Exercise 13.2

1. If $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$, find $P(A \cap B)$ if A and B are independent events.

Sol. Because, A and B are independent events;

$$\therefore \quad P(A \cap B) = P(A)\, P(B) = \frac{3}{5} \times \frac{1}{5} = \frac{3}{25}.$$

2. Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Sol. Let E_1 : first card drawn is black

E_2 : second card drawn is black

then $\qquad P(E_1) = \frac{26}{52}$

[$\because$ We know that there are 26 black cards in a pack of 52 cards]

Since the first card drawn is not replaced (given: without replacement), there are 25 (= 26 – 1) black cards in a pack of 51 (= 52 – 1) cards.

$P(E_2, E_1)$ *i.e.,* probability that second card is black known that first card is black $= \frac{25}{51}$

$\therefore$ Required probability $= P(E_1 \cap E_2) = P(E_1 \text{ and } E_2)$ *i.e.,* probability that both cards are black.

$$= P(E_1)\, P(E_2\, E_1) = \frac{26}{52} \times \frac{25}{51} = \frac{25}{102}.$$

3. A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

Sol. Given: The box contains a total of 15 oranges out of which 12 are good and 3 are bad. The box is approved for sale if all the three oranges drawn, (one by one), without replacement are good ones.

Let E_1 : first orange drawn is good

E_2 : second orange drawn is good

E_3 : third orange drawn is good

then $P(E_1) = \frac{12}{15}$, $P(E_2 / E_1) = \frac{11}{14}$ and $P(E_3 / E_1E_2)$

i.e., probability that third orange drawn is good when both the first and oranges drawn are good = $\frac{10}{13}$

$\therefore$ Required probability = $P(E_1 \cap E_2 \cap E_3)$

$= P(E_1 \text{ and } E_2 \text{ and } E_3) = P(E_1)\, P(E_2 / E_1)\, P(E_3 / E_1E_2)$

$= \frac{12}{15} \times \frac{11}{14} \times \frac{10}{13} = \frac{44}{91}.$

4. A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.

Sol. The sample space for the random experiment 'a fair coin and an unbiased die are tossed' is

S = {(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2) (T, 3), (T, 4), (T, 5), (T, 6)}

$n(S) = 2 \times 6 = 12$

A : head appears on the coin

and B : 3 appears on the die

$\therefore$ A = {(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6)}

B = {(H, 3), (T, 3)}

$A \cap B$ = {(H, 3)}

$\therefore$ $n(A) = 6$, $n(B) = 2$, $n(A \cap B) = 1$

$\therefore$ $P(A) = \frac{n(A)}{n(S)} = \frac{6}{12} = \frac{1}{2}$, $P(B) = \frac{2}{12} = \frac{1}{6}$, $P(A \cap B) = \frac{1}{12}$

Since, $P(A \cap B) = \frac{1}{12} = \frac{1}{2} \times \frac{1}{6} = P(A)\, P(B)$;

therefore the events A and B are independent.

5. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'the number is even', and B be the event, 'the number is red'. Are A and B independent?

Sol. The sample space is

$S = \{\underbrace{1, 2, 3}_{\text{red}}, \underbrace{4, 5, 6}_{\text{green}}\}$ $\qquad \therefore \quad n(S) = 6$

Event A : the number is even and B : the number is red

$\Rightarrow$ A = {2, 4, 6} and B : {1, 2, 3} $\quad A \cap B = \{2\}$

$\therefore$ $n(A) = 3$, $n(B) = 3$, $n(A \cap B) = 1$

$\therefore$ $P(A) = \frac{3}{6} = \frac{1}{2}$, $P(B) = \frac{3}{6} = \frac{1}{2}$, $P(A \cap B) = \frac{1}{6}$

Since, $\frac{1}{6} \neq \frac{1}{2} \times \frac{1}{2}$, Therefore, $P(A \cap B) \neq P(A)\, P(B)$.

Therefore the events A and B are not independent.

6. Let E and F be events with $P(E) = \frac{3}{5}$, $P(F) = \frac{3}{10}$ and $P(E \cap F) = \frac{1}{5}$. Are E and F independent?

Sol. Here $P(E \cap F) = \frac{1}{5}$ and $P(E)\, P(F) = \frac{3}{5} \times \frac{3}{10} = \frac{9}{50}$

Since, $\frac{1}{5} \neq \frac{9}{50}$, therefore $P(E \cap F) \neq P(E)\, P(F)$.

Therefore the events E and F are not independent.

7. Given that the events A and B are such that $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are (*i*) mutually exclusive (*ii*) independent.

Sol. Given: $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$, $P(B) = p$

We know that $\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

$\therefore \quad \frac{3}{5} = \frac{1}{2} + p - P(A \cap B)$

$\Rightarrow P(A \cap B) = p + \frac{1}{2} - \frac{3}{5} = p - \frac{1}{10}$

(*i*) If A and B are mutually exclusive, then $A \cap B = \phi$ so that

$P(A \cap B) = 0 \quad \Rightarrow \quad p - \frac{1}{10} = 0 \quad \therefore \quad p = \frac{1}{10}$

(*ii*) If A and B are independent, then $P(A \cap B) = P(A)\, P(B)$

$\Rightarrow \quad p - \frac{1}{10} = \frac{1}{2}p \quad \Rightarrow \quad p - \frac{1}{2}p = 10 \quad \Rightarrow \quad \frac{1}{2}p = \frac{1}{10}$

$\therefore \quad p = \frac{1}{5}$.

8. Let A and B be independent events with P(A) = 0.3 and P(B) = 0.4. Find

(*i*) $P(A \cap B)$ (*ii*) $P(A \cup B)$ (*iii*) $P(A / B)$ (*iv*) $P(B / A)$.

Sol. Given: $P(A) = 0.3$ and $P(B) = 0.4$

and A and B are independent events

$\Rightarrow \quad P(A \cap B) = P(A) \,.\, P(B) \qquad ...(i)$

(*i*) $\therefore \quad P(A \cap B) = P(A)\, P(B) = 0.3 \times 0.4 = 0.12$.

(*ii*) and $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A)\, P(B) \qquad$ [By (*i*)]

$= 0.3 + 0.4 - 0.3 \times 0.4 = 0.7 - 0.12 = 0.58$.

(*iii*) Again $P(A / B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\, P(B)}{P(B)}$ [By (*i*)] $= P(A) = 0.3$.

(*iv*) Also $P(B / A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)\, P(A)}{P(A)}$ [By (*i*)] $= P(B) = 0.4$.

9. If A and B are two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{8}$, find P(not A and not B).

Sol. **Given:** P(A) = $\frac{1}{4}$, P(B) = $\frac{1}{2}$ and P(A ∩ B) = $\frac{1}{8}$

Here P(A ∩ B) = $\frac{1}{8} = \frac{1}{4} \times \frac{1}{2}$ = P(A) P(B)

⇒ A and B are independent events

⇒ A′ and B′ are independent events

⇒ P(A′ ∩ B′) = P(A′) P(B′) ...(*i*)

Now, P(not A and not B) = P(A′ and B′) = P(A′ ∩ B′)

= P(A′) P(B′) [By (*i*)]

= [1 – P(A)] [1 – P(B)]

$= \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{2}\right) = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$.

10. Events A and B are such that P(A) = $\frac{1}{2}$, P(B) = $\frac{7}{12}$ and P(not A or not B) = $\frac{1}{4}$. State whether A and B are independent?

Sol. **Given:** P(A) = $\frac{1}{2}$, P(B) = $\frac{7}{12}$ and

P(not A or not B) = $\frac{1}{4}$ ⇒ P(A′ ∪ B′) = $\frac{1}{4}$

⇒ P(A ∩ B)′ = $\frac{1}{4}$ (By De Morgan's Law)

⇒ 1 – P(A ∩ B) = $\frac{1}{4}$ [∵ P(A′) = 1 – P(A)]

⇒ P(A ∩ B) = 1 – $\frac{1}{4} = \frac{3}{4}$

Also, P(A) P(B) = $\frac{1}{2} \times \frac{7}{12} = \frac{7}{24}$

Since, P(A ∩ B) $\left(= \frac{3}{4}\right)$ ≠ P(A) P(B) $\left(= \frac{7}{24}\right)$, the events A and B are not independent.

11. Given two independent events A and B such that P(A) = 0.3, P(B) = 0.6. Find

(*i*) P(A and B) **(*ii*) P(A and not B)**

(*iii*) P(A or B) **(*iv*) P(neither A nor B).**

Sol. **Given:** A and B are independent events such that

P(A) = 0.3, P(B) = 0.6

(*i*) ∴ P(A and B) = P(A ∩ B) = P(A) P(B) = 0.3 × 0.6 = 0.18

(*ii*) P(A and not B) = P(A ∩ B′) = P(A) P(B′)

[∵ A and B are independent ⇒ A and B′ are independent]

= P(A) [1 – P(B)] = 0.3(1 – 0.6) = 0.3 × 0.4 = 0.12.

(*iii*) P(A or B) = P(A ∪ B) = P(A) + P(B) – P(A ∩ B)

= P(A) + P(B) – P(A) P(B)

[∵ A and B are independent events]

$= 0.3 + 0.6 - 0.3 \times 0.6 = 0.9 - 0.18 = 0.72.$

(*iv*) P(neither A nor B) $= P(A' \cap B') = P(A') P(B')$

[$\because$ A and B are independent $\Rightarrow$ A′ and B′ are independent]

$= [1 - P(A)] [1 - P(B)] = (1 - 0.3) (1 - 0.6) = 0.7 \times 0.4 = 0.28.$

12. A die is tossed thrice. Find the probability of getting an odd number at least once.

Sol. Let S be the sample space of tossing a dice.

Therefore, $S = \{1, 2, 3, 4, 5, 6\} \Rightarrow n(S) = 6$

Let E be the event of getting an odd number on a dice.

$\therefore \quad E = \{1, 3, 5\} \Rightarrow n(E) = 3$

$\Rightarrow \quad P(E) = \frac{3}{6}$ ...(*i*)

Let event E_1: an odd number on first toss

E_2: an odd number on second toss

E_3: an odd number on third toss

then we know from common sense that E_1, E_2, E_3 are independent events and hence E_1', E_2', E_3' are also independent.

Now $P(E_1) = P(E_2) = P(E_3) = \frac{3}{6} = \frac{1}{2}$ [By (*i*)]

$\therefore$ P(an odd number **at least** once)

= 1 – P(an odd number on none of the three dice)

$= 1 - P(E_1' \text{ and } E_2' \text{ and } E_3')$

$= 1 - P(E_1') P(E_2') P(E_3')$

$= 1 - \left(1-\frac{1}{2}\right)\left(1-\frac{1}{2}\right)\left(1-\frac{1}{2}\right) = 1 - \left(\frac{1}{2}\right)^3 = \frac{7}{8}.$

13. Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that

(*i*) both balls are red.

(*ii*) first ball is black and second is red.

(*iii*) one of them is black and other is red.

Sol. Since the two balls are drawn with replacement, the two draws are independent.

Number of black balls = 10, Number red balls = 8

Total number of balls = 10 + 8 = 18

(*i*) P(both balls are red)

= P(first ball is red and second ball is red)

= P(first ball is red) × P(second ball is red)

$= \frac{8}{18} \times \frac{8}{18} = \frac{4}{9} \times \frac{4}{9} = \frac{16}{81}.$

(*ii*) P(first ball is black and second is red)

= P(first ball is black) × P(second ball is red)

$= \frac{10}{18} \times \frac{8}{18} = \frac{5}{9} \times \frac{4}{9} = \frac{20}{81}.$

(*iii*) P(one of the balls is black and other is red)

= P('first ball is black and second is red' or 'first ball is red and second is black')

= P(first ball is black and second is red) + P(first ball is red and second is black)

= P(first ball is black) × P(second ball is red) + P(first ball is red) × P(second ball is black)

$$= \frac{10}{18} \times \frac{8}{18} + \frac{8}{18} \times \frac{10}{18} = \frac{5}{9} \times \frac{4}{9} + \frac{4}{9} \times \frac{5}{9}$$

$$= \frac{20}{81} + \frac{20}{81} = \frac{40}{81}.$$

14. Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

(*i*) the problem is solved

(*ii*) exactly one of them solves the problem.

Sol. Let event E: A solves the problem

and F: B solves the problem

then $P(E) = \frac{1}{2}$ and $P(F) = \frac{1}{3}$ (given)

(*i*) P(the problem is solved)

= P(at least one solves the problem)

= 1 – Probability that neither A nor B solves the problem

= 1 – P(E′ ∩ F′) = 1 – P(E′ and F′) = 1 – P(E′) P(F′)

[∵ Events E and F are independent (given) ⇒ E′ and F′ are also independent]

= 1 – [1 – P(E)] [1 – P(F)]

$$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 1 - \frac{1}{2} \times \frac{2}{3} = 1 - \frac{1}{3} = \frac{2}{3}.$$

(*ii*) P(exactly one of them solves the problem)

= P('A solves and B does not solve' or 'B solves and A does not solve')

= P(E ∩ F′) + P(F ∩ E′) = P(E) P(F′) + P(F) P(E′)

(∵ Events are independent)

$$= \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{3}\left(1 - \frac{1}{2}\right)$$

$$= \frac{1}{2} \times \frac{2}{3} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{3} + \frac{1}{6} = \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2}.$$

15. One card is drawn at random from a well shuffled deck of 52 cards. In which of the following cases are the events E and F independent?

(*i*) E: 'the card drawn is a spade'
F: 'the card drawn is an ace'

(*ii*) E: 'the card drawn is black'
F: 'the card drawn is a king'

(*iii*) E: 'the card drawn is a king or queen'
F: 'the card drawn is a queen or jack'.

Sol. (*i*) E: the card is a spade

F: the card is an ace

$\Rightarrow$ E $\cap$ F: $\Rightarrow$ the common cards in E and F

$\Rightarrow$ the card is ace of spade $\Rightarrow$ $n(E \cap F) = 1$

$\therefore \quad P(E) = \frac{13}{52} = \frac{1}{4}, \; P(F) = \frac{4}{52} = \frac{1}{13}$ [$\because$ We know that in a pack of cards, there are 13 spade cards and 4 aces]

Now, $P(E \cap F) = \frac{1}{52} = \frac{1}{4} \times \frac{1}{13} = P(E)\,P(F)$

$\Rightarrow$ E and F are independent.

(*ii*) E: the card is black

F: the card is a king

$\Rightarrow$ E $\cap$ F: $\Rightarrow$ the common cards in E and F

$\Rightarrow$ the card is a black king $\Rightarrow$ $n(E \cap F) = 2$

[$\because$ There are 2 black kings in a pack of 52 cards]

$\therefore \quad P(E) = \frac{26}{52} = \frac{1}{2}, \; P(F) = \frac{4}{52} = \frac{1}{13}$

$P(E \cap F) = \frac{2}{52} = \frac{1}{26} = \frac{1}{2} \times \frac{1}{13} = P(E)\,P(F)$

$\Rightarrow$ E and F are independent.

(*iii*) E: the card drawn is a king or queen

F: the card drawn is a queen or jack

$\Rightarrow$ E $\cap$ F: $\Rightarrow$ the common cards in E and F $\Rightarrow$ the card drawn is a queen

$\therefore \quad P(E) = \frac{4+4}{52} = \frac{2}{13}, \; P(F) = \frac{4+4}{52} = \frac{2}{13}$ [$\because$ We know that in a pack of 52 cards there are 4 jacks, 4 queens and 4 kings]

$P(E \cap F) = \frac{4}{52} = \frac{1}{13}$ Also, $P(E)\,P(F) = \frac{2}{13} \times \frac{2}{13} = \frac{4}{169}$

Since, $P(E \cap F) \neq P(E)\,P(F)$, the events E and F are not independent.

16. In a hostel, 60% of the students read Hindi news paper, 40% read English news paper and 20% read both Hindi and English news papers. A student is selected at random.

(*a*) Find the probability that she reads neither Hindi nor English news papers.

(*b*) If she reads Hindi news paper, find the probability that she reads English news paper.

(*c*) If she reads English news paper, find the probability that she reads Hindi news paper.

Sol. Let Event A: a student reads Hindi newspaper

B: a student reads English newspaper

Given: 60% students read Hindi newspaper, 40% read English newspaper and 20% read both.

Therefore, $P(A) = \frac{60}{100} = \frac{3}{5}$, $P(B) = \frac{40}{100} = \frac{2}{5}$,

$$P(A \cap B) = P(A \text{ and } B) = \frac{20}{100} = \frac{1}{5}$$

(*a*) Required probability = Probability that she reads neither newspaper = $P(A' \cap B') = P(A \cup B)'$

(De-Morgan's Law)

$$= 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - \left[\frac{3}{5} + \frac{2}{5} - \frac{1}{5}\right] = 1 - \frac{4}{5} = \frac{1}{5}.$$

Remark: Here $P(A' \cap B') \neq P(A')\, P(B')$ because events A and B are not independent

$$\left(\because\ P(A)\,.\,P(B) = \frac{3}{5} \times \frac{2}{5} = \frac{6}{25} \neq P(A \cap B)\left(= \frac{1}{5}\right)\right)$$

and hence A′ and B′ are not independent.

(*b*) Required probability = $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{5}}{\frac{3}{5}} = \frac{1}{3}$.

(*c*) Required probability = $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{5}}{\frac{2}{5}} = \frac{1}{2}$.

Choose the correct answer in Exercises 16 and 17:

17. The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

(A) 0 **(B) $\frac{1}{3}$** **(C) $\frac{1}{12}$** **(D) $\frac{1}{36}$.**

Sol. When a pair of dice is rolled, the sample space is

$S = \{(x, y) : x, y \in \{1, 2, 3, 4, 5, 6\}\}$. $\therefore$ $n(S) = 6 \times 6 = 36$

Let event E: an even prime number on each die

Since, 2 is the only even prime number, $E = \{(2, 2)\}$

$\therefore$ $n(E) = 1$

Required probability = $P(E) = \frac{1}{36}$ $\Rightarrow$ (D) is the correct option.

18. Two events A and B will be independent, if

(A) A and B are mutually exclusive

(B) P(A′B′) = [1 – P(A)] [1 – P(B)]

(C) P(A) = P(B) **(D) P(A) + P(B) = 1.**

Sol. A and B are independent

$\Rightarrow$ A′ and B′ are independent

$\Rightarrow \quad P(A' \cap B') = P(A')\, P(B') = [1 - P(A)]\, [1 - P(B)]$

$\therefore$ (B) is the correct option.