NCERT Class 12 Maths

Solutions

Chapter - 9

Exercise 9.3

In each of the Exercises 1 to 5, form a differential equation representing the given family of curves by eliminating arbitrary constants *a* and *b*.

1. $\frac{x}{a} + \frac{y}{b} = 1$

Sol. Equation of the given family of curves is $\frac{x}{a} + \frac{y}{b} = 1$ ...(*i*)

Here there are two arbitrary constants *a* and *b*. So we shall differentiate both sides of (*i*) two times w.r.t. *x*.

From (*i*), $\frac{1}{a} . 1 + \frac{1}{b} \frac{dy}{dx} = 0$ or $\frac{1}{a} = -\frac{1}{b}\frac{dy}{dx}$ ...(*ii*)

Again diff. (*ii*) w.r.t. *x*, $0 = -\frac{1}{b}\frac{d^2y}{dx^2}$

Multiplying both sides by $-b$, $\frac{d^2y}{dx^2} = 0.$

Which is the required D.E.

Remark. We need not eliminate *a* and *b* because they have already got eliminated in the process of differentiation.

2. $\boldsymbol{y^2 = a(b^2 - x^2)}$

Sol. Equation of the given family of curves is

$$y^2 = a(b^2 - x^2) \quad ...(i)$$

Here there are two arbitrary constants *a* and *b*. So, we are to differentiate (*i*) twice w.r.t. *x*.

From (*i*), $2y \frac{dy}{dx} = a(0 - 2x) = -2ax.$

Dividing by 2, $y\frac{dy}{dx} = -ax$ ...(*ii*)

Again differentiating both sides of (*ii*) w.r.t. *x*,

$$y\frac{d^2y}{dx^2} + \frac{dy}{dx} . \frac{dy}{dx} = -a \quad \text{or} \quad y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -a \quad ...(iii)$$

Putting this value of $-a$ from (*iii*) in (*ii*), (To eliminate *a*, as *b* is already absent in both (*ii*) and (*iii*)), we have

$$y\frac{dy}{dx} = \left(y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right) x \quad \text{or} \quad xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = y\frac{dy}{dx}$$

$$\text{or} \quad xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0.$$

3. $\boldsymbol{y = ae^{3x} + be^{-2x}}$

Sol. Equation of the family of curves is

$$y = a\,e^{3x} + b\,e^{-2x} \quad ...(i)$$

Here are two arbitrary constants *a* and *b*.

From (*i*), $\frac{dy}{dx} = 3\,ae^{3x} - 2\,be^{-2x}$ ...(*ii*)

Again differentiating both sides of (*ii*), w.r.t. *x*,

$$\frac{d^2y}{dx^2} = 9\,ae^{3x} + 4\,be^{-2x} \quad ...(iii)$$

Let us eliminate *a* and *b* from (*i*), (*ii*) and (*iii*).

Equation (*ii*) – 3 × eqn. (*i*) gives (To eliminate *a*),

$$\frac{dy}{dx} - 3y = -5\ be^{-2x} \qquad ...(iv)$$

Again Eqn. (*iii*) – 3 × eqn (*ii*) gives (again to eliminate *a*)

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 10\ be^{-2x} \qquad ...(v)$$

Now Eqn. (*v*) + 2 × eqn. (*iv*) gives (To eliminate *b*)

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2\left(\frac{dy}{dx} - 3y\right) = 10\ be^{-2x} - 10\ be^{-2x}$$

or $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2\frac{dy}{dx} - 6y = 0$$

or $$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

which is the required D.E.

4. $\boldsymbol{y = e^{2x}(a + bx)}$

Sol. Equation of the given family of curves is

$$y = e^{2x}(a + bx) \qquad ...(i)$$

Here are two arbitrary constants *a* and *b*.

From (*i*), $$\frac{dy}{dx} = \left(\frac{d}{dx}e^{2x}\right)(a + bx) + e^{2x}\frac{d}{dx}(a + bx)$$

or $$\frac{dy}{dx} = 2\ e^{2x}(a + bx) + e^{2x} \cdot b$$

or $$\frac{dy}{dx} = 2y + be^{2x} \qquad ...(ii)$$

(By (*i*))

Again differentiating both sides of (*ii*), w.r.t. *x*

$$\frac{d^2y}{dx^2} = 2\frac{dy}{dx} + 2\ be^{2x} \qquad ...(iii)$$

Let us eliminate *b* from eqns. (*ii*) and (*iii*), (as *a* is already absent in both (*ii*) and (*iii*))

From eqn. (*ii*) $\frac{dy}{dx} - 2y = be^{2x}$

Putting this value of be^{2x} in (*iii*), we have

$$\frac{d^2y}{dx^2} = 2\frac{dy}{dx} + 2\left(\frac{dy}{dx} - 2y\right) \Rightarrow \frac{d^2y}{dx^2} = 2\frac{dy}{dx} + 2\frac{dy}{dx} - 4y$$

or $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

which is the required D.E.

5. $\boldsymbol{y = e^{x}(a \cos x + b \sin x)}$

Sol. Equation of family of curves is

$$y = e^x(a \cos x + b \sin x) \qquad ...(i)$$

$$\therefore \frac{dy}{dx} = \left(\frac{d}{dx}e^x\right)(a \cos x + b \sin x) + e^x(-a \sin x + b \cos x)$$

or $\dfrac{dy}{dx} = e^x (a \cos x + b \sin x) + e^x (-a \sin x + b \cos x)$

or $\dfrac{dy}{dx} = y + e^x (-a \sin x + b \cos x)$ ...(*ii*)

(By (*i*))

Again differentiating both sides of eqn. (*ii*), w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x (-a \sin x + b \cos x) + e^x (-a \cos x - b \sin x)$$

or $\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} + \left(\dfrac{dy}{dx} - y\right) - e^x (a \cos x + b \sin x)$

(By (*ii*))

or $\dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dx} - y - y$

(By (*i*))

or $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$ which is the required D.E.

6. Form the differential equation of the family of circles touching the *y*-axis at the origin.

Sol. Clearly, a circle which touches y-axis at the origin must have its centre on x-axis.

[$\because$ x-axis being at right angles to tangent y-axis is the normal or line of radius of the circle.]

$\therefore$ **The centre of circle is $(r, 0)$ where r is the radius of the circle.**

$\therefore$ Equation of required circles is

$(x - r)^2 + (y - 0)^2 = r^2$ $\qquad [(x - \alpha)^2 + (y - \beta)^2 = r^2]$

or $x^2 + r^2 - 2rx + y^2 = r^2$

or $x^2 + y^2 = 2rx$ ...(*i*)

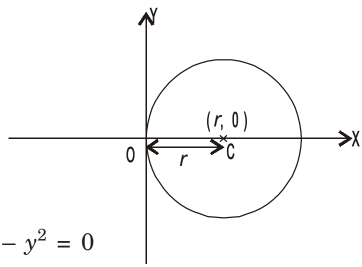
where r is the only arbitrary constant.

$\therefore$ Differentiating both sides of (*i*) only once w.r.t. x, we have

$$2x + 2y\frac{dy}{dx} = 2r \qquad ...(ii)$$

To eliminate r, putting the value of $2r$ from (*ii*) in (*i*),

$$x^2 + y^2 = \left(2x + 2y\frac{dy}{dx}\right)x$$



or $x^2 + y^2 = 2x^2 + 2xy\dfrac{dy}{dx}$

or $-2xy\dfrac{dy}{dx} - x^2 + y^2 = 0$

Multiplying by -1, $2xy\dfrac{dy}{dx} + x^2 - y^2 = 0$

or $2xy\dfrac{dy}{dx} + x^2 = y^2$ which is the required D.E.

Remark. The above question can also be stated as : **Form the D.E. of the family of circles passing through the origin and having centres on *x*-axis.**

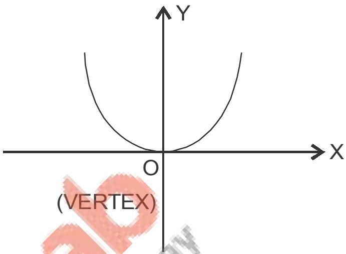
7. Find the differential equation of the family of parabolas having vertex at origin and axis along positive *y*-axis.

Sol. We know that equation of parabolas having vertex at origin and axis along positive y-axis is $\boldsymbol{x^2 = 4ay}$ ...(i)

Here a is the only arbitrary constant. So differentiating both sides of Eqn. (i) only once w.r.t. x, we have

$$2x = 4a \frac{dy}{dx} \quad ...(ii)$$

To eliminate a, putting



$4a = \frac{x^2}{y}$ from (i) in (ii), we have

$$2x = \frac{x^2}{y} \frac{dy}{dx}$$

$$\Rightarrow \quad 2xy = x^2 \frac{dy}{dx}$$

Dividing both sides by x, $2y = x \frac{dy}{dx}$

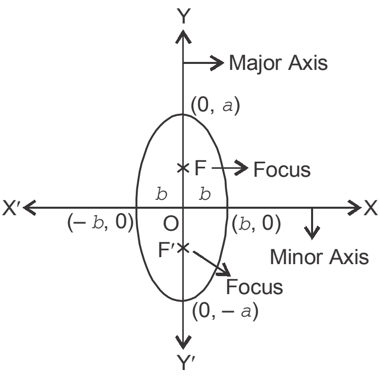
$$\Rightarrow \quad - x \frac{dy}{dx} + 2y = 0$$

$$\Rightarrow \quad x \frac{dy}{dx} - 2y = 0$$ which is the required D.E.

8. Form the differential equation of family of ellipses having foci on *y*-axis and centre at the origin.

Sol. We know that equation of ellipses having foci on y-axis *i.e.*, vertical ellipses with major axis as y-axis is

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1 \quad ...(i)$$



Here a and b are two arbitrary constants.

So we shall differentiate eqn. (i) twice w.r.t. x.

Differentiating both sides of (i) w.r.t. x, we have

$$\frac{1}{a^2} 2y \frac{dy}{dx} + \frac{1}{b^2} 2x = 0$$

or
$$\frac{2}{a^2} y \frac{dy}{dx} = - \frac{2}{b^2} x$$

Dividing both sides by 2,

$$\frac{1}{a^2}\ y\ \frac{dy}{dx} = \frac{-1}{b^2}\ x \qquad \text{...}(ii)$$

Again differentiating both sides of (*ii*) w.r.t. *x*, we have

$$\frac{1}{a^2}\left[y\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{dy}{dx}\right] = \frac{-1}{b^2} \qquad \text{...}(iii)$$

To eliminate *a* and *b*, putting this value of $\frac{-1}{b^2}$ from (*iii*) in (*ii*), the required differential equation is

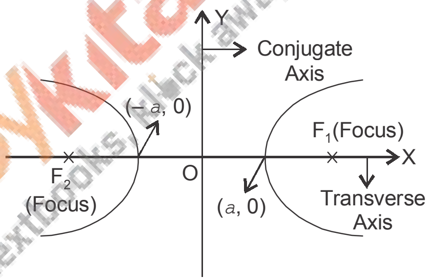
$$\frac{1}{a^2}\ y\ \frac{dy}{dx} = \frac{1}{a^2}\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right] x$$

Multiplying both sides by a^2, $y\ \frac{dy}{dx} = xy\ \frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2$

or $\quad xy\ \frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\ \frac{dy}{dx} = 0$

which is the required differential equation.

9. Form the differential equation of the family of hyperbolas having foci on *x*-axis and centre at the origin.



Sol. We know that equation of hyperbolas having foci on *x*-axis and centre at origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{...}(i)$$

Here *a* and *b* are two arbitrary constants. So we shall differentiate eqn. (*i*) twice w.r.t. *x*.

From (*i*), $\frac{1}{a^2}\ .\ 2x - \frac{1}{b^2}\ .\ 2y\ \frac{dy}{dx} = 0$ or $\frac{2}{a^2}\ x = \frac{2}{b^2}\ y\ \frac{dy}{dx}$

Dividing both sides by 2, $\quad \frac{1}{a^2}\ x = \frac{1}{b^2}\ y\ \frac{dy}{dx} \qquad \text{...}(ii)$

Again differentiating both sides of (*ii*), w.r.t. *x*,

$$\frac{1}{a^2}\ .\ 1 = \frac{1}{b^2}\ .\left[y\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{dy}{dx}\right]$$

or $$\frac{1}{a^2} = \frac{1}{b^2}\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right] \qquad \text{...}(iii)$$

Dividing eqn. (*iii*) by eqn. (*ii*), we have (To eliminate *a* and *b*)

$$\frac{1}{x} = \frac{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2}{y\frac{dy}{dx}}$$

Cross-multiplying, $x\left(y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right) = y\frac{dy}{dx}$

or $$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

which is the required differential equation.

10. Form the differential equation of the family of circles having centres on y-axis and radius 3 units.

Sol. We know that on y-axis, $x = 0$.

$\therefore$ Centre of the circle on y-axis is $(0, \beta)$.

$\therefore$ Equation of the circle having centre on y-axis and radius 3 units is

$$(x - 0)^2 + (y - \beta)^2 = 3^2 \qquad [(x - \alpha)^2 + (y - \beta)^2 = r^2]$$

or $$x^2 + (y - \beta)^2 = 9 \qquad ...(i)$$

Here β is the only arbitrary constant. So we shall differentiate both sides of eqn. (i) only once w.r.t. x,

From (i), $2x + 2(y - \beta)\frac{d}{dx}(y - \beta) = 0$

or $2x + 2(y - \beta)\frac{dy}{dx} = 0$

or $2(y - \beta)\frac{dy}{dx} = -2x \qquad \therefore \qquad y - \beta = \frac{-2x}{2\frac{dy}{dx}} = \frac{-x}{\frac{dy}{dx}}$

Putting this value of $(y - \beta)$ in (i) (To eliminate β), we have

$$x^2 + \frac{x^2}{\left(\frac{dy}{dx}\right)^2} = 9$$

L.C.M. $= \left(\frac{dy}{dx}\right)^2$. Multiplying both sides by this L.C.M.,

$$x^2\left(\frac{dy}{dx}\right)^2 + x^2 = 9\left(\frac{dy}{dx}\right)^2$$

$\Rightarrow x^2\left(\frac{dy}{dx}\right)^2 - 9\left(\frac{dy}{dx}\right)^2 + x^2 = 0$ or $(x^2 - 9)\left(\frac{dy}{dx}\right)^2 + x^2 = 0$

which is the required differential equation.

11. Which of the following differential equation has $y = c_1 e^x + c_2 e^{-x}$ as the general solution?

(A) $\frac{d^2y}{dx^2} + y = 0$ **(B)** $\frac{d^2y}{dx^2} - y = 0$

(C) $\frac{d^2y}{dx^2} + 1 = 0$ (D) $\frac{d^2y}{dx^2} - 1 = 0$

Sol. **Given:** $y = c_1 e^x + c_2 e^{-x}$ ...(i)

$\therefore \frac{dy}{dx} = c_1 e^x + c_2 e^{-x} (-1) = c_1 e^x - c_2 e^{-x}$

$\therefore \frac{d^2y}{dx^2} = c_1 e^x - c_2 e^{-x} (-1) = c_1 e^x + c_2 e^{-x}$

or $\frac{d^2y}{dx^2} = y$ [By (i)]

or $\frac{d^2y}{dx^2} - y = 0$ which is differential equation given in option (B)

$\therefore$ Option (B) is the correct answer.

12. Which of the following differential equations has $y = x$ as one of its particular solutions?

(A) $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = x$ (B) $\frac{d^2y}{dx^2} + x \frac{dy}{dx} + xy = x$

(C) $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 0$ (D) $\frac{d^2y}{dx^2} + x \frac{dy}{dx} + xy = 0$

Sol. **Given:** $y = x$

$\therefore \frac{dy}{dx} = 1$ and $\frac{d^2y}{dx^2} = 0$

These values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ clearly satisfy the D.E. of option (C).

[$\because$ L.H.S. of D.E. of option (C) = $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy$

$= 0 - x^2 (1) + x (x) = - x^2 + x^2 = 0$ = R.H.S. of option (C)]

$\therefore$ Option (C) is the correct answer.

Exercise 9.4 *(Page No. 395-397)*

For each of the differential equations in Exercises 1 to 4, find the general solution:

1. $\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$

Sol. The given differential equation is

$$\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x} \quad \text{or} \quad dy = \frac{1 - \cos x}{1 + \cos x} dx.$$

Integrating both sides, $\int dy = \int \frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} dx$

or $y = \int \tan^2 \frac{x}{2} dx = \int \left(\sec^2 \frac{x}{2} - 1\right) dx = \frac{\tan \frac{x}{2}}{\frac{1}{2}} - x + c$