# NCERT Class 12 Maths Solutions

# Chapter - 7

# Integrals

## Exercise 7.9

**Evaluate the definite integrals in Exercises 1 to 11:**

**Result.** If $\int f(x)\ dx = \phi(x)$, then $\int_a^b f(x)\ dx = \phi(b) - \phi(a)$ ...(*i*)

(This is known as **Second Fundamental Theorem**).

**1.** $\int_{-1}^{1} (x+1)\ dx$

**Sol.** $\int_{-1}^{1} (x+1)\ dx = \left(\frac{x^2}{2}+x\right)_{-1}^{1} = \phi(b) - \phi(a)$

(By Second Fundamental Theorem given in Eqn. (*i*) page 496)

$$= \left(\frac{1^2}{2}+1\right) - \left(\frac{(-1)^2}{2}-1\right) = \frac{1}{2} + 1 - \left(\frac{1}{2}-1\right)$$

$$= \frac{1}{2} + 1 - \frac{1}{2} + 1 = 2.$$

**Remark.** [Constant $c$ will never occur in the value of a definite integral because $c$ in the value of $\phi(b)$ gets cancelled with $c$ in $\phi(a)$ when we subtract them to get $\phi(b) - \phi(a)$].

**2.** $\int_{2}^{3} \frac{1}{x}\ dx$

**Sol.** $\int_{2}^{3} \frac{1}{x}\ dx = (\log|x|)_2^3 = \phi(b) - \phi(a) = \log|3| - \log|2|$

$$= \log 3 - \log 2 = \log \frac{3}{2}. \quad [\because\ |x| = x \text{ if } x \ge 0]$$

**3.** $\int_{1}^{2} (4x^3 - 5x^2 + 6x + 9)\ dx$

**Sol.** $\int_{1}^{2} (4x^3 - 5x^2 + 6x + 9)\ dx = \left(4\frac{x^4}{4} - 5\frac{x^3}{3} + 6\frac{x^2}{2} + 9x\right)_1^2$

$$= \left(x^4 - \frac{5}{3}x^3 + 3x^2 + 9x\right)_1^2$$

$$= \left[2^4 - \frac{5}{3}(2)^3 + 3(2)^2 + 9(2)\right] - \left[1 - \frac{5}{3} + 3 + 9\right]$$

$$= \left(16 - \frac{40}{3} + 12 + 18\right) - \left(13 - \frac{5}{3}\right)$$

$$= 46 - \frac{40}{3} - \left(13 - \frac{5}{3}\right) = 46 - \frac{40}{3} - 13 + \frac{5}{3}$$

$$= 33 - \frac{40}{3} + \frac{5}{3} = \frac{99 - 40 + 5}{3} = \frac{104 - 40}{3} = \frac{64}{3}.$$

**4.** $\int_{0}^{\frac{\pi}{4}} \sin 2x\ dx$

**Sol.** $\int_{0}^{\frac{\pi}{4}} \sin 2x\ dx = \left(\frac{-\cos 2x}{2}\right)_0^{\frac{\pi}{4}} = \frac{-\cos\frac{\pi}{2}}{2} - \left(\frac{-\cos 0}{2}\right)$

$$= \frac{-0}{2} - \left(\frac{-1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}.$$

**5.** $\int_0^{\frac{\pi}{2}} \cos 2x \ dx$

**Sol.** $\int_0^{\frac{\pi}{2}} \cos 2x \ dx = \left(\frac{\sin 2x}{2}\right)_0^{\frac{\pi}{2}} = \frac{\sin \pi}{2} - \frac{\sin 0}{2}$

$= \frac{0}{2} - \frac{0}{2} = 0$

$[\because \ \sin \pi = \sin 180° = \sin (180° - 0°) = \sin 0 = 0]$

**6.** $\int_4^5 e^x \ dx$

**Sol.** $\int_4^5 e^x \ dx = \left(e^x\right)_4^5 = e^5 - e^4 = e^4 (e - 1).$

**7.** $\int_0^{\frac{\pi}{4}} \tan x \ dx$

**Sol.** $\int_0^{\frac{\pi}{4}} \tan x \, dx = \left(\log |\sec x|\right)_0^{\frac{\pi}{4}}$

$= \log \left|\sec \frac{\pi}{4}\right| - \log |\sec 0| = \log |\sqrt{2}| - \log |1|$

$= \log \sqrt{2} - \log 1 = \log 2^{1/2} - 0 = \frac{1}{2} \log 2.$

**8.** $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \ dx$

**Sol.** $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \ dx = \left(\log |\operatorname{cosec} x - \cot x|\right)_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \log \left|\operatorname{cosec} \frac{\pi}{4} - \cot \frac{\pi}{4}\right| - \log \left|\operatorname{cosec} \frac{\pi}{6} - \cot \frac{\pi}{6}\right|$

$= \log |\sqrt{2} - 1| - \log |2 - \sqrt{3}|$

$= \log (\sqrt{2} - 1) - \log (2 - \sqrt{3})$ $\quad [\because \ |x| = x \text{ if } x \geq 0]$

$= \log \left(\frac{\sqrt{2} - 1}{2 - \sqrt{3}}\right).$

**9.** $\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$

**Sol.** $\int_0^1 \frac{dx}{\sqrt{1 - x^2}} = \left(\sin^{-1} x\right)_0^1$ $\quad \left[\because \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}\right]$

$= \sin^{-1} 1 - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$ $\quad \left[\because \ \sin \frac{\pi}{2} = 1 \text{ and } \sin 0 = 0\right]$

**10.** $\int_0^1 \frac{dx}{1 + x^2}$

**Sol.** $\int_0^1 \frac{dx}{1+x^2} = \left(\tan^{-1} x\right)_0^1 \qquad \left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

$= \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$

$\left[\because \quad \tan\frac{\pi}{4} = 1 \text{ and } \tan 0 = 0\right]$

**11.** $\int_2^3 \frac{dx}{x^2 - 1}$

**Sol.** $\int_2^3 \frac{1}{x^2-1}\,dx = \int_2^3 \frac{1}{x^2-1^2}\,dx$

$= \left(\frac{1}{2(1)}\log\left|\frac{x-1}{x+1}\right|\right)_2^3 \left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$

$= \frac{1}{2}\log\left|\frac{3-1}{3+1}\right| - \frac{1}{2}\log\left|\frac{2-1}{2+1}\right| = \frac{1}{2}\log\left|\frac{1}{2}\right| - \frac{1}{2}\log\left|\frac{1}{3}\right|$

$= \frac{1}{2}\left(\log\frac{1}{2} - \log\frac{1}{3}\right) \qquad [\because \ |x| = x \text{ if } x \geq 0]$

$= \frac{1}{2}\left[\log\left(\frac{\frac{1}{2}}{\frac{1}{3}}\right)\right] = \frac{1}{2}\log\frac{3}{2}.$

**Evaluate the definite integrals in Exercises 12 to 20:**

**12.** $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$

**Sol.** $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2x)\,dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2x)\,dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{1}{2}\sin\pi - \left(0 + \frac{1}{2}\sin 0\right)\right] = \frac{1}{2}\left[\frac{\pi}{2} + 0 - 0\right]$

$= \frac{\pi}{4}. \ [\because \quad \sin\pi = \sin 180° = \sin(180° - 0°) = \sin 0 = 0]$

**13.** $\int_2^3 \frac{x\,dx}{x^2+1}$

**Sol.** $\int_2^3 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_2^3 \frac{2x}{x^2+1}\,dx$

$= \frac{1}{2}\left(\log|x^2+1|\right)_2^3. \qquad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$

(Here $f(x) = x^2 + 1$ and $f'(x) = 2x$)

$= \frac{1}{2}$ (log | 10 | – log | 5 |) $= \frac{1}{2}$ (log 10 – log 5)

$= \frac{1}{2} \log \frac{10}{5} = \frac{1}{2} \log 2.$

**14.** $\int_0^1 \frac{2x+3}{5x^2+1}\, dx$

**Sol.** $\int_0^1 \frac{2x+3}{5x^2+1}\, dx = \int_0^1 \left( \frac{2x}{5x^2+1} + \frac{3}{5x^2+1} \right) dx$

$= \int_0^1 \frac{2x}{5x^2+1}\, dx + 3 \int_0^1 \frac{dx}{5x^2+1}$

$= \frac{1}{5} \int_0^1 \frac{10x}{5x^2+1}\, dx + 3 \int_0^1 \frac{dx}{(\sqrt{5}x)^2 + 1^2}$

$= \frac{1}{5} \left( \log |5x^2+1| \right)_0^1 + 3 \cdot \frac{1}{1} \frac{\left( \tan^{-1}\left( \frac{\sqrt{5}x}{1} \right) \right)_0^1}{\sqrt{5} \rightarrow \text{Coefficient of } x}$

$\left[ \because \int \frac{f'(x)}{f(x)} dx = \log |f(x)| \text{ and } \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \right]$

$= \frac{1}{5}$ (log 6 – log 1) + $\frac{3}{\sqrt{5}}$ ($\tan^{-1} \sqrt{5} - \tan^{-1} 0$)

$= \frac{1}{5} \log 6 + \frac{3}{\sqrt{5}} \tan^{-1} \sqrt{5}.$

**15.** $\int_0^1 x\, e^{x^2}\, dx$

**Sol.** To evaluate $\int_0^1 x\, e^{x^2}\, dx$

Let us first evaluate $\int x\, e^{x^2}\, dx$

$= \frac{1}{2} \int e^{x^2} (2x\, dx)$ ...(*i*)

Put $x^2 = t$. Therefore $2x = \frac{dt}{dx}$ $\therefore$ $2x\, dx = dt$

$\therefore$ From (*i*), $\int x\, e^{x^2}\, dx = \frac{1}{2} \int e^t\, dt = \frac{1}{2} e^t$

**Putting** $t = x^2$, $= \frac{1}{2} e^{x^2}$ ...(*ii*)

$\therefore$ The given integral $\int_0^1 x\, e^{x^2}\, dx = \frac{1}{2} \left( e^{x^2} \right)_0^1$ [By (*ii*)]

$= \frac{1}{2} (e^1 - e^0) = \frac{1}{2} (e - 1).$

**Note.** Please note that limits 0 and 1 specified in the given integral are limits for $x$.

Therefore after substituting $x^2 = t$ and evaluating the integral, we must put back $t = x^2$ and only then use $\int_a^b f(x)\, dx = \phi(b) - \phi(a)$.

**Remark.** In the next Exercise 7.10 we shall also learn to change the limits of integration from values of $x$ to values of $t$ and then we may use our discretion even here also.

**16.** $\displaystyle\int_1^2 \frac{5x^2}{x^2+4x+3}\, dx$

**Sol.** $\displaystyle\int_1^2 \frac{5x^2}{x^2+4x+3}\, dx = \int_1^2 \frac{5x^2}{(x+1)(x+3)}$ ...(*i*)

[$\because$ $x^2 + 4x + 3 = x^2 + 3x + x + 3$ $= x(x + 3) + 1(x + 3) = (x + 1)(x + 3)$]

The integrand $\dfrac{5x^2}{(x+1)(x+3)}$ is a rational function and degree of numerator = degree of denominator.

So let us apply long division.

$$(x + 1)(x + 3) = x^2 + 4x + 3 \;\big)\; 5x^2 \;\big(\, 5$$
$$\underline{\begin{array}{r} 5x^2 + 20x + 15 \\ - \quad - \quad - \end{array}}$$
$$-\,20x - 15$$

$$\therefore \quad \frac{5x^2}{(x+1)(x+3)} = 5 + \frac{(-20x-15)}{(x+1)(x+3)}$$

Putting this value in (*i*),

$$\int_1^2 \frac{5x^2}{x^2+4x+3}\, dx = \int_1^2 \left(5 + \frac{(-20x-15)}{(x+1)(x+3)}\right) dx$$

$$= \int_1^2 5\, dx + \int_1^2 \frac{-20x-15}{(x+1)(x+3)}\, dx = 5\,(x)_1^2 + \mathbf{I}$$

$= 5(2 - 1) + \mathbf{I} = 5 + \mathbf{I}$ ...(*ii*)

where $\text{I} = \displaystyle\int_1^2 \frac{-20x-15}{(x+1)(x+3)}\, dx$

Let integrand of I $= \dfrac{-20x-15}{(x+1)(x+3)} = \dfrac{\text{A}}{x+1} + \dfrac{\text{B}}{x+3}$ ...(*iii*)

(Partial Fractions)

Multiplying both sides by L.C.M. = $(x + 1)(x + 3)$,

$$-20x - 15 = \text{A}(x + 3) + \text{B}(x + 1)$$
$$= \text{A}x + 3\text{A} + \text{B}x + \text{B}$$

Comparing coefficients of $x$ and constant terms on both sides, we have

**Coefficients of $x$:** A + B = – 20 ...(*iv*)

**Constant terms:** 3A + B = – 15 ...(*v*)

Subtracting (*iv*) and (*v*), $-2A = -5$. Therefore $A = \frac{5}{2}$.

Putting $A = \frac{5}{2}$ in (*iv*), $\frac{5}{2} + B = -20 \Rightarrow B = -20 - \frac{5}{2}$

or $\quad B = \frac{-40-5}{2} = \frac{-45}{2}$

Putting these values of A and B in (*iii*),

$$\frac{-20x-15}{(x+1)(x+3)} = \frac{\frac{5}{2}}{x+1} - \frac{\frac{45}{2}}{x+3}$$

$$\therefore \quad I = \int_1^2 \frac{-20x-15}{(x+1)(x+3)}\,dx = \frac{5}{2}\int_1^2 \frac{1}{x+1}\,dx - \frac{45}{2}\int_1^2 \frac{1}{x+3}\,dx$$

$$= \frac{5}{2}\left(\log|x+1|\right)_1^2 - \frac{45}{2}\left(\log|x+3|\right)_1^2$$

$$= \frac{5}{2}\left(\log|3| - \log|2|\right) - \frac{45}{2}\left(\log|5| - \log|4|\right)$$

$$= \frac{5}{2}\log\frac{3}{2} - \frac{45}{2}\log\frac{5}{4} \qquad [\because\ |x| = x \text{ if } x \ge 0]$$

$$= \frac{5}{2}\left(\log\frac{3}{2} - 9\log\frac{5}{4}\right)$$

Putting this value of I in (*ii*),

$$\int_1^2 \frac{5x^2}{x^2+4x+3}\,dx = 5 + \frac{5}{2}\left(\log\frac{3}{2} - 9\log\frac{5}{4}\right) = 5 - \frac{5}{2}\left(9\log\frac{5}{4} - \log\frac{3}{2}\right)$$

**17.** $\int_0^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2)\,dx$

**Sol.** $\int_0^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2)\,dx = 2\int_0^{\frac{\pi}{4}} \sec^2 x\,dx + \int_0^{\frac{\pi}{4}} x^3\,dx + 2\int_0^{\frac{\pi}{4}} 1\,dx$

$$= 2\,(\tan x)_0^{\frac{\pi}{4}} + \left(\frac{x^4}{4}\right)_0^{\frac{\pi}{4}} + 2\,(x)_0^{\frac{\pi}{4}}$$

$$= 2\left(\tan\frac{\pi}{4} - \tan 0\right) + \frac{\left(\frac{\pi}{4}\right)^4}{4} - 0 + 2\left(\frac{\pi}{4} - 0\right)$$

$$= 2(1-0) + \frac{\left(\frac{\pi^4}{256}\right)}{4} + \frac{2\pi}{4} \qquad = 2 + \frac{\pi^4}{1024} + \frac{\pi}{2}.$$

**18.** $\int_0^{\pi} \left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right) dx$

**Sol.** $\int_0^{\pi} \left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right) dx = \int_0^{\pi} \left[\left(\frac{1-\cos x}{2}\right) - \left(\frac{1+\cos x}{2}\right)\right] dx$

$$\left(\because\ \sin^2\theta = \frac{1-\cos 2\theta}{2} \text{ and } \cos^2\theta = \frac{1+\cos 2\theta}{2}\right)$$

$$= \int_0^{\pi} \left( \frac{1 - \cos x - 1 - \cos x}{2} \right) dx = \int_0^{\pi} \frac{-2\cos x}{2}\, dx$$

$$= -\int_0^{\pi} \cos x \; dx = -\left(\sin x\right)_0^{\pi} = -(\sin \pi - \sin 0) = -(0 - 0) = 0.$$

$[\because \; \sin \pi = \sin 180° = \sin (180° - 0) = \sin 0 = 0]$

**19.** $\displaystyle \int_0^2 \frac{6x + 3}{x^2 + 4}\, dx$

**Sol.** $\displaystyle \int_0^2 \frac{6x+3}{x^2+4}\, dx = \int_0^2 \frac{6x}{x^2+4}\, dx + 3\int_0^2 \frac{1}{x^2+4}\, dx$

$$= 3\int_0^2 \frac{2x}{x^2+4}\, dx + 3\,\frac{1}{2}\left(\tan^{-1}\frac{x}{2}\right)_0^2$$

$$= 3\left(\log |x^2 + 4|\right)_0^2 + \frac{3}{2}\,(\tan^{-1} 1 - \tan^{-1} 0)$$

$$\left[\because \int \frac{f'(x)}{f(x)}\, dx = \log |f(x)| \text{ and } \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= 3\,(\log 8 - \log 4) + \frac{3}{2}\left(\frac{\pi}{4} - 0\right) \quad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$= 3 \log \frac{8}{4} + \frac{3\pi}{8} = 3 \log 2 + \frac{3\pi}{8}.$$

**20.** $\displaystyle \int_0^1 \left( x\, e^x + \sin \frac{\pi x}{4} \right) dx$

**Sol.** $\displaystyle \int_0^1 \left( xe^x + \sin\frac{\pi x}{4} \right) dx = \int_0^1 \underset{\text{I}\;\;\text{II}}{x\, e^x}\, dx + \int_0^1 \sin\frac{\pi x}{4}\, dx$

Applying Product Rule on first definite integral,

$$\left(\text{I} \int \text{II}\, dx\right)_0^1 - \int_0^1 \left( \frac{d}{dx}(\text{I}) \int \text{II}\, dx \right) dx$$

$$= \left(x\, e^x\right)_0^1 - \int_0^1 1 \,.\, e^x\, dx - \frac{\left(\cos\frac{\pi x}{4}\right)_0^1}{\frac{\pi}{4} \to \text{Coefficient of } x \text{ in } \frac{\pi x}{4}}$$

$$= e^1 - 0 - \int_0^1 e^x\, dx - \frac{4}{\pi}\left[\cos\frac{\pi}{4} - \cos 0\right] = e - \left(e^x\right)_0^1 - \frac{4}{\pi}\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= e - (e - e^0) - \frac{4}{\pi\sqrt{2}} + \frac{4}{\pi}$$

$$= e - e + 1 - \frac{2 \,.\, 2}{\pi\sqrt{2}} + \frac{4}{\pi} = 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}.$$

**Choose the correct answer in Exercises 21 and 22:**

**21.** $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ **equals**

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

**Sol.** $\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = \left(\tan^{-1} x\right)_1^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1} 1$

$= \frac{\pi}{3} - \frac{\pi}{4}$ $\left[\because \tan\frac{\pi}{3} = \sqrt{3} \text{ and } \tan\frac{\pi}{4} = 1\right]$

$= \frac{4\pi - 3\pi}{12} = \frac{\pi}{12}$

$\therefore$ Option (D) is the correct answer.

**22.** $\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$ **equals**

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{12}$ (C) $\frac{\pi}{24}$ (D) $\frac{\pi}{4}$

**Sol.** $\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} = \int_0^{\frac{2}{3}} \frac{dx}{(3x)^2+2^2} = \left[\frac{1}{2}\frac{\tan^{-1}\frac{3x}{2}}{3 \to \text{Coefficient of } x \text{ in } 3x}\right]$

$= \frac{1}{6}\left[\tan^{-1}\frac{3x}{2}\right]_0^{\frac{2}{3}} = \frac{1}{6}\left[\tan^{-1}\left(\frac{3}{2}\times\frac{2}{3}\right) - \tan^{-1} 0\right]$

$= \frac{1}{6}(\tan^{-1} 1 - \tan^{-1} 0) = \frac{1}{6}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{24}$

$\left[\because \tan\frac{\pi}{4} = 1 \text{ and } \tan 0 = 0\right]$

$\therefore$ Option (C) is the correct answer.