NCERT Class 12 Maths

Solutions

Chapter - 7

Exercise 7.11

By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 6:

1. $\int_0^{\frac{\pi}{2}} \cos^2 x \; dx$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$ ...(i)

$\therefore \quad I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right) dx$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

or $I = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$ ...(ii)

Adding Eqns. (i) and (ii),

$$2I = \int_0^{\frac{\pi}{2}} (\cos^2 x + \sin^2 x)\, dx = \int_0^{\frac{\pi}{2}} 1 \, dx = (x)_0^{\frac{\pi}{2}}$$

$\Rightarrow \quad 2I = \frac{\pi}{2} \quad \Rightarrow \quad I = \frac{\pi}{4}.$

2. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, dx$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, dx$ ...(i)

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} \, dx$$

$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

or $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \, dx$ ...(ii)

Adding Eqns. (i) and (ii), we have

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\right) dx = \int_0^{\frac{\pi}{2}} 1 \, dx$$

$\Rightarrow 2I = (x)_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2} \quad \Rightarrow \quad I = \frac{\pi}{4}.$

3. $\int_0^{\frac{\pi}{2}} \frac{\sin^{3/2} x \, dx}{\sin^{3/2} x + \cos^{3/2} x}$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} \, dx$ ...(i)

Changing x to $\frac{\pi}{2} - x$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^{3/2}\left(\frac{\pi}{2} - x\right)}{\sin^{3/2}\left(\frac{\pi}{2} - x\right) + \cos^{3/2}\left(\frac{\pi}{2} - x\right)}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x} \qquad ...(ii)$$

Adding Eqns. (*i*) and (*ii*),

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{3/2} x + \cos^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x}\, dx = \int_0^{\frac{\pi}{2}} 1\, dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4}.$$

4. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^5 x\, dx}{\sin^5 x + \cos^5 x}$

Sol. Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x + \cos^5 x}\, dx \qquad ...(i)$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{\cos^5\left(\frac{\pi}{2} - x\right)}{\sin^5\left(\frac{\pi}{2} - x\right) + \cos^5\left(\frac{\pi}{2} - x\right)}\, dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

or $\quad I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\cos^5 x + \sin^5 x}\, dx \qquad ...(ii)$

Adding Eqns. (*i*) and (*ii*), we have

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{\cos^5 x}{\sin^5 x + \cos^5 x} + \frac{\sin^5 x}{\cos^5 x + \sin^5 x}\right) dx$$

$$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x + \sin^5 x}{\sin^5 x + \cos^5 x}\, dx = \int_0^{\frac{\pi}{2}} 1\, dx = (x)_0^{\frac{\pi}{2}}$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} \qquad \Rightarrow \quad I = \frac{\pi}{4}.$$

5. $\displaystyle\int_{-5}^{5} |x+2|\, dx$

Sol. Let $I = \displaystyle\int_{-5}^{5} |x+2|\, dx \qquad ...(i)$

We can evaluate this integral only if we can get rid of the modulus.

Putting expression within modulus equal to 0, we have

$x + 2 = 0$, *i.e.*, $x = -2 \in (-5, 5)$

$\therefore$ From (*i*), $I = \displaystyle\int_{-5}^{5} |x+2|\, dx$

$$= \int_{-5}^{-2} |x+2|\, dx + \int_{-2}^{5} |x+2|\, dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \text{ where } a < c < b\right]$$

$$= \int_{-5}^{-2} -(x+2)\ dx + \int_{-2}^{5} (x+2)\ dx$$

[$\because$ On $(-5, -2)$, $x < -2 \Rightarrow x + 2 < 0$
$\Rightarrow |x + 2| = -(x + 2)$ and on $(-2, 5)$; $x > -2$
$\Rightarrow x + 2 > 0 \Rightarrow |x + 2| = x + 2$, by definition of modulus function]

$$= -\left(\frac{x^2}{2} + 2x\right)_{-5}^{-2} + \left(\frac{x^2}{2} + 2x\right)_{-2}^{5}$$

$$= -\left[\left(\frac{4}{2} - 4\right) - \left(\frac{25}{2} - 10\right)\right] + \left[\left(\frac{25}{2} + 10\right) - \left(\frac{4}{2} - 4\right)\right]$$

$$= -\left[-2 - \frac{5}{2}\right] + \left[\frac{45}{2} + 2\right] = 2 + \frac{5}{2} + \frac{45}{2} + 2$$

$$= 4 + \frac{50}{2} = 4 + 25 = 29.$$

6. $\int_2^8 |x - 5|\ dx$

Sol. We know by definition of modulus function, that

$$|x - 5| = \begin{cases} x - 5 & \text{if } x - 5 \ge 0, i.e., x \ge 5 \quad ...(i) \\ -(x-5) = 5 - x, & \text{if } x < 5 \quad ...(ii) \end{cases}$$

$$\therefore \int_2^8 |x - 5|\ dx = \int_2^5 |x - 5|\ dx + \int_5^8 |x - 5|\ dx$$

$$= \int_2^5 (5 - x)\ dx + \int_5^8 (x - 5)\ dx = \left(5x - \frac{x^2}{2}\right)_2^5 + \left(\frac{x^2}{2} - 5x\right)_5^8$$

[By (ii)] [By (i)]

$$= \left(25 - \frac{25}{2}\right) - (10 - 2) + (32 - 40) - \left(\frac{25}{2} - 25\right)$$

$$= 25 - \frac{25}{2} - 8 - 8 - \frac{25}{2} + 25 = 34 - \frac{50}{2} = 34 - 25 = 9$$

By using the properties of definite integrals, evaluate the integrals in Exercises 7 to 11:

7. $\int_0^1 x(1 - x)^n\ dx$

Sol. Let $I = \int_0^1 x(1 - x)^n\ dx$

$$\therefore \quad I = \int_0^1 (1 - x)\ (1 - (1 - x))^n\ dx \left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx\right]$$

or $\quad I = \int_0^1 (1 - x)\ (1 - 1 + x)^n\ dx$

or $\quad I = \int_0^1 (1-x)\ x^n\ dx = \int_0^1 (x^n - x^{n+1})\ dx$

$$= \left(\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right)_0^1 = \frac{1}{n+1} - \frac{1}{n+2} - (0-0)$$

$$= \frac{n+2-n-1}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}.$$

8. $\displaystyle\int_0^{\frac{\pi}{4}} \log(1+\tan x)\ dx$

Sol. Let $I = \displaystyle\int_0^{\frac{\pi}{4}} \log(1+\tan x)\ dx$ ...(*i*)

Changing x to $\frac{\pi}{4} - x$ $\quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$I = \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right] dx = \int_0^{\frac{\pi}{4}} \log\left[1+\frac{1-\tan x}{1+\tan x}\right] dx$$

$$\left[\because \tan\left(\frac{\pi}{4}-x\right) = \frac{\tan\frac{\pi}{4} - \tan x}{1+\tan\frac{\pi}{4}\tan x} = \frac{1-\tan x}{1+\tan x}\right]$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan x}\right) dx \qquad ...(ii)$$

Adding Eqns. (*i*) and (*ii*), we have

$$2I = \int_0^{\frac{\pi}{4}} \left[\log(1+\tan x) + \log\left(\frac{2}{1+\tan x}\right)\right] dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left[(1+\tan x)\frac{2}{(1+\tan x)}\right] dx = \int_0^{\frac{\pi}{4}} \log 2\ dx$$

or $\quad 2I = (\log 2)\ [x]_0^{\frac{\pi}{4}} = \frac{\pi}{4}\log 2 \quad$ Dividing by 2, $I = \frac{\pi}{8}\log 2.$

9. $\displaystyle\int_0^2 x\sqrt{2-x}\ dx$

Sol. Let $I = \displaystyle\int_0^2 x\sqrt{2-x}\ dx$

Changing x to $2-x$ $\quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$I = \int_0^2 (2-x)\sqrt{2-(2-x)}\ dx$$

$$= \int_0^2 (2-x)\sqrt{x}\ dx = \int_0^2 (2x^{1/2} - x^{3/2})\ dx$$

$$= \left[2 \cdot \frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2}\right]_0^2 = \left(\frac{4}{3} \cdot 2^{3/2} - \frac{2}{5} \cdot 2^{5/2}\right) - (0 - 0)$$

$$= \frac{4}{3} \times 2\sqrt{2} - \frac{2}{5} \times 4\sqrt{2} = \left(\frac{8}{3} - \frac{8}{5}\right)\sqrt{2}$$

$(\because\ 2^{3/2} = (2^{1/2})^3 = (\sqrt{2})^3 = \sqrt{2}\ \sqrt{2}\ \sqrt{2} = 2\sqrt{2}$

and $2^{5/2} = (2^{1/2})^5 = (\sqrt{2})^5 = \sqrt{2}\ \sqrt{2}\ \sqrt{2}\ \sqrt{2}\ \sqrt{2} = 2.2.\sqrt{2}$

$$= 4\sqrt{2}\ ) = \frac{16\sqrt{2}}{15}.$$

10. $\int_0^{\frac{\pi}{2}} (2 \log \sin x - \log \sin 2x)\ dx$

Sol. Let $I = \int_0^{\pi/2} (2 \log \sin x - \log \sin 2x)\, dx$

$$= \int_0^{\pi/2} (\log \sin^2 x - \log \sin 2x)\, dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin^2 x}{\sin 2x}\right) dx = \int_0^{\pi/2} \log\left(\frac{\sin^2 x}{2 \sin x \cos x}\right) dx$$

or $\quad I = \int_0^{\pi/2} \log\left(\frac{1}{2} \tan x\right) dx \qquad \ldots(i)$

$\therefore \quad I = \int_0^{\pi/2} \log\left(\frac{1}{2} \tan\left(\frac{\pi}{2} - x\right) dx\right) \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$

or $\quad I = \int_0^{\pi/2} \log\left(\frac{1}{2} \cot x\right) dx \qquad \ldots(ii)$

Adding Eqns. (*i*) and (*ii*),

$$2I = \int_0^{\pi/2} \left[\log\left(\frac{1}{2} \tan x\right) + \log\left(\frac{1}{2} \cot x\right)\right] dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log\left(\frac{1}{2} \tan x \frac{1}{2} \cot x\right) dx = \int_0^{\pi/2} \log \frac{1}{4}\, dx = \log \frac{1}{4}\, (x)_0^{\pi/2}$$

$$= (\log 1 - \log 4)\frac{\pi}{2} = -\frac{\pi}{2} \log 4 \qquad (\because\ \log 1 = 0)$$

$$\therefore \quad I = -\frac{\pi}{4} \log 4 = -\frac{\pi}{4} \log 2^2 = -\frac{2\pi}{4} \log 2 = -\frac{\pi}{2} \log 2.$$

11. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\ dx$

Sol. Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\ dx$ or $I = 2 \int_0^{\frac{\pi}{2}} \sin^2 x\ dx \qquad \ldots(i)$

[$\because$ For $f(x) = \sin^2 x$, $f(-x) = \sin^2(-x) = (-\sin x)^2 = \sin^2 x = f(x)$

$\therefore\ f(x)$ is an even function of x and hence

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\ dx]$$

$$\therefore \quad I = 2\int_{0}^{\frac{\pi}{2}} \sin^2\left(\frac{\pi}{2} - x\right) dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\text{or} \quad I = 2\int_{0}^{\frac{\pi}{2}} \cos^2 x\ dx \qquad ...(ii)$$

Adding Eqns. (i) and (ii), we have

$$2I = 2\int_{0}^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x)\ dx$$

$$\text{or} \quad 2I = 2\int_{0}^{\frac{\pi}{2}} 1\ dx = 2\ (x)_{0}^{\frac{\pi}{2}} = 2\ .\ \frac{\pi}{2} = \pi \ \therefore \quad I = \frac{\pi}{2}.$$

Using properties of definite integrals, evaluate the following integrals in Exercises 12 to 18:

12. $\displaystyle\int_0^{\pi} \frac{x\,dx}{1+\sin x}$

Sol. Let $\displaystyle I = \int_0^{\pi} \frac{x}{1+\sin x}\ dx \qquad ...(i)$

Changing x to $\pi - x$, $\displaystyle I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}\ dx$

$$\text{or} \quad I = \int_0^{\pi} \frac{\pi - x}{1+\sin x}dx \quad ...(ii) \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

Adding Eqns. (i) and (ii), we have

$$2I = \int_0^{\pi}\left(\frac{x}{1+\sin x} + \frac{\pi - x}{1+\sin x}\right) dx = \int_0^{\pi} \frac{x + \pi - x}{1+\sin x}\ dx$$

$$= \int_0^{\pi} \frac{\pi}{1+\sin x}\ dx = \pi \int_0^{\pi} \frac{1}{1+\sin x}\ dx$$

$$\text{or} \quad 2I = 2\pi \int_0^{\pi/2} \frac{dx}{1+\sin x}$$

$$\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(2a-x) = f(x)\right]$$

$$= 2\pi \int_0^{\pi/2} \frac{dx}{1+\sin\left(\frac{\pi}{2} - x\right)} \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= 2\pi \int_0^{\pi/2} \frac{dx}{1+\cos x}$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \frac{dx}{2\cos^2\frac{x}{2}} = \frac{\pi}{2}\int_0^{\pi/2} \sec^2\frac{x}{2}\ dx \quad = \frac{\pi}{2}\left[\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2}$$

$$= \pi \left(\tan \frac{\pi}{4} - \tan 0 \right) = \pi(1 - 0) = \pi.$$

13. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \ dx$

Sol. Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \ dx$

Here Integrand $f(x) = \sin^7 x$

$\therefore \ f(-x) = \sin^7 (-x) = (-\sin x)^7 = -\sin^7 x = -f(x)$

$\therefore \ f(x)$ is an odd function of x.

$$\therefore \quad I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \ dx = 0.$$

$$\left[\because \text{ If } f(x) \text{ is an odd function of } x, \text{ then } \int_{-a}^{a} f(x)\, dx = 0 \right]$$

14. $\int_0^{2\pi} \cos^5 x \ dx$

Sol. $\int_0^{2\pi} \cos^5 x \ dx = 2 \int_0^{\pi} \cos^5 x \ dx$

$$\left[\because \int_0^{2a} f(x)\, dx = \int_0^{a} f(x)\, dx, \text{ if } f(2a - x) = f(x) \right]$$

Here $f(x) = \cos^5 x \quad \therefore \quad f(2\pi - x) = \cos^5 (2\pi - x) = \cos^5 x$

$= f(x) = 2(0) = 0$

$\left[\because \int_0^{2a} f(x) \ dx = 0, \text{ if } f(2a - x) = -f(x). \text{ Here } f(x) = \cos^5 x \right.$

$\left. \therefore \ f(\pi - x) = \cos^5 (\pi - x) = (-\cos x)^5 = -\cos^5 x = -f(x) \right]$

Alternatively. To evaluate $\int_0^{2\pi} \cos^5 x \ dx$, put $\sin x = t$.

Remark. In fact $\int_0^{2\pi} \cos^n x \ dx$ or $\int_0^{\pi} \cos^n x \ dx$ for all positive **odd integers** n is equal to zero.

This is a very important result for I.I.T. Entrance Examination.

15. $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} \ dx$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} \ dx \qquad ...(i)$

Changing x to $\frac{\pi}{2} - x$ in integrand of (i),

$$\left[\because \int_0^{a} f(x)\, dx = \int_0^{a} f(a - x)\, dx \right]$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)} \ dx = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \cos x \sin x} \ dx$$

$$= -\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx \quad ...(ii)$$

Adding equations (*i*) and (*ii*), we have $2\text{I} = 0$ or $\text{I} = 0.$

16. $\int_0^{\pi} \log (1 + \cos x)\, dx$

Sol. Let $\text{I} = \int_0^{\pi} \log (1 + \cos x)\, dx \quad ...(i)$

$\therefore \quad \text{I} = \int_0^{\pi} \log (1 + \cos (\pi - x))\, dx \quad \left[\because \int_0^{a} f(x)\, dx = \int_0^{a} f(a - x)\, dx\right]$

or $\text{I} = \int_0^{\pi} \log (1 - \cos x)\, dx \quad ...(ii)$

Adding Eqns. (*i*) and (*ii*), we have

$$2\text{I} = \int_0^{\pi} [\log (1 + \cos x) + \log (1 - \cos x)]\, dx$$

$$= \int_0^{\pi} \log ((1 + \cos x)(1 - \cos x))\, dx = \int_0^{\pi} \log (1 - \cos^2 x)\, dx$$

$\Rightarrow 2\text{I} = \int_0^{\pi} \log \sin^2 x\, dx = 2 \int_0^{\pi} \log \sin x\, dx \quad (\because \log m^n = n \log m)$

Dividing by 2, $\text{I} = \int_0^{\pi} \log \sin x\, dx = 2 \int_0^{\frac{\pi}{2}} \log \sin x\, dx \quad ...(iii)$

$\left[\because \text{ For } f(x) = \log \sin x,\ f(\pi - x) = \log \sin (\pi - x) = \log \sin x = f(x) \text{ and if } f(2a - x) = f(x); \text{ then } \int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx\right]$

$\therefore \quad \text{I} = 2 \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx \quad \left[\because \int_0^{a} f(x)\, dx = \int_0^{a} f(a - x)\, dx\right]$

or $\text{I} = 2 \int_0^{\frac{\pi}{2}} \log \cos x\, dx \quad ...(iv)$

Adding Eqns. (*iii*) and (*iv*), we have

$$2\text{I} = 2 \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x)\, dx$$

Dividing by 2, $\text{I} = \int_0^{\frac{\pi}{2}} (\log \sin x \cos x)\, dx$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{2 \sin x \cos x}{2}\right) dx = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right) dx$$

or $\text{I} = \int_0^{\frac{\pi}{2}} (\log \sin 2x - \log 2)\, dx$

or $\text{I} = \int_0^{\frac{\pi}{2}} \log \sin 2x\, dx - \int_0^{\frac{\pi}{2}} \log 2\, dx$

or $\text{I} = \int_0^{\frac{\pi}{2}} \log \sin 2x\, dx - \log 2\, (x)_0^{\frac{\pi}{2}}$

or $\quad I = \int_0^{\frac{\pi}{2}} \log \sin 2x \; dx - \frac{\pi}{2} \log 2$

or $\quad I = I_1 - \frac{\pi}{2} \log 2$ ...(*v*)

where $\quad I_1 = \int_0^{\frac{\pi}{2}} \log \sin 2x \; dx$ ...(*vi*)

Put $2x = t$ to make I_1 look as I given by (*iii*)

$\therefore \quad 2 = \frac{dt}{dx} \quad$ or $\quad 2\, dx = dt \quad$ or $\quad dx = \frac{dt}{2}$

To change the limits: When $x = 0, \quad t = 2x = 0$

When $x = \frac{\pi}{2}, \; t = 2x = \pi$

$\therefore \quad$ From (*vi*), $I_1 = \int_0^{\pi} \log \sin t \; \frac{dt}{2} = \frac{1}{2} \int_0^{\pi} \log \sin t \; dt$

or $\qquad I_1 = \frac{1}{2} \times 2 \int_0^{\frac{\pi}{2}} \log \sin t \, dt$

(For reason see Explanation within brackets below Eqn. (*iii*))

or $\quad I_1 = \int_0^{\frac{\pi}{2}} \log \sin t \; dt = \int_0^{\frac{\pi}{2}} \log \sin x \, dx \left[\because \int_a^b f(t)\, dt = \int_a^b f(x)\, dx \right]$

or $\quad I_1 = \frac{I}{2}$ [By Eqn. (*iii*)]

Putting this value of I_1 in Eqn. (*v*), $I = \frac{I}{2} - \frac{\pi}{2} \log 2$

Multiplying by L.C.M. = 2, $2I = I - \pi \log 2$

or $\quad 2I - I = -\pi \log 2 \quad$ or $\quad I = -\pi \log 2$.

17. $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} \; dx$

Sol. Let $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} \; dx$ ...(*i*)

$\therefore \quad I = \int_0^a \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{a - (a - x)}} \; dx = \int_0^a \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{x}} \; dx$ ...(*ii*)

Adding Eqns. (*i*) and (*ii*), we have

$$2I = \int_0^a \left(\frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} + \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{x}} \right) dx = \int_0^a \left(\frac{\sqrt{x} + \sqrt{a - x}}{\sqrt{x} + \sqrt{a - x}} \right) dx$$

or $2I = \int_0^a 1 \; dx = (x)_0^a = a \;\; \therefore \quad I = \frac{a}{2}$.

18. $\int_0^4 |x-1|\, dx$

Sol. Let $I = \int_0^4 |x-1|\, dx$ ...(*i*)

Putting the expression $(x - 1)$ within modulus equal to zero, we have

$x = 1 \in (0, 4)$

$\therefore$ From (*i*), $I = \int_0^4 |x-1|\, dx = \int_0^1 |x-1|\, dx + \int_1^4 |x-1|\, dx$

$= -\int_0^1 (x-1)\, dx + \int_1^4 (x-1)\, dx$

[$\because$ On (0, 1); $x < 1 \Rightarrow x - 1 < 0$ and hence $|x - 1| = -(x - 1)$ and on (1, 4), $x > 1 \Rightarrow x - 1 > 0$ and hence $|x - 1| = (x - 1)$ by definition of modulus function]

$$= -\left(\frac{x^2}{2} - x\right)_0^1 + \left(\frac{x^2}{2} - x\right)_1^4 = -\left(\left(\frac{1}{2} - 1\right) - 0\right) + \left(\frac{16}{2} - 4 - \left(\frac{1}{2} - 1\right)\right)$$

$$= \frac{-1}{2} + 1 + 8 - 4 - \frac{1}{2} + 1 \qquad = 6 - \frac{2}{2} = 6 - 1 = 5.$$

19. Show that $\int_0^a f(x)\, g(x)\, dx = 2\int_0^a f(x)\, dx$, if f and g are defined as $f(x) = f(a - x)$ and $g(x) + g(a - x) = 4$.

Sol. Given: $f(x) = f(a - x)$ ...(*i*)

and $g(x) + g(a - x) = 4$ ...(*ii*)

Let $I = \int_0^a f(x)\, g(x)\, dx$ ...(*iii*)

$\therefore$ $I = \int_0^a f(a-x)\, g(a-x)\, dx$ $\left[\because \int_0^a F(x)\, dx = \int_0^a F(a-x)\, dx\right]$

Putting $f(a - x) = f(x)$ from (*i*),

$I = \int_0^a f(x)\, g(a-x)\, dx$ ...(*iv*)

Adding Eqns. (*iii*) and (*iv*), we have

$2I = \int_0^a (f(x)\, g(x) + f(x)\, g(a-x))\, dx = \int_0^a f(x)\, (g(x) + g(a-x))\, dx$

or $2I = \int_0^a f(x)\,(4)\, dx$ [By (*ii*)] $= 4\int_0^a f(x)\, dx$

Dividing by 2, $I = 2\int_0^a f(x)\, dx$ = R.H.S.

Choose the correct answer in Exercises 20 and 21:

20. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)\, dx$ is

(A) 0 **(B) 2** **(C) π** **(D) 1**

Sol. Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)\, dx$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \; dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos x \; dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^5 x \; dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \, dx$$

$$= 0 + 0 + 0 + \left(x \right)_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2} - \left(\frac{-\pi}{2} \right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

[$\because$ Each of the three functions x^3, $x \cos x$ and $\tan^5 x$ is an odd function of x as $f(-x) = -f(x)$ for each of them and $\int_{-a}^{a} f(x)\, dx = 0$ for each odd function $f(x)$]

$\therefore$ Option (C) is the correct option.

21. The value of $\int_0^{\frac{\pi}{2}} \log \left(\frac{4 + 3 \sin x}{4 + 3 \cos x} \right) dx$ is

(A) 2 **(B) $\frac{3}{4}$** **(C) 0** **(D) – 2**

Sol. Let $I = \int_0^{\frac{\pi}{2}} \log \left(\frac{4 + 3 \sin x}{4 + 3 \cos x} \right) dx$ ...(*i*)

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \log \left(\frac{4 + 3 \sin \left(\frac{\pi}{2} - x \right)}{4 + 3 \cos \left(\frac{\pi}{2} - x \right)} \right) dx$$

or $I = \int_0^{\frac{\pi}{2}} \log \left(\frac{4 + 3 \cos x}{4 + 3 \sin x} \right) dx$ ...(*ii*)

Adding Eqns. (*i*) and (*ii*), we get

$$2I = \int_0^{\frac{\pi}{2}} \left[\log \left(\frac{4 + 3 \sin x}{4 + 3 \cos x} \right) + \log \left(\frac{4 + 3 \cos x}{4 + 3 \sin x} \right) \right] dx$$

$$= \int_0^{\frac{\pi}{2}} \log \left[\frac{4 + 3 \sin x}{4 + 3 \cos x} \cdot \frac{4 + 3 \cos x}{4 + 3 \sin x} \right] dx = \int_0^{\frac{\pi}{2}} \log 1 \; dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0 \; dx = 0 \quad \Rightarrow \quad I = \frac{0}{2} = 0.$$