when } x = 1 \qquad ...(i)$$

It looks to be a homogeneous differential equation because each coefficient of dx and dy is of same degree (here 1).

From (i), $(x + y)\, dy = -(x - y)\, dx$

$$\therefore \quad \frac{dy}{dx} = \frac{-(x-y)}{x+y} = \frac{y-x}{y+x} = \frac{x\left(\frac{y}{x}-1\right)}{x\left(\frac{y}{x}+1\right)}$$

or $$\frac{dy}{dx} = \frac{\frac{y}{x}-1}{\frac{y}{x}+1} = f\left(\frac{y}{x}\right) \qquad ...(ii)$$

$\therefore$ Given differential equation is homogeneous.

Put $\dfrac{y}{x} = v$. Therefore $y = vx$.

$$\therefore \quad \frac{dy}{dx} = v\,.\,1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$$

Putting these values in eqn. (ii), $\quad v + x\dfrac{dv}{dx} = \dfrac{v-1}{v+1}$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{v-1-v(v+1)}{v+1} = \frac{v-1-v^2-v}{v+1} = \frac{-v^2-1}{v+1}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{(v^2+1)}{v+1} \qquad \therefore \quad x(v+1)\,dv = -(v^2+1)\,dx$$

Separating variables, $\frac{v+1}{v^2+1}\,dv = -\frac{dx}{x}$

$$\therefore \quad \int \frac{v}{v^2+1}\,dv + \int \frac{1}{v^2+1}\,dv = -\int \frac{1}{x}\,dx$$

$$\Rightarrow \quad \frac{1}{2}\int \frac{2v}{v^2+1}\,dv + \tan^{-1} v = -\log x + c$$

$$\Rightarrow \quad \frac{1}{2}\log(v^2+1) + \tan^{-1} v = -\log x + c \left[\because \int \frac{f'(v)}{f(v)}\,dv = \log f(v)\right]$$

Putting $v = \frac{y}{x}$, $\frac{1}{2}\log\left(\frac{y^2}{x^2}+1\right) + \tan^{-1}\frac{y}{x} = -\log x + c$

$$\Rightarrow \quad \frac{1}{2}\log\left(\frac{y^2+x^2}{x^2}\right) + \tan^{-1}\frac{y}{x} = -\log x + c$$

$$\Rightarrow \quad \frac{1}{2}[\log(x^2+y^2) - \log x^2] + \tan^{-1}\frac{y}{x} = -\log x + c$$

$$\Rightarrow \quad \frac{1}{2}\log(x^2+y^2) - \frac{1}{2}\,2\log x + \tan^{-1}\frac{y}{x} = -\log x + c$$

$$\Rightarrow \quad \frac{1}{2}\log(x^2+y^2) + \tan^{-1}\frac{y}{x} = c \qquad ...(iii)$$

To find c: Given: $y = 1$ when $x = 1$.

Putting $x = 1$ and $y = 1$ in (*iii*), $\frac{1}{2}\log 2 + \tan^{-1} 1 = c$

or $c = \frac{1}{2}\log 2 + \frac{\pi}{4}$ $\left(\because \tan\frac{\pi}{4} = 1 \Rightarrow \tan^{-1} 1 = \frac{\pi}{4}\right)$

Putting this value of c in (*iii*),

$$\frac{1}{2}\log(x^2+y^2) + \tan^{-1}\frac{y}{x} = \frac{1}{2}\log 2 + \frac{\pi}{4}$$

Multiplying by 2,

$$\log(x^2+y^2) + 2\tan^{-1}\frac{y}{x} = \log 2 + \frac{\pi}{2}$$

which is the required particular solution.

12. $x^2\,dy + (xy + y^2)\,dx = 0$; $y = 1$ when $x = 1$

Sol. The given differential equation is

$$x^2\,dy + (xy+y^2)\,dx = 0 \quad \text{or} \quad x^2\,dy = -y(x+y)\,dx$$

$$\therefore \quad \frac{dy}{dx} = -\frac{y(x+y)}{x^2} = -\frac{yx\left(1+\frac{y}{x}\right)}{x^2}$$

or $\dfrac{dy}{dx} = -\dfrac{y}{x}\left(1+\dfrac{y}{x}\right) = F\left(\dfrac{y}{x}\right)$ ...(*i*)

$\therefore$ The given differential equation is homogeneous.

Put $\dfrac{y}{x} = v$, *i.e.*, $y = vx$

Differentiating w.r.t. x, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Putting these values of $\dfrac{y}{x}$ and $\dfrac{dy}{dx}$ in differential equation (*i*),

we have $v + x\dfrac{dv}{dx} = -v(1+v) = -v - v^2$

Transposing v to R.H.S., $x\dfrac{dv}{dx} = -v^2 - 2v$

or $x\dfrac{dv}{dx} = -v(v+2)$ $\qquad x\,dv = -v(v+2)\,dx$

or $\dfrac{dv}{v(v+2)} = -\dfrac{dx}{x}$

Integrating both sides, $\displaystyle\int \frac{1}{v(v+2)}\,dv = -\int \frac{1}{x}\,dx$

or $\dfrac{1}{2}\displaystyle\int \frac{2}{v(v+2)}\,dv = -\log|x|$ or $\dfrac{1}{2}\displaystyle\int \frac{(v+2)-v}{v(v+2)}\,dv = -\log|x|$

Separating terms

or $\displaystyle\int\left(\frac{1}{v} - \frac{1}{v+2}\right)dv = -2\log|x|$

or $\log|v| - \log|v+2| = \log x^{-2} + \log|c|$

or $\log\left|\dfrac{v}{v+2}\right| = \log|cx^{-2}|$

$\therefore \left|\dfrac{v}{v+2}\right| = \left|\dfrac{c}{x^2}\right| \qquad \therefore \dfrac{v}{v+2} = \pm\dfrac{c}{x^2}$

Replacing v to $\dfrac{y}{x}$, we have

$\dfrac{\frac{y}{x}}{\frac{y}{x}+2} = \pm\dfrac{c}{x^2}$ or $\dfrac{y}{y+2x} = \pm\dfrac{c}{x^2}$

or $x^2y = C(y + 2x)$

where $C = \pm c$ ...(*ii*)

To find C

Put $x = 1$ and $y = 1$ (given) in eqn. (*ii*), $1 = 3C$ $\therefore C = \dfrac{1}{3}$

Putting $C = \dfrac{1}{3}$ in eqn. (*ii*), required particular solution is

$$x^2y = \frac{1}{3}(y + 2x) \quad \text{or} \quad 3x^2y = y + 2x.$$

13. $\left[x \sin^2\left(\frac{y}{x}\right) - y\right] dx + x\, dy = 0;\ y = \frac{\pi}{4}$ **when** $x = 1$

Sol. The given differential equation is

$$\left(x \sin^2 \frac{y}{x} - y\right) dx + x\, dy = 0;\ y = \frac{\pi}{4},\ x = 1$$

$$\Rightarrow \quad x\, dy = -\left(x \sin^2 \frac{y}{x} - y\right) dx$$

Dividing by dx, $\quad x \frac{dy}{dx} = -x \sin^2 \frac{y}{x} + y$

Dividing by x, $\quad \frac{dy}{dx} = -\sin^2 \frac{y}{x} + \frac{y}{x} \quad ...(i)$

$$= F\left(\frac{y}{x}\right)$$

$\therefore$ The given differential equation is homogeneous.

Put $\frac{y}{x} = v \quad \therefore\ y = vx \quad \therefore\ \frac{dy}{dx} = v \cdot 1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$

Putting these values in differential equation (i), we have

$$v + x\frac{dv}{dx} = -\sin^2 v + v \ \Rightarrow\ x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \quad x\, dv = -\sin^2 v\, dx$$

Separating variables, $\quad \frac{dv}{\sin^2 v} = -\frac{dx}{x}$

Integrating, $\quad \int \operatorname{cosec}^2 v\, dv = -\int \frac{1}{x}\, dx$

$\Rightarrow \quad -\cot v = -\log |x| + c$

Dividing by -1, $\quad \cot v = \log |x| - c$

Putting $v = \frac{y}{x}$, $\quad \cot \frac{y}{x} = \log |x| - c \quad ...(ii)$

To find $\boldsymbol{c}$ **:** $y = \frac{\pi}{4}$ when $x = 1$ (given)

Putting $x = 1$ and $y = \frac{\pi}{4}$ in (ii), $\cot \frac{\pi}{4} = \log 1 - c$

or $\quad 1 = 0 - c \quad$ or $\quad c = -1$

Putting $c = -1$ in (ii), required particular solution is

$$\cot \frac{y}{x} = \log |x| + 1 = \log |x| + \log e = \log |ex|.$$

14. $\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0;\ y = 0$ **when** $x = 1$

Sol. The given differential equation is

$$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec} \frac{y}{x} = 0;\ y = 0 \text{ when } x = 1$$

or $$\frac{dy}{dx} = \frac{y}{x} - \text{cosec}\ \frac{y}{x} = f\left(\frac{y}{x}\right) \qquad ...(i)$$

$\therefore$ Given differential equation (i) is homogeneous.

Put $\frac{y}{x} = v \quad \therefore \quad y = vx \qquad \therefore \quad \frac{dy}{dx} = v\ .\ 1 + x\ \frac{dv}{dx}$

Putting these values in differential equation (i),

$$v + x\ \frac{dv}{dx} = v - \text{cosec}\ v \quad \Rightarrow \quad x\ \frac{dv}{dx} = \frac{-1}{\sin v}$$

$\therefore \quad x \sin v\ dv = -\ dx$

Separating variables, $\quad \sin v\ dv = -\ \frac{dx}{x}$

Integrating both sides, $\quad \int \sin v\ dv = -\int \frac{1}{x}\ dx$

$$-\cos v = -\log |x| + c$$

Dividing by – 1, $\quad \cos v = \log |x| - c$

Putting $v = \frac{y}{x}$, $\quad \cos \frac{y}{x} = \log |x| - c \qquad ...(ii)$

To find *c*: Given: $y = 0$ when $x = 1$

$\therefore$ From (ii), $\cos 0 = \log 1 - c \quad$ or $\quad 1 = 0 - c = -c$

$\therefore \quad c = -1$

Putting $c = -1$ in (ii), $\cos \frac{y}{x} = \log |x| + 1 = \log |x| + \log e$

$\Rightarrow \quad \cos \frac{y}{x} = \log |ex|$ which is the required particular solution.

15. $\mathbf{2xy + y^2 - 2x^2 \frac{dy}{dx} = 0;\ y = 2}$ **when** $\mathbf{x = 1}$

Sol. The given differential equation is

$$2xy + y^2 - 2x^2 \frac{dy}{dx} = 0;\ y = 2 \text{ when } x = 1. \qquad ...(i)$$

The given differential equation looks to be homogeneous because each coefficient of dx and dy is of same degree (2 here).

From (i), $-\ 2x^2 \frac{dy}{dx} = -\ 2xy - y^2 \quad \therefore \quad \frac{dy}{dx} = \frac{-\ 2xy}{-\ 2x^2} - \frac{y^2}{-\ 2x^2}$

or $$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{2}\left(\frac{y}{x}\right)^2 = F\left(\frac{y}{x}\right) \qquad ...(ii)$$

$\therefore$ The given differential equation is homogeneous.

Put $\frac{y}{x} = v \quad \therefore \quad y = vx \quad \therefore \quad \frac{dy}{dx} = v\ .\ 1 + x\ \frac{dv}{dx} = v + x\ \frac{dv}{dx}$

Putting these values in differential equation (ii), we have

$$v + x\ \frac{dv}{dx} = v + \frac{1}{2}\ v^2 \quad \Rightarrow \quad x\ \frac{dv}{dx} = \frac{1}{2}\ v^2 \quad \Rightarrow 2x\ dv = v^2\ dx$$

Separating variables, $\quad 2\ \frac{dv}{v^2} = \frac{dx}{x}$

Integrating both sides, $2 \int v^{-2}\, dv = \int \frac{1}{x}\, dx$

$\Rightarrow \quad 2\, \frac{v^{-1}}{-1} = \log |x| + c \quad \Rightarrow \quad \frac{-2}{v} = \log |x| + c$

Putting $v = \frac{y}{x}$, $\quad \frac{-2}{\left(\frac{y}{x}\right)} = \log |x| + c$

or $\frac{-2x}{y} = \log |x| + c$ ...(*iii*)

To find *c*: Given: $y = 2$, when $x = 1$.

$\therefore$ From (*iii*), $\frac{-2}{2} = \log 1 + c$ or $-1 = c$

Putting $c = -1$ in (*iii*), the required particular solution is

$$-\frac{2x}{y} = \log |x| - 1$$

$\Rightarrow \quad y(\log |x| - 1) = -2x \qquad \Rightarrow \quad y = \frac{-2x}{\log|x| - 1}$

$\Rightarrow \qquad y = \frac{-2x}{-(1 - \log|x|)} \quad \Rightarrow \quad y = \frac{2x}{1 - \log|x|}$.

16. Choose the correct answer:

A homogeneous differential equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution:

(A) $y = vx$ (B) $v = yx$ (C) $x = vy$ (D) $x = v$

Sol. We know that a homogeneous differential equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by the substitution $\frac{x}{y} = v$ *i.e.*, $x = vy$.

$\therefore$ Option (C) is the correct answer.

17. Which of the following is a homogeneous differential equation?

(A) $(4x + 6y + 5)\, dy - (3y + 2x + 4)\, dx = 0$

(B) $(xy)\, dx - (x^3 + y^3)\, dy = 0$ (C) $(x^3 + 2y^2)\, dx + 2xy\, dy = 0$

(D) $y^2\, dx + (x^2 - xy - y^2)\, dy = 0$

Sol. Out of the four given options; option (D) is the only option in which all coefficients of dx and dy are of **same degree** (here 2). It may be noted that xy is a term of second degree.

Hence differential equation in option (D) is **Homogeneous differential equation.**