Exercise 8.2

1. Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.

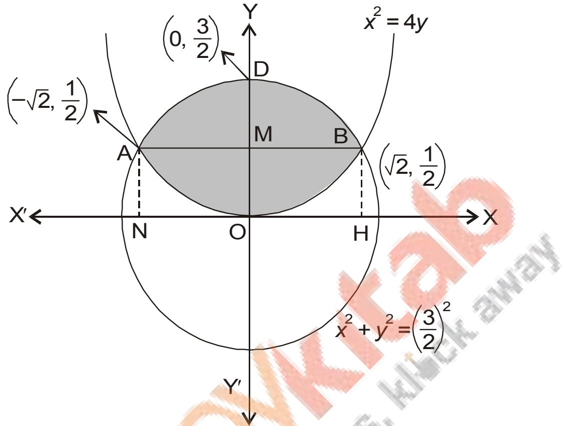
Sol. Step I. Let us draw graphs and shade the region of integration.

Given: Equation of the circle is $4x^2 + 4y^2 = 9$

Dividing by 4, $\quad x^2 + y^2 = \frac{9}{4} = \left(\frac{3}{2}\right)^2$ ...(*i*)

We know that this equation (*i*) represents a circle whose centre is (0, 0) and radius $\frac{3}{2}$ $(x^2 + y^2 = r^2)$

Equation of parabola is $x^2 = 4y$ ...(*ii*)



(eqn. (*ii*) represents an upward parabola symmetrical about *y*-axis)

Step II. Let us solve eqns. of circle (*i*) and parabola (*ii*) for *x* and *y* to find their points of intersection.

Putting $x^2 = 4y$ from (*ii*) in (*i*), we have $4y + y^2 = \frac{9}{4}$

Multiplying by L.C.M. (= 4),

$$16y + 4y^2 = 9 \quad \text{or} \quad 4y^2 + 16y - 9 = 0$$

$$\Rightarrow \quad 4y^2 + 18y - 2y - 9 = 0 \Rightarrow 2y(2y + 9) - 1(2y + 9) = 0$$

$$\Rightarrow \quad (2y + 9)(2y - 1) = 0$$

$\therefore$ Either $\quad 2y + 9 = 0 \quad$ or $\quad 2y - 1 = 0$

$\Rightarrow \quad 2y = -9 \quad$ or $\quad 2y = 1$

$\Rightarrow \quad y = -\frac{9}{2} \quad$ or $\quad y = \frac{1}{2}$

For $y = -\frac{9}{2}$, from (*i*) $x^2 = 4y = 4\left(-\frac{9}{2}\right) = -18$

which is impossible because square of a real number can never be negative.

For $y = \frac{1}{2}$, from (i), $\quad x^2 = 4y = 4 \times \frac{1}{2} = 2$

$\therefore \quad x = \pm \sqrt{2}$

$\therefore$ Points of intersections of circle (i) and parabola (ii) are

$$A\left(-\sqrt{2}, \frac{1}{2}\right) \text{ and } B\left(\sqrt{2}, \frac{1}{2}\right).$$

Step III. Area OBM = Area between parabola (ii) and y-axis

$$= \left| \int_0^{\frac{1}{2}} x \, dy \right|$$

($\because$ at O, $y = 0$ and at B, $y = \frac{1}{2}$)

From (ii), putting $x = \sqrt{4y} = 2\sqrt{y} = 2\, y^{\frac{1}{2}}$,

$$\text{Area OBM} = \left| \int_0^{\frac{1}{2}} 2y^{\frac{1}{2}} \, dy \right| = 2 \,.\, \frac{\left(y^{\frac{3}{2}}\right)_0^{\frac{1}{2}}}{\frac{3}{2}}$$

$$= 2 \,.\, \frac{2}{3} \left[\left(\frac{1}{2}\right)^{\frac{3}{2}} - 0 \right] = \frac{4}{3} \; \frac{1}{2} \sqrt{\frac{1}{2}} \quad \left[\because \; x^{\frac{3}{2}} = x\sqrt{x}\,\right]$$

$$= \frac{2}{3} \; \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{3} \qquad ...(iii) \qquad \left| \because \; \frac{x}{\sqrt{x}} = \sqrt{x} \right.$$

Step IV. Now area BDM = Area between circle (i) and y-axis

$$= \left| \int_{\frac{1}{2}}^{\frac{3}{2}} x \, dy \right| \quad \left[\because \text{ At point B, } y = \frac{1}{2} \text{ and at point D, } y = \frac{3}{2}\right]$$

From (i), putting $x^2 = \left(\frac{3}{2}\right)^2 - y^2$ i.e., $x = \sqrt{\left(\frac{3}{2}\right)^2 - y^2}$,

$$= \left| \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\left(\frac{3}{2}\right)^2 - y^2} \, dy \right| = \left[\frac{y}{2} \sqrt{\left(\frac{3}{2}\right)^2 - y^2} + \frac{\left(\frac{3}{2}\right)^2}{2} \sin^{-1} \frac{y}{\left(\frac{3}{2}\right)} \right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$\left[\because \int \sqrt{a^2 - y^2} \, dy = \frac{y}{2} \sqrt{a^2 - y^2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a}\right]$$

$$= \frac{3}{4} \sqrt{\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2} + \frac{9}{8} \sin^{-1} \left(\frac{\frac{3}{2}}{\frac{3}{2}}\right) - \left[\frac{1}{4} \sqrt{\frac{9}{4} - \frac{1}{4}} + \frac{9}{8} \sin^{-1} \left(\frac{\frac{1}{2}}{\frac{3}{2}}\right) \right]$$

$$= \left(\frac{3}{4}\times 0\right) + \frac{9}{8}\sin^{-1} 1 - \left[\frac{1}{4}\sqrt{\frac{8}{4}} + \frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$= \frac{9}{8} \times \frac{\pi}{2} - \frac{1}{4}\sqrt{2} - \frac{9}{8}\sin^{-1}\frac{1}{3}$$

$$= \frac{9\pi}{16} - \frac{\sqrt{2}}{4} - \frac{9}{8}\sin^{-1}\frac{1}{3} \qquad ...(iv)$$

Step V. $\therefore$ Required shaded area (of circle (i) which is interior to parabola (ii)) = Area AOBDA

= 2(Area OBD) = 2[Area OBM + Area MBD]

$$= 2\left[\frac{\sqrt{2}}{3} + \left(\frac{9\pi}{16} - \frac{\sqrt{2}}{4} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right)\right]$$

(By (iii)) (By (iv))

$$= 2\left[\sqrt{2}\left(\frac{1}{3} - \frac{1}{4}\right) + \frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$= 2\sqrt{2}\left(\frac{4-3}{12}\right) + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}$$

$$= \left(\frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}\right) = \frac{\sqrt{2}}{6} + \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right)$$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\cos^{-1}\frac{1}{3} \textbf{ sq. units.}$$

Ans $\left(\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right)$

Remark: $= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\sqrt{1-\frac{1}{9}}$ $\quad (\because \cos^{-1} x = \sin^{-1}\sqrt{1-x^2})$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\sqrt{\frac{8}{9}} = \left(\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right) \text{ sq. units.}$$

Note: The equation $(x-\alpha)^2 + (y-\beta)^2 = r^2$ represents a circle whose centre is (α, β) and radius is r.

2. Find the area bounded by the curves $(x - 1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.

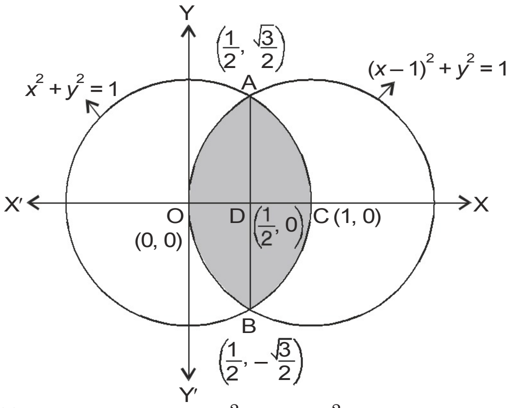
Sol. The equations of the two circles are

$$x^2 + y^2 = 1 \qquad ...(i)$$

and $$(x - 1)^2 + y^2 = 1 \qquad ...(ii)$$

The first circle has centre at the origin and radius 1. The second circle has centre at (1, 0) and radius 1. Both are symmetrical about the x-axis. Circle (i) is symmetrical about y-axis also.

For points of intersections of circles (*i*) and (*ii*), let us solve equations (*i*) and (*ii*) for *x* and *y*.



From (*i*), $\quad y^2 = 1 - x^2$

Putting $y^2 = 1 - x^2$ in eqn. (*ii*), $\quad (x - 1)^2 + 1 - x^2 = 1$

or $\quad x^2 + 1 - 2x + 1 - x^2 = 1$

or $\quad -2x + 1 = 0 \quad \therefore \quad x = \frac{1}{2}$

Putting $x = \frac{1}{2}$, $y^2 = 1 - x^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore \quad y = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$

$\therefore$ The two points of intersections of circles (*i*) and (*ii*) are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

From (*i*), $\quad y^2 = 1 - x^2$;

$\therefore \quad y = \sqrt{1 - x^2}$ in first quadrant.

From (*ii*), $\quad y^2 = 1 - (x - 1)^2$ and

$\therefore \quad y = \sqrt{1 - (x - 1)^2}$ in first quadrant.

Required area OACBO (area enclosed between the two circles) (shown shaded)

= 2 × Area OAC

= 2 [Area OAD + Area DAC]

$$= 2\left[\int_0^{1/2} y \text{ of circle } (ii)\, dx + \int_{1/2}^{1} y \text{ of circle } (i)\, dx\right]$$

$$= 2\left[\int_0^{1/2} \sqrt{1 - (x-1)^2}\, dx + \int_{1/2}^{1} \sqrt{1 - x^2}\, dx\right]$$

$$= 2\left[\left\{\frac{(x-1)\sqrt{1-(x-1)^2}}{2} + \frac{1}{2}\sin^{-1}(x-1)\right\}_0^{1/2} + \left\{\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x\right\}_{1/2}^{1}\right]$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \left\{-\frac{1}{2}\sqrt{\frac{3}{4}} + \sin^{-1}\left(-\frac{1}{2}\right)\right\} - \{\sin^{-1}(-1)\} + \sin^{-1} 1 - \left\{\frac{1}{2}\sqrt{\frac{3}{4}} + \sin^{-1}\frac{1}{2}\right\}$$

$$= -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} \qquad = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{ sq. units.}$$

3. Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.

Sol. Equation of the given curve is $y = x^2 + 2$ ...(*i*)

or $x^2 = y - 2$

It is an upward parabola ($\because$ An equation of the form $x^2 = ky$, $k > 0$ represents an upward parabola).

Eqn. (*i*) contains only even powers of x and hence remains unchanged on changing x to $-x$ in (*i*).

$\therefore$ The parabola (*i*) is symmetrical about y-axis.

Parabola (*i*) meets y-axis (its line of symmetry) *i.e.* $x = 0$ in (0, 2) [put $x = 0$ in (*i*) to get $y = 2$]

$\therefore$ Vertex of the parabola is (0, 2).

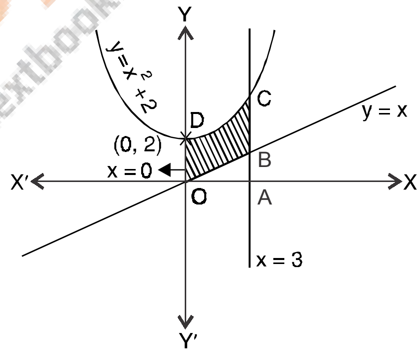
Equation of the given line is $y = x$ ...(*ii*)

We know that it is a straight line passing through the origin and having slope 1 *i.e.*, making an angle of 45° with x-axis.

Table of values for the line $y = x$

x	0	1	2
y	0	1	2

Also the required area is given to be bounded by the vertical lines $x = 0$ to $x = 3$.



$\therefore$ Limits of integration are given to be $x = 0$ to $x = 3$.

Area bounded by parabola (*i*) namely $y = x^2 + 2$, the x-axis and the ordinates $x = 0$ to $x = 3$ is the area OACD and

$$= \int_0^3 y\,dx = \int_0^3 (x^2 + 2)\,dx$$

$$= \left(\frac{x^3}{3} + 2x\right)_0^3 = (9 + 6) - 0 = 15 \qquad ...(iii)$$

Area bounded by line (*ii*) namely $y = x$, the x-axis and the ordinates $x = 0$, $x = 3$ is area OAB and $= \int_0^3 y\,dx = \int_0^3 x\,dx = \left(\dfrac{x^2}{2}\right)_0^3$

$$= \frac{9}{2} - 0 = \frac{9}{2} \qquad ...(iv)$$

$\therefore$ Required area (shown shaded) *i.e.,* area OBCD

= area OACD – area OAB

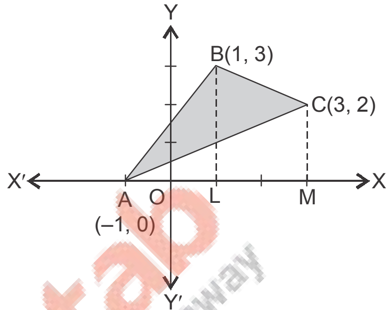
= Area given by (*iii*) – Area given by (*iv*)

$= 15 - \dfrac{9}{2} = \dfrac{21}{2}$ sq. units.

Remark: On solving Eqns (*i*) and (*ii*) for x we get imaginary values of x and hence curves (*i*) and (*ii*) don't intersect.

4. Using integration, find the area of the region bounded by the triangle whose vertices are (– 1, 0), (1, 3) and (3, 2).

Sol. Given: Vertices of triangle are A(– 1, 0), B(1, 3) and C(3, 2).



$\therefore$ Equation of line AB is

$$y - 0 = \frac{3-0}{1-(-1)}(x - (-1))$$

$$\left(\boldsymbol{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}\right)$$

or $\quad y = \dfrac{3}{2}(x + 1)$

$\therefore$ Area of ΔABL = Area bounded by this line AB and x-axis

$$= \left|\int_{-1}^{1} y\,dx\right|$$

($\because$ At point A, $x = -1$ and at point B, $x = 1$)

$$= \left|\int_{-1}^{1} \frac{3}{2}(x+1)\,dx\right| = \frac{3}{2}\left|\int_{-1}^{1}(x+1)\,dx\right|$$

$$= \frac{3}{2}\left|\left(\frac{x^2}{2}+x\right)_{-1}^{1}\right| = \frac{3}{2}\left[\left(\frac{1}{2}+1\right)-\left(\frac{1}{2}-1\right)\right]$$

$$= \frac{3}{2}\left(\frac{3}{2}-\left(-\frac{1}{2}\right)\right) = \frac{3}{2}\left(\frac{3}{2}+\frac{1}{2}\right) = \frac{3}{2}\cdot\frac{4}{2} = 3 \qquad ...(i)$$

Again equation of line BC is

$$y - 3 = \frac{2-3}{3-1}(x - 1)$$

$$\Rightarrow y - 3 = -\frac{1}{2}(x - 1) \Rightarrow y = 3 - \left(\frac{x-1}{2}\right) = \frac{6-x+1}{2}$$

$$\Rightarrow \quad y = \frac{7-x}{2} = \frac{1}{2}(7 - x)$$

$\therefore$ Area of trapezium BLMC = Area bounded by line BC and x-axis

$$= \left| \int_1^3 y\,dx \right| = \left| \int_1^3 \frac{1}{2}(7-x)\,dx \right|$$

$$= \frac{1}{2}\left(7x - \frac{x^2}{2}\right)_1^3 = \frac{1}{2}\left[21 - \frac{9}{2} - \left(7 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left(21 - \frac{9}{2} - 7 + \frac{1}{2}\right) = \frac{1}{2}\left(\frac{42-9-14+1}{2}\right) = \frac{1}{4}(20)$$

$$= 5 \qquad ...(ii)$$

Again equation of line AC is

$$y - 0 = \frac{2-0}{3-(-1)}(x-(-1)) \Rightarrow y = \frac{2}{4}(x+1)$$

$$\Rightarrow \quad y = \frac{1}{2}(x+1)$$

$\therefore$ Area of ΔACM = Area bounded by line AC and x-axis

$$= \left| \int_{-1}^3 y\,dx \right| = \left| \int_{-1}^3 \frac{1}{2}(x+1)\,dx \right| = \frac{1}{2}\left(\frac{x^2}{2} + x\right)_{-1}^3$$

$$= \frac{1}{2}\left[\frac{9}{2} + 3 - \left(\frac{1}{2} - 1\right)\right] = \frac{1}{2}\left[\frac{9}{2} + 3 - \frac{1}{2} + 1\right]$$

$$= \frac{1}{2}\left[\frac{9+6-1+2}{2}\right] = \frac{16}{4} = 4 \qquad ...(iii)$$

We can observe from the figure that required area of ΔABC

= Area of ΔABL + Area of Trapezium BLMC – Area of ΔACM

= 3 + 5 – 4 = 4 sq. units.

By (*i*) By (*ii*) By (*iii*)

5. Using integration, find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

Sol. Equation of one side of triangle is $y = 2x + 1$ ...(*i*)

Equation of second side of triangle is $y = 3x + 1$ ...(*ii*)

Third side of triangle is $x = 4$. ...(*iii*)

It is a line parallel to y-axis at a distance 4 to right of y-axis.

Let us solve (*i*) and (*ii*) for x and y.

Eqn. (*ii*) – eqn. (*i*)

gives $x = 0$.

Put $x = 0$ in (*i*), $y = 1$.

$\therefore$ Point of intersection of lines (*i*) and (*ii*) is A(0, 1)

Putting $x = 4$ from (*iii*) in (*i*), $y = 8 + 1 = 9$

$\therefore$ Point of intersection of lines (*i*) and (*iii*) is B(4, 9).

Putting $x = 4$ from (*iii*) in (*ii*), $y = 12 + 1 = 13$.

∴ Point of intersection of lines (*ii*) and (*iii*) is C(4, 13).

Area between line (*ii*) *i.e.,* line AC and *x*-axis

$$= \int_0^4 y\,dx = \int_0^4 (3x+1)\,dx \quad \text{[By } (ii)\text{]}$$

$$= \left(\frac{3x^2}{2} + x\right)_0^4$$

$= 24 + 4 = 28$ sq. units ...(*iv*)

Area between line (*i*) *i.e.,* line AB and *x*-axis

$$= \int_0^4 y\,dx = \int_0^4 (2x+1)\,dx$$

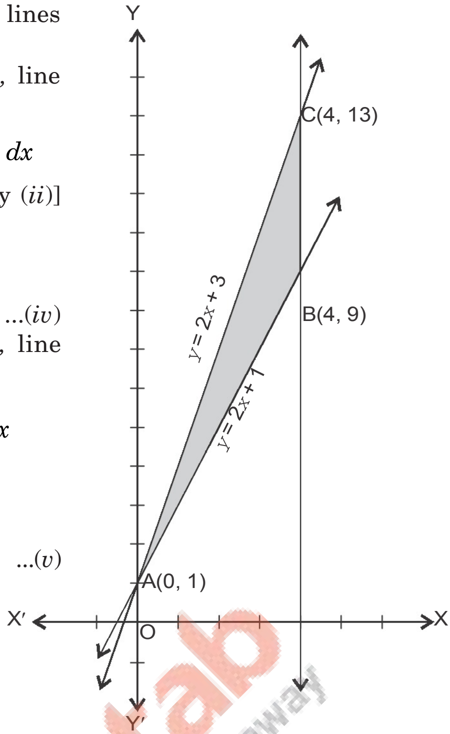
$$= \left(x^2 + x\right)_0^4 \quad \text{[By } (i)\text{]}$$

$= 16 + 4 = 20$ sq. units ...(*v*)

∴ Area of triangle ABC = Area given by (*iv*)

– Area given by (*v*)

$= 28 - 20 = 8$ sq. units.



6. Choose the correct answer:

Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

(A) $2(\pi - 2)$ **(B) $\pi - 2$** **(C) $2\pi - 1$** **(D) $2(\pi + 2)$.**

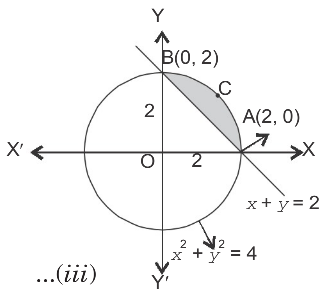
Sol. **Step I.** Equation of circle is $x^2 + y^2 = 4 = 2^2$ ...(*i*)

∴ $y^2 = 2^2 - x^2$

∴ $y = \sqrt{2^2 - x^2}$ ...(*ii*)

for arc AB of the circle in first quadrant.

We know that eqn. (*i*) represents a circle whose centre is origin and radius is 2.



Equation of the line is $x + y = 2$ ...(*iii*)

Table of values

x	0	2
y	2	0

∴ Graph of equation (*iii*) is the straight line joining the points (0, 2) and (2, 0).

The region for required area is shown as shaded in the figure.

Step II. From the graphs of circle (*i*) and straight line (*iii*), it is clear that points of intersections of circle (*i*) and straight line (*iii*) are A(2, 0) and B(0, 2).

Step III. Area OACB, bounded by circle (*i*) and coordinate axes in first quadrant

$$= \left| \int_0^2 y\, dx \right| = \int_0^2 \sqrt{2^2 - x^2}\, dx \quad (\because \text{ From } (ii),\ y = \sqrt{2^2 - x^2}\ )$$

$$= \left(\frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2} \right)_0^2$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]$$

$$= \left(\frac{2}{2}\sqrt{4 - 4} + 2\sin^{-1} 1 \right) - (0 + 2\sin^{-1} 0)$$

$$= 0 + 2\left(\frac{\pi}{2}\right) - 2(0) = \pi \qquad \text{...}(iv)$$

Step IV. Area of triangle OAB, bounded by straight line (*iii*) and co-ordinate axes

$$= \left| \int_0^2 y\, dx \right| = \left| \int_0^2 (2 - x)\, dx \right| \qquad (\because \text{ From } (iii),\ y = 2 - x)$$

$$= \left(2x - \frac{x^2}{2} \right)_0^2 = (4 - 2) - (0 - 0) = 2 \qquad \text{...}(v)$$

Step V. $\therefore$ Required shaded area

= Area OACB given by (*iv*) – Area of triangle OAB by (*v*)

= $(\pi - 2)$ sq. units.

$\therefore$ Option (B) is the correct answer.

7. Choose the correct answer:

Area lying between the curves $y^2 = 4x$ and $y = 2x$ is

(A) $\frac{2}{3}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{3}{4}$.

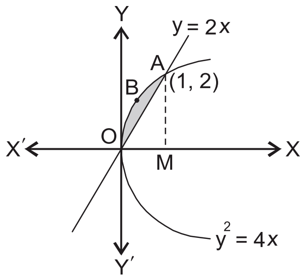
Sol. Step I. Equation of one curve (parabola) is

$y^2 = 4x$ ...(*i*)

$\therefore \quad y = \sqrt{4x} = 2\sqrt{x} = 2x^{\frac{1}{2}}$ ...(*ii*)

for arc of the parabola in first quadrant.

We know that eqn. (*i*) represents a rightward parabola symmetrical about *x*-axis.



Equation of second curve (line) is $y = 2x$ ...(*iii*)

We know that $y = 2x$ represents a straight line passing through the origin.

We are required to find the area of the shaded region.

II. Let us solve (*i*) and (*iii*) for *x* and *y*.

Putting $y = 2x$ from (*iii*) in (*i*), we have

$$4x^2 = 4x \quad \Rightarrow \quad 4x^2 - 4x = 0 \quad \Rightarrow \quad 4x(x - 1) = 0$$

$\therefore$ Either $4x = 0$ or $x - 1 = 0$

i.e., $x = \frac{0}{4} = 0$ or $x = 1$

When $x = 0$, from (*ii*), $y = 0$ $\therefore$ point is O(0, 0)

When $x = 1$, from (*ii*), $y = 2x = 2$ $\therefore$ point is A(1, 2)

$\therefore$ Points of intersections of circle (*i*) and line (*ii*) are O(0, 0) and A(1, 2).

III. Area OBAM = Area bounded by parabola (*i*) and *x*-axis

$$= \left| \int_0^1 y\, dx \right| = \left| \int_0^1 2x^{\frac{1}{2}}\, dx \right| \quad [\because \text{ From } (ii)\ y = 2x^{\frac{1}{2}}]$$

$$= 2\frac{\left(x^{\frac{3}{2}}\right)_0^1}{\frac{3}{2}} = \frac{4}{3}(1 - 0) = \frac{4}{3} \qquad \ldots(iv)$$

IV. Area of ΔOAM = Area of bounded by line (*iii*) and *x*-axis

$$= \left| \int_0^1 y\, dx \right| = \left| \int_0^1 2x\, dx \right| \qquad (\because \text{ From } (iii)\ y = 2x)$$

$$= 2\left(\frac{x^2}{2}\right)_0^1 = \left(x^2\right)_0^1 = 1 - 0 = 1 \qquad \ldots(v)$$

V. $\therefore$ Required shaded area OBA

= Area OBAM – Area of Δ OAM

$= \frac{4}{3} - 1 = \frac{4-3}{3} = \frac{1}{3}$ sq. units.

(By (*iv*)) (By (*v*))

$\therefore$ Option (B) is the correct answer.