NCERT Class 12 Maths

Solutions

Chapter - 7

Integrals

Exercise 7.10

Evaluate the integrals in Exercises 1 to 8 using substitution:

1. $\int_0^1 \frac{x}{x^2+1}\,dx$

Sol. Let I = $\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx$ ...(*i*)

Put $x^2 + 1 = t$. Therefore $2x = \frac{dt}{dx} \Rightarrow 2x\,dx = dt.$

To change the limits of integration from values of *x* to values of *t*.

When $x = 0,\ t = 0 + 1 = 1$

When $x = 1,\ t = 1 + 1 = 2$

$\therefore$ From (*i*), $I = \frac{1}{2}\int_1^2 \frac{dt}{t} = \frac{1}{2}\left(\log|t|\right)_1^2 = \frac{1}{2}(\log|2| - \log|1|)$

$$= \frac{1}{2}\ (\log 2 - \log 1) = \frac{1}{2}\ (\log 2 - 0) = \frac{1}{2}\ \log 2.$$

2. $\int_0^{\frac{\pi}{2}} \sqrt{\sin \phi}\ \cos^5\phi\ \ d\phi$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \sqrt{\sin \phi}\ \ \cos^5 \phi\ d\phi$ ...(*i*)

Put $\sin \phi = t$.

($\because$ one factor of integrand is $\cos^5 \phi$ where $n = 5$ is odd.)

$\therefore\ \cos \phi = \frac{dt}{d\phi}$ *i.e.,* $\cos \phi\ d\phi = dt$.

To change the limits of integration from ϕ to t

When $\phi = 0$, $t = \sin \phi = \sin 0 = 0$

When $\phi = \frac{\pi}{2}$, $t = \sin \phi = \sin \frac{\pi}{2} = 1$

Now Integrand $\sqrt{\sin \phi}\ \ \cos^5 \phi = \sqrt{\sin \phi}\ \ \cos^4 \phi \cos \phi$

$$= \sqrt{\sin \phi}\ \ (\cos^2 \phi)^2 \cos \phi = \sqrt{\sin \phi}\ \ (1 - \sin^2 \phi)^2 \cos \phi$$

$\therefore$ From (*i*), $I = \int_0^{\frac{\pi}{2}} \sqrt{\sin \phi}\ (1 - \sin^2 \phi)^2 \cos \phi\ d\phi$

$$= \int_0^1 \sqrt{t}\ (1 - t^2)^2\ \ dt = \int_0^1 t^{1/2}\ (1 + t^4 - 2t^2)\ \ dt$$

$$= \int_0^1 \left(t^{\frac{1}{2}} + t^{\frac{1}{2}+4} - 2t^{\frac{1}{2}+2} \right) dt = \int_0^1 (t^{1/2} + t^{9/2} - 2t^{5/2})\ dt$$

$$= \int_0^1 t^{1/2}\ \ dt + \int_0^1 t^{9/2}\ \ dt - 2\int_0^1 t^{5/2}\ \ dt$$

$$= \frac{(t^{3/2})_0^1}{\frac{3}{2}} + \frac{(t^{11/2})_0^1}{\frac{11}{2}} - 2\frac{(t^{7/2})_0^1}{\frac{7}{2}}$$

$$= \frac{2}{3}\ (1 - 0) + \frac{2}{11}\ (1 - 0) - \frac{4}{7}\ (1 - 0)$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7} = \frac{2(77) + 2(21) - 4(33)}{3(11)(7)}$$

$$= \frac{154 + 42 - 132}{231} = \frac{196 - 132}{231} = \frac{64}{231}.$$

3. $\int_0^1 \sin^{-1}\left(\frac{2x}{1 + x^2}\right)\ dx$

Sol. Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{1 + x^2}\right)\ dx$ ...(*i*)

Put $x = \tan \theta$. $\therefore\ \frac{dx}{d\theta} = \sec^2 \theta \Rightarrow dx = \sec^2 \theta\ d\theta$

To change the limits of integration

When $x = 0$, $\tan \theta = 0 = \tan 0 \quad \Rightarrow \quad \theta = 0$

When $x = 1$, $\tan \theta = 1 = \tan \frac{\pi}{4} \quad \Rightarrow \quad \theta = \frac{\pi}{4}$

$\therefore$ From (*i*), $I = \int_0^{\frac{\pi}{4}} \left(\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\right) \sec^2\theta \, d\theta$

$= \int_0^{\frac{\pi}{4}} (\sin^{-1}(\sin 2\theta)) \sec^2\theta \, d\theta = \int_0^{\frac{\pi}{4}} 2\theta \sec^2\theta \, d\theta$

$= 2 \int_0^{\frac{\pi}{4}} \underset{\text{I}}{\theta} \underset{\text{II}}{\sec^2\theta} \, d\theta$

Applying Product Rule of Integration

$\left(\int_a^b \text{I} \,.\, \text{II}\, dx = \left(\text{I} \int \text{II}\right)_a^b - \int_a^b \left(\frac{d}{dx}(\text{I}) \int \text{II}\, dx\right) dx\right)$

$= 2\left[(\theta \,.\, \tan\theta)_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 1 \,.\, \tan\theta \, d\theta\right]$

$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} - 0 - \int_0^{\frac{\pi}{4}} \tan\theta\, d\theta\right] = 2\left[\frac{\pi}{4} - (\log \sec\theta)_0^{\frac{\pi}{4}}\right]$

$= 2\left[\frac{\pi}{4} - \left(\log\sec\frac{\pi}{4} - \log\sec 0\right)\right] = 2\left[\frac{\pi}{4} - (\log\sqrt{2} - \log 1)\right]$

$= \frac{\pi}{2} - 2\log 2^{1/2} \quad (\because \log 1 = 0)$

$= \frac{\pi}{2} - 2 \,.\, \frac{1}{2} \log 2 = \frac{\pi}{2} - \log 2.$

4. $\int_0^2 x\sqrt{x+2}\, dx$

Sol. Let $I = \int_0^2 x\sqrt{x+2}\, dx$ ...(*i*)

Put $\sqrt{\text{Linear}} = t$, *i.e.*, $\sqrt{x+2} = t$. Therefore $x + 2 = t^2$.

$\therefore \frac{dx}{dt} = 2t \quad \Rightarrow \quad dx = 2t\, dt$

To change the limits of Integration

When $x = 0$, $t = \sqrt{x+2} = \sqrt{2}$

When $x = 2$, $t = \sqrt{x+2} = \sqrt{2+2} = \sqrt{4} = 2.$

$\therefore$ From (*i*), $I = \int_{\sqrt{2}}^2 (t^2 - 2)\; t \,.\, 2t\, dt$

$[\because \; x + 2 = t^2 \;\Rightarrow\; x = t^2 - 2]$

$= 2\int_{\sqrt{2}}^2 t^2(t^2-2)\, dt = 2\int_{\sqrt{2}}^2 (t^4 - 2t^2)\, dt$

$= 2\left[\left(\frac{t^5}{5}\right)_{\sqrt{2}}^2 - 2\left(\frac{t^3}{3}\right)_{\sqrt{2}}^2\right] = 2\left[\frac{1}{5}(2^5 - (\sqrt{2})^5) - \frac{2}{3}(2^3 - (\sqrt{2})^3)\right]$

$$= 2\left[\frac{1}{5}(32 - 4\sqrt{2}) - \frac{2}{3}(8 - 2\sqrt{2})\right]\left[\because \ (\sqrt{2})^3 = \sqrt{2}.\sqrt{2}.\sqrt{2} = 2\sqrt{2},\right.$$

$$\left.\text{and } (\sqrt{2})^5 = \sqrt{2}.\sqrt{2}.\sqrt{2}.\sqrt{2}.\sqrt{2} = 4\sqrt{2}\right]$$

$$= 2\left[\frac{32}{5} - \frac{4\sqrt{2}}{5} - \frac{16}{3} + \frac{4\sqrt{2}}{3}\right] = 2\left[\frac{96 - 12\sqrt{2} - 80 + 20\sqrt{2}}{15}\right]$$

$$= \frac{2}{15}(16 + 8\sqrt{2}) = \frac{16}{15}(2 + \sqrt{2}) = \frac{16}{15}(\sqrt{2}.\sqrt{2} + \sqrt{2})$$

$$= \frac{16\sqrt{2}}{15}(\sqrt{2} + 1).$$

5. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x}\, dx$

Sol. Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x}\, dx = -\int_0^{\frac{\pi}{2}} \frac{-\sin x}{1 + \cos^2 x}\, dx$ ...(*i*)

Put $\cos x = t$. Therefore $-\sin x = \dfrac{dt}{dx} \Rightarrow -\sin x\, dx = dt.$

To change the limits of Integration.

When $x = 0$, $t = \cos 0 = 1$, When $x = \dfrac{\pi}{2}$, $t = \cos\dfrac{\pi}{2} = 0$

$\therefore$ From (*i*), $I = -\displaystyle\int_1^0 \frac{dt}{1 + t^2} = -\int_1^0 \frac{1}{t^2 + 1}\, dt$

$$= -\left(\tan^{-1} t\right)_1^0 = -(\tan^{-1} 0 - \tan^{-1} 1) = -\left(0 - \frac{\pi}{4}\right)$$

$$\left[\because \ \tan 0 = 0 \Rightarrow \tan^{-1} 0 = 0 \text{ and } \tan\frac{\pi}{4} = 1 \Rightarrow \tan^{-1} 1 = \frac{\pi}{4}\right] = \frac{\pi}{4}.$$

6. $\displaystyle\int_0^2 \frac{dx}{x + 4 - x^2}$

Sol. $\displaystyle\int_0^2 \frac{dx}{4 + x - x^2} = \int_0^2 \frac{dx}{-x^2 + x + 4} = \int_0^2 \frac{dx}{-(x^2 - x - 4)}$

(Making coeff. of x^2 numerically unity)

Completing squares by adding and subtracting

$$\left(\frac{1}{2}\text{coeff. of } x\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}. = \int_0^2 \frac{dx}{-\left[x^2 - x + \frac{1}{4} - \frac{1}{4} - 4\right]}$$

$$= \int_0^2 \frac{dx}{-\left[\left(x - \frac{1}{2}\right)^2 - \frac{17}{4}\right]} = \int_0^2 \frac{dx}{\frac{17}{4} - \left(x - \frac{1}{2}\right)^2} = \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}$$

$$= \frac{1}{2 \times \frac{\sqrt{17}}{2}} \left[\log \left| \frac{\frac{\sqrt{17}}{2} + \left(x - \frac{1}{2}\right)}{\frac{\sqrt{17}}{2} - \left(x - \frac{1}{2}\right)} \right| \right]_0^2 \quad \left(\because \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a + x}{a - x} \right| \right)$$

$$= \frac{1}{\sqrt{17}} \left[\log \left| \frac{\sqrt{17} + 2x - 1}{\sqrt{17} - 2x + 1} \right| \right]_0^2$$

$$= \frac{1}{\sqrt{17}} \left[\log \left| \frac{\sqrt{17} + 3}{\sqrt{17} - 3} \right| - \log \left| \frac{\sqrt{17} - 1}{\sqrt{17} + 1} \right| \right]$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{\sqrt{17} + 3}{\sqrt{17} - 3} \times \frac{\sqrt{17} + 1}{\sqrt{17} - 1} \right) = \frac{1}{\sqrt{17}} \log \frac{20 + 4\sqrt{17}}{20 - 4\sqrt{17}}$$

($\because$ $(\sqrt{17} + 3)(\sqrt{17} + 1) = 17 + \sqrt{17} + 3\sqrt{17} + 3 = 20 + 4\sqrt{17}$. Similarly $(\sqrt{17} - 3)(\sqrt{17} - 1) = 20 - 4\sqrt{17}$)

$$= \frac{1}{\sqrt{17}} \log \frac{4(5 + \sqrt{17})}{4(5 - \sqrt{17})} = \frac{1}{\sqrt{17}} \log \frac{5 + \sqrt{17}}{5 - \sqrt{17}}$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{5 + \sqrt{17}}{5 - \sqrt{17}} \times \frac{5 + \sqrt{17}}{5 + \sqrt{17}} \right) = \frac{1}{\sqrt{17}} \log \frac{(5 + \sqrt{17})^2}{25 - 17}$$

$$= \frac{1}{\sqrt{17}} \log \frac{42 + 10\sqrt{17}}{8} = \frac{1}{\sqrt{17}} \log \frac{21 + 5\sqrt{17}}{4}.$$

7. $\displaystyle\int_{-1}^{1} \frac{dx}{x^2 + 2x + 5}$

Sol. Let $\displaystyle I = \int_{-1}^{1} \frac{dx}{x^2 + 2x + 5} = \int_{-1}^{1} \frac{dx}{x^2 + 2x + 1 + 4}$ (To complete squares)

$$= \int_{-1}^{1} \frac{1}{(x + 1)^2 + 2^2} dx \quad ...(i)$$

Put $x + 1 = t$. $\quad \therefore \dfrac{dx}{dt} = 1 \quad \Rightarrow \quad dx = dt$

To change the limits of Integration

When $x = -1$, $t = -1 + 1 = 0$

When $x = 1$, $t = 1 + 1 = 2$

$\therefore$ From (*i*), $\displaystyle I = \int_0^2 \frac{1}{t^2 + 2^2} dt = \frac{1}{2} \left(\tan^{-1} \frac{t}{2} \right)_0^2$

$$\left[\because \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \right]$$

$$= \frac{1}{2} \left[\tan^{-1} \frac{2}{2} - \tan^{-1} \frac{0}{2} \right] = \frac{1}{2} (\tan^{-1} 1 - \tan^{-1} 0)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}. \qquad \left[\because \tan\frac{\pi}{4} = 1 \text{ and } \tan 0 = 0\right]$$

8. $\displaystyle\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x}\, dx$

Sol. Let I = $\displaystyle\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x}\, dx$ ...(*i*)

[Type $\int (f(x) + g(x))\; e^{ax}\, dx$. Put $ax = t$ and it will become $\int (f(t) + f'(t))\; e^t\, dt = e^t f(t)$]

Put $2x = t$ $\therefore\ 2 = \frac{dt}{dx} \Rightarrow 2dx = dt \Rightarrow dx = \frac{dt}{2}$

To change the limits of Integration

When $x = 1,\ t = 2x = 2$, When $x = 2,\ t = 2x = 4$

$$\therefore \text{ From } (i),\ \text{I} = \int_2^4 \left(\frac{1}{\frac{t}{2}} - \frac{1}{2\left(\frac{t}{2}\right)^2}\right) e^t\, \frac{dt}{2} \quad \left[\because\ 2x = t \Rightarrow x = \frac{t}{2}\right]$$

$$\therefore\ \text{I} = \int_2^4 \left(\frac{2}{t} - \frac{2}{t^2}\right) e^t\, \frac{dt}{2} = \int_2^4 \frac{1}{2}.2\left(\frac{1}{t} - \frac{1}{t^2}\right) e^t\, dt$$

$$= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t\, dt = \int_2^4 (f(t) + f'(t))\, e^t\, dt$$

$$\left(\text{Here } f(t) = \frac{1}{t} = t^{-1} \text{ and therefore } f'(t) = (-1)t^{-2} = \frac{-1}{t^2}\right)$$

$$= \left(e^t f(t)\right)_2^4 = \left(\frac{e^t}{t}\right)_2^4 = \frac{e^4}{4} - \frac{e^2}{2} = \frac{e^4 - 2e^2}{4} = \frac{e^2\,(e^2 - 2)}{4}.$$

Choose the correct answer in Exercises 9 and 10.

9. The value of the integral $\displaystyle\int_{\frac{1}{3}}^1 \frac{(x - x^3)^{1/3}}{x^4}\, dx$ is

(A) 6 **(B) 0** **(C) 3** **(D) 4**

Sol. Let I = $\displaystyle\int_{\frac{1}{3}}^1 \frac{(x - x^3)^{1/3}}{x^4}\, dx$

$$= \int_{\frac{1}{3}}^1 \frac{\left[x^3\left(\frac{x}{x^3} - 1\right)\right]^{1/3}}{x^4}\, dx = \int_{\frac{1}{3}}^1 \frac{(x^3)^{1/3}\left(\frac{1}{x^2} - 1\right)^{1/3}}{x^4}$$

$$= \int_{\frac{1}{3}}^1 \frac{x\,(x^{-2} - 1)^{1/3}}{x^4}\, dx = \int_{\frac{1}{3}}^1 (x^{-2} - 1)^{1/3}\; x^{-3}\, dx$$

$$\text{I} = \frac{-1}{2} \int_{\frac{1}{3}}^1 (x^{-2} - 1)^{1/3}\,(-2x^{-3})\; dx \qquad ...(i)$$

Put $x^{-2} - 1 = t$

Therefore $-2x^{-3} = \frac{dt}{dx}$ $\Rightarrow$ $-2x^{-3}\,dx = dt$

To change the limits of Integration

When $x = \frac{1}{3}$, $t = x^{-2} - 1 = \left(\frac{1}{3}\right)^{-2} - 1$

$= (3^{-1})^{-2} - 1 = 3^2 - 1 = 9 - 1 = 8$

When $x = 1$, $t = 1^{-2} - 1 = 1 - 1 = 0$

$\therefore$ From (i), $I = \frac{-1}{2}\int_8^0 t^{1/3}\,dt = \frac{-1}{2}\left(\frac{t^{4/3}}{\frac{4}{3}}\right)_8^0$

$= \frac{-1}{2}\cdot\frac{3}{4}[0 - 8^{4/3}] = \frac{-3}{8}[-(2^3)^{4/3}] = \frac{-3}{8}(-2^4) = \frac{3}{8}\times 16 = 6$

$\therefore$ Option (A) is the correct answer.

10. If $f(x) = \int_0^x t\sin t\,dt$, then $f'(x)$ is

(A) $\cos x + x\sin x$

(B) $x\sin x$

(C) $x\cos x$

(D) $\sin x + x\cos x$

Sol. $f(x) = \int_0^x t\sin t\,dt$
I II

Applying Product Rule of Integration

$$\left[\int_a^b \text{I}\,.\,\text{II}\,dx = \left(\text{I}\int \text{II}\,dx\right)_a^b - \int_a^b \frac{d}{dx}(\text{I})\int \text{II}\,dx\,dx\right]$$

$\Rightarrow$ $f(x) = \left(t(-\cos t)\right)_0^x - \int_0^x 1(-\cos t)\,dt$

$= -x\cos x - 0 + \int_0^x \cos t\,dt = -x\cos x + (\sin t)_0^x$

$= -x\cos x + \sin x - \sin 0 = -x\cos x + \sin x$

$\therefore$ $f'(x) = -(x(-\sin x) + (\cos x)1) + \cos x$

$= x\sin x - \cos x + \cos x = x\sin x$

$\therefore$ Option (B) is the correct answer.

OR

$f(x) = \int_0^x \sin t\,dt$ $\therefore$ $f'(x) = (t\sin t)_0^x$

[$\because$ Derivative operator and integral operator cancel with each other]

$= x\sin x - 0 = x\sin x$.