NCERT Class 12 Maths

Solutions

Chapter - 11

Exercise 11.2

1. Show that the three lines with direction cosines

$$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$$

are mutually perpendicular.

Sol. Given: Direction cosines of three lines are

$$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; = l_1, m_1, n_1, \qquad \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; = l_2, m_2, n_2$$

and $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13} = l_3, m_3, n_3$

For first two lines;

$$l_1l_2 + m_1m_2 + n_1n_2 = \frac{12}{13}\left(\frac{4}{13}\right) + \left(\frac{-3}{13}\right)\left(\frac{12}{13}\right) + \left(\frac{-4}{13}\right)\left(\frac{3}{13}\right)$$

$$= \frac{48}{169} - \frac{36}{169} - \frac{12}{169} = \frac{48 - 36 - 12}{169} = \frac{0}{169} = 0$$

$\therefore$ The first two lines are perpendicular to each other.

For second and third line,

$$l_2l_3 + m_2m_3 + n_2n_3$$

$$= \frac{4}{13}\left(\frac{3}{13}\right) + \frac{12}{13}\left(\frac{-4}{13}\right) + \frac{3}{13}\left(\frac{12}{13}\right) = \frac{12}{169} - \frac{48}{169} + \frac{36}{169}$$

$$= \frac{12 - 48 + 36}{169} = \frac{0}{169} = 0$$

$\therefore$ Second and third lines are perpendicular to each other.

For first and third line,

$$l_1l_3 + m_1m_3 + n_1n_3$$

$$= \frac{12}{13}\left(\frac{3}{13}\right) + \left(\frac{-3}{13}\right)\left(\frac{-4}{13}\right) + \left(\frac{-4}{13}\right)\left(\frac{12}{13}\right) = \frac{36}{169} + \frac{12}{169} - \frac{48}{169}$$

$$= \frac{36 + 12 - 48}{169} = \frac{0}{169} = 0$$

$\therefore$ First and third line are also perpendicular to each other.

$\therefore$ The three given lines are mutually perpendicular.

2. Show that the line through the points (1, – 1, 2), (3, 4, – 2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

Sol. We know that direction ratios of the line joining the points A(1, – 1, 2) and B(3, 4, – 2) are $x_2 - x_1, y_2 - y_1, z_2 - z_1$

i.e., $3 - 1, 4 - (-1), -2 - 2 = 2, 5, -4 = a_1, b_1, c_1$ (say)

Again, direction ratios of the line joining the points C(0, 3, 2) and D(3, 5, 6) are $x_2 - x_1, y_2 - y_1, z_2 - z_1$

i.e., $3 - 0, 5 - 3, 6 - 2 = 3, 2, 4 = a_2, b_2, c_2$ (say)

For these lines AB and CD,

$$a_1a_2 + b_1b_2 + c_1c_2 = 2(3) + 5(2) + (-4)(4)$$
$$= 6 + 10 - 16 = 0$$

$\therefore$ Given line AB is perpendicular to given line CD.

3. Show that the line through the points (4, 7, 8), (2, 3, 4) is parallel to the line through the points (– 1, – 2, 1), (1, 2, 5).

Sol. We know that direction ratios of the line joining the points A(4, 7, 8) and B(2, 3, 4) are $x_2 - x_1, y_2 - y_1, z_2 - z_1$

i.e., $2 - 4, 3 - 7, 4 - 8$ *i.e.,* $-2, -4, -4 = a_1, b_1, c_1$ (say)

Again, direction ratios of the line joining the points C(– 1, – 2, 1) and D(1, 2, 5) are $1 - (-1), 2 - (-2), 5 - 1 = 2, 4, 4 = a_2, b_2, c_2$ (say)

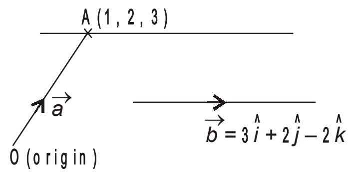
For these lines AB and CD,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \qquad \left(\text{as } \frac{-2}{2} = \frac{-4}{4} = \frac{-4}{4} \ (= -1 \text{ each})\right)$$

$\therefore$ Given line AB is parallel to given line CD.

4. Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector

$$\mathbf{3\hat{i} + 2\hat{j} - 2\hat{k}}.$$



Sol. A point on the required line is

$$A(1, 2, 3) = (x_1, y_1, z_1)$$

i.e., Position vector of a point on the required line is

$$\vec{a} = \overrightarrow{OA} = (1, 2, 3) = \hat{i} + 2\hat{j} + 3\hat{k}.$$

The required line is parallel to the vector $\vec{b} = 3\hat{i} + 2\hat{j} - 2\hat{k}$ (and hence direction ratios of the required line are coefficient of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{b}$ *i.e.,* $3, 2, -2 = a, b, c$)

$\therefore$ Vector equation of required line is

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{a}} + \lambda \vec{\boldsymbol{b}} \quad i.e., \ \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$$

where λ is a real number.

Remark. Also cartesian equation of the required line in this Q. No. 4 is

$$\frac{\boldsymbol{x - x_1}}{\boldsymbol{a}} = \frac{\boldsymbol{y - y_1}}{\boldsymbol{b}} = \frac{\boldsymbol{z - z_1}}{\boldsymbol{c}} \quad i.e., \ \frac{x-1}{3} = \frac{y-2}{2} = \frac{z-3}{-2}$$

5. Find the equation of the line in vector and in cartesian form that passes through the point with position vector $\mathbf{2\hat{i} - \hat{j} + 4\hat{k}}$ and is in the direction $\mathbf{\hat{i} + 2\hat{j} - \hat{k}}$.

Sol. Position vector of a point on the required line is

$$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k} = (2, -1, 4) = (x_1, y_1, z_1)$$

The required line is in the direction of the vector

$$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$$

($\Rightarrow$ direction ratios of required line are coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{b}$ *i.e.,* $1, 2, -1 = a, b, c$)

$\therefore$ Equation of the required line in vector form is $\vec{\boldsymbol{r}} = \vec{\boldsymbol{a}} + \lambda \vec{\boldsymbol{b}}$

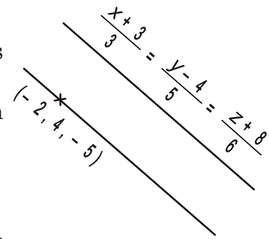
i.e., $$\vec{r} = (2\hat{i} - \hat{j} + 4\hat{k}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$$

where λ is a real number and equation of line in cartesian form is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad i.e., \ \frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}.$$

6. Find the cartesian equation of the line which passes through the point (– 2, 4, – 5) and parallel to the line given by

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}.$$



Sol. Given: A point on the required line is $(-2, 4, -5) = (x_1, y_1, z_1)$.

Equations of the given line in cartesian form are

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$$

(It is standard form because coefficients of x, y, z are unity each)

$\therefore$ Direction ratios (D.R.'s) of the given line are its denominators 3, 5, 6 and hence d.r.'s of the required parallel line are also 3, 5, 6 = a, b, c.

$\therefore$ Eqations of the required line are

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \quad i.e., \quad \frac{x-(-2)}{3} = \frac{y-4}{5} = \frac{z-(-5)}{6}$$

$$i.e., \quad \frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}.$$

7. The cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. Write its vector form.

Sol. Given: The cartesian equation of a line is

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$$

$$i.e \quad \frac{x-5}{3} = \frac{y-(-4)}{7} = \frac{z-6}{2}$$

compairing the given equation with the standard form

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

we have $x_1 = 5$, $y_1 = -4$, $z_1 = 6$; $a = 3$, $b = 7$, $c = 2$

Hence the given line passes through the point

$$\vec{a} = (x_1, y_1, z_1) = (5, -4, 6) = 5\hat{i} - 4\hat{j} + 6\hat{k}$$

and is parallel (or collinear) with the vector

$$\vec{b} = a\hat{i} + b\hat{j} + c\hat{k} = 3\hat{i} + 7\hat{j} + 2\hat{k}$$

$\therefore$ Vector equation of the given line is $\vec{r} = \vec{a} + \lambda\vec{b}$

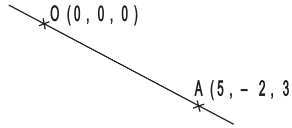
$$\text{i.e } \vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$$

8. Find the vector and cartesian equations of the line that passes through the origin and (5, – 2, 3).

Sol. Vector equation of the line

$\vec{a}$ = Position vector of a point here O (say) on the line

$= (0, 0, 0) = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$



$\vec{b}$ = A vector along the line

$= \overrightarrow{OA}$ = Position vector of point A – Position vector of point O

$= (5, -2, 3) - (0, 0, 0) = (5, -2, 3) = 5\hat{i} - 2\hat{j} + 3\hat{k}$

$\therefore$ Vector equation of the line is $\vec{r} = \vec{a} + \lambda \vec{b}$

i.e., $\vec{r} = \vec{0} + \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$ i.e. $\vec{r} = \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$.

Cartesian equation of the line

Direction ratios of line OA are 5 – 0, – 2 – 0, 3 – 0

i.e., 5, – 2, 3 | $x_2 - x_1, y_2 - y_1, z_2 - z_1 = a, b, c$

A point on the line O is $(0, 0, 0) = (x_1, y_1, z_1)$.

$\therefore$ Cartesian equation of the line is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad i.e., \quad \frac{x - 0}{5} = \frac{y - 0}{-2} = \frac{z - 0}{3}$$

i.e., $\frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$.

Remark. In the solution of the above question we can also take:

$\vec{a}$ = Position vector of point A = (5, – 2, 3) = $5\hat{i} - 2\hat{j} + 3\hat{k}$ for vector form and point A as $(x_1, y_1, z_1) = (5, -2, 3)$ for Cartesian form.

The equation of line in vector form is $\vec{r} = \vec{a} + \lambda \vec{b}$

i.e., $\vec{r} = 5\hat{i} - 2\hat{j} + 3\hat{k} + \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$

and equation of line in cartesian form is $\frac{x - 5}{5} = \frac{y + 2}{-2} = \frac{z - 3}{3}$.

9. Find vector and cartesian equations of the line that passes through the points (3, – 2, – 5) and (3, – 2, 6).

Sol. Vector Equation

Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A(3, – 2, – 5) and B(3, – 2, 6).

$\therefore \quad \vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$\therefore$ A vector along the line = $\overrightarrow{AB}$ = position vector of point B – position vector of point A

$= \vec{b} - \vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k} - 3\hat{i} + 2\hat{j} + 5\hat{k} = 11\hat{k}$

Vector equation of the line is

$\vec{r} = \vec{a} + \lambda \overrightarrow{AB}$ *i.e.*, $\vec{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + \lambda(11\hat{k})$

i.e., $\vec{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + 11\lambda\hat{k}$.

Note. Another vector equation for the same line is

$\vec{r} = \vec{b} + \lambda \overrightarrow{AB}$ *i.e.*, $\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k} + 11\lambda\hat{k}$.

Cartesian Equation

Direction ratios of line AB are 3 – 3, – 2 + 2, 6 + 5

i.e., 0, 0, 11 | $x_2 - x_1, y_2 - y_1, z_2 - z_1$

$\therefore$ Equations of the line are $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$

i.e., $\dfrac{x-3}{0} = \dfrac{y+2}{0} = \dfrac{z+5}{11}$.

10. Find the angle between the following pairs of lines:

(*i*) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ **and**

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

(*ii*) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ **and**

$\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$**.**

Sol. (*i*) **Given:** Equation of one line is

$\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$

Comparing with $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$,

$\vec{a_1} = 2\hat{i} - 5\hat{j} + \hat{k}$ and a vector along the line is

$\vec{b_1} = 3\hat{i} + 2\hat{j} + 6\hat{k}$ ...(*i*)

(It may be noted that vector $\vec{a_1}$ is the position vector of a point on the line and not a vector along the line).

Given: Equation of second line is

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

Comparing with $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ we have

$\vec{a_2} = 7\hat{i} - 6\hat{k}$ and a vector along the second line is

$\vec{b_2} = \hat{i} + 2\hat{j} + 2\hat{k}$ ...(*ii*)

Let θ be the angle between the two lines.

We know that $\cos\theta = \dfrac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$

$$= \frac{3(1)+2(2)+6(2)}{\sqrt{9+4+36}\sqrt{1+4+4}} = \frac{3+4+12}{\sqrt{49}\sqrt{9}}$$

$$\cos\theta = \frac{19}{7(3)} = \frac{19}{21} \qquad \therefore \qquad \theta = \cos^{-1}\frac{19}{21}.$$

(*ii*) Comparing the equations of the two given lines with $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ we have $\vec{b_1} = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b_2} = 3\hat{i} - 5\hat{j} - 4\hat{k}$.

Let θ be the angle between the two lines

$$\therefore \quad \cos\theta = \frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}||\vec{b_2}|} = \frac{(1)(3)+(-1)(-5)+(-2)(-4)}{\sqrt{1+1+4}\sqrt{9+25+16}}$$

$$= \frac{3+5+8}{\sqrt{6}\sqrt{50}}$$

$$= \frac{16}{\sqrt{300}} = \frac{16}{\sqrt{3\times100}} = \frac{16}{10\sqrt{3}}$$

or $\cos\theta = \dfrac{8}{5\sqrt{3}}$ $\quad\therefore\quad \theta = \cos^{-1}\dfrac{8}{5\sqrt{3}}$.

11. Find the angle between the following pairs of lines:

(*i*) $\dfrac{x-2}{2} = \dfrac{y-1}{5} = \dfrac{z+3}{-3}$ **and** $\dfrac{x+2}{-1} = \dfrac{y-4}{8} = \dfrac{z-5}{4}$

(*ii*) $\dfrac{x}{2} = \dfrac{y}{2} = \dfrac{z}{1}$ **and** $\dfrac{x-5}{4} = \dfrac{y-2}{1} = \dfrac{z-3}{8}$.

Sol. (*i*) **Given:** Equation of one line is $\dfrac{x-2}{2} = \dfrac{y-1}{5} = \dfrac{z+3}{-3}$

(It is standard form because coefficients of x, y, z are unity each)

$\therefore$ Denominators 2, 5, – 3 are direction ratios of this line *i.e.,* a vector along the line is

$$\vec{b_1} = (2, 5, -3) = 2\hat{i} + 5\hat{j} - 3\hat{k} \qquad ...(i)$$

Given: Equation of second line is $\dfrac{x+2}{-1} = \dfrac{y-4}{8} = \dfrac{z-5}{4}$

(It is also standard form)

$\therefore$ Denominators – 1, 8, 4 are direction ratios of this line *i.e.,* a vector along the line is

$$\vec{b_2} = (-1, 8, 4) = -\hat{i} + 8\hat{j} + 4\hat{k} \quad ...(ii)$$

Let θ be the angle between the two given lines.

We know that $\cos \theta = \dfrac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$

$$= \frac{2(-1) + 5(8) + (-3)(4)}{\sqrt{4 + 25 + 9}\,\sqrt{1 + 64 + 16}} \quad \text{(From } (i) \text{ and } (ii))$$

$$\Rightarrow \cos \theta = \frac{-2 + 40 - 12}{\sqrt{38}\,\sqrt{81}} = \frac{26}{9\sqrt{38}} \Rightarrow \theta = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right).$$

(*ii*) **Given:** Equation of one line is

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1} \quad \text{(Standard Form)}$$

∴ Denominators 2, 2, 1 are direction ratios of this line *i.e.,* a vector along this line is

$$\vec{b_1} = (2, 2, 1) = 2\hat{i} + 2\hat{j} + \hat{k} \quad ...(i)$$

Given: Equation of second line is

$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8} \quad \text{(Standard Form)}$$

∴ Denominators 4, 1, 8 are direction ratios of this line *i.e.,* a vector along this line is

$$\vec{b_2} = (4, 1, 8) = 4\hat{i} + \hat{j} + 8\hat{k} \quad ...(ii)$$

Let θ be the angle between the two lines.

We know that $\cos \theta = \dfrac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$

$$= \frac{2(4) + 2(1) + 1(8)}{\sqrt{4 + 4 + 1}\,\sqrt{16 + 1 + 64}} = \frac{8 + 2 + 8}{\sqrt{9}\,\sqrt{81}} = \frac{18}{3 \times 9} = \frac{2}{3}$$

$$\therefore \qquad \theta = \cos^{-1}\frac{2}{3}.$$

12. Find the values of p so that the lines $\dfrac{1-x}{3} = \dfrac{7y-14}{2p} = \dfrac{z-3}{2}$ and $\dfrac{7-7x}{3p} = \dfrac{y-5}{1} = \dfrac{6-z}{5}$ are at right angles.

Sol. Let us put the equations of these lines in standard form (*i.e.,* making coeff. of x, y, z unity in each of them)

The first line can be written as

$$-\frac{(x-1)}{3} = \frac{7(y-2)}{2p} = \frac{z-3}{2} \quad \text{or} \quad \frac{x-1}{-3} = \frac{y-2}{\left(\frac{2p}{7}\right)} = \frac{z-3}{2}$$

$\therefore$ direction ratios of this line are $-3, \frac{2p}{7}, 2 = a_1, b_1, c_1$.

And the equation of 2nd line can be written as

$$\frac{-7(x-1)}{3p} = \frac{y-5}{1} = \frac{-(z-6)}{5} \quad \text{or} \quad \frac{x-1}{-\frac{3p}{7}} = \frac{y-5}{1} = \frac{z-6}{-5}$$

$\therefore$ The direction ratios of 2nd line are $\frac{-3p}{7}, 1, -5 = a_2, b_2, c_2$.

$\because$ The two lines are perpendicular, therefore

$$\boldsymbol{a_1a_2 + b_1b_2 + c_1c_2 = 0}$$

$$\Rightarrow -3\left(\frac{-3p}{7}\right) + \left(\frac{2p}{7}\right)(1) + 2 \times (-5) = 0$$

$$\Rightarrow \frac{9p}{7} + \frac{2p}{7} - 10 = 0 \quad \Rightarrow \frac{11p}{7} = 10 \quad \Rightarrow p = \frac{70}{11}.$$

13. Show that the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are perpendicular to each other.

Sol. Given: Equation of one line is

$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1} \qquad \text{(Standard form)}$$

Direction ratios of this line are its denominators 7, – 5, 1

$$= a_1, b_1, c_1 \quad (\Rightarrow \vec{b_1} = 7\hat{i} - 5\hat{j} + \hat{k})$$

Given: Equation of second line is $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ (Standard form)

Direction ratios of this line are its denominators 1, 2, 3

$$= a_2, b_2, c_2 \quad (\Rightarrow \vec{b_2} = \hat{i} + 2\hat{j} + 3\hat{k})$$

$$\vec{b_1} \cdot \vec{b_2} = a_1a_2 + b_1b_2 + c_1c_2 = 7(1) + (-5)(2) + 1(3)$$
$$= 7 - 10 + 3 = 0$$

$\therefore$ The two given lines are perpendicular to each other.

14. Find the shortest distance between the lines

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \text{ and}$$
$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k}).$$

Sol. Comparing the equations of the given lines with

$$\vec{r} = \vec{a_1} + \lambda \vec{b_1} \text{ and } \vec{r} = \vec{a_2} + \mu \vec{b_2}, \text{ we have}$$

$$\vec{a_1} = \hat{i} + 2\hat{j} + \hat{k}, \quad \vec{b_1} = \hat{i} - \hat{j} + \hat{k}$$

and $\vec{a_2} = 2\hat{i} - \hat{j} - \hat{k}, \quad \vec{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$

We know **that the S.D. between the two skew lines is given by**

$$d = \frac{|(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|} \quad ...(i)$$

Now $\vec{a_2} - \vec{a_1} = (2\hat{i} - \hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} - 3\hat{j} - 2\hat{k}$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = (-2-1)\hat{i} - (2-2)\hat{j} + (1+2)\hat{k}$$

$$= -3\hat{i} - 0\hat{j} + 3\hat{k}$$

$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(-3)^2 + 0^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$

Again $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (\hat{i} - 3\hat{j} - 2\hat{k}) \cdot (-3\hat{i} + 3\hat{k})$

$$= (1)(-3) + (-3)(0) + (-2)(3) = -9$$

Putting these values in eqn. (*i*),

$$\text{S.D. } (d) = \frac{|-9|}{3\sqrt{2}} = \frac{9}{3\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

15. Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}.$$

Sol. Equation of one line is $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$

Comparing with $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$, we have

$$x_1 = -1, y_1 = -1, z_1 = -1; a_1 = 7, b_1 = -6, c_1 = 1$$

$\therefore$ vector form of this line is $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$

where $\vec{a_1} = (x_1, y_1, z_1) = (-1, -1, -1) = -\hat{i} - \hat{j} - \hat{k}$

and $\vec{b_1} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k} = 7\hat{i} - 6\hat{j} + \hat{k}$

Equation of second line is $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

Comparing with $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$; we have

$$x_2 = 3, y_2 = 5, z_2 = 7; a_2 = 1, b_2 = -2, c_2 = 1$$

$\therefore$ vector form of this second line is $\vec{r} = \vec{a_2} + \mu \vec{b_2}$

where $\vec{a_2} = (x_2, y_2, z_2) = (3, 5, 7) = 3\hat{i} + 5\hat{j} + 7\hat{k}$

and $\vec{b_2} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k} = \hat{i} - 2\hat{j} + \hat{k}$

we know that S.D. between two skew lines is given by

$$d = \frac{|(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|} \quad \dots (i)$$

Now $\vec{a_2} - \vec{a_1} = 3\hat{i} + 5\hat{j} + 7\hat{k} - (-\hat{i} - \hat{j} - \hat{k})$

$$= 4\hat{i} + 6\hat{j} + 8\hat{k}$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$= (-6 + 2)\hat{i} - (7 - 1)\hat{j} + (-14 + 6)\hat{k}$

$= -4\hat{i} - 6\hat{j} - 8\hat{k}$

$\therefore \left|\vec{b_1} \times \vec{b_2}\right| = \sqrt{(-4)^2 + (-6)^2 + (-8)^2}$

$$= \sqrt{16 + 36 + 64} = \sqrt{116}$$

again $(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2}) = 4(-4) + 6(-6) + 8(-8)$

$$= -16 - 36 - 64 = -116$$

Putting these values in eqn. (*i*),

S.D. $(d) = \frac{|-116|}{\sqrt{116}} = \frac{116}{\sqrt{116}} = \sqrt{116}$

$$= \sqrt{4 \times 29} = 2\sqrt{29}$$

16. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \text{ and}$$

$$\vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k}).$$

Sol. Equation of the first line is

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) = \vec{a_1} + \lambda \vec{b_1}$$

Comparing, $\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b_1} = \hat{i} - 3\hat{j} + 2\hat{k}$

Equation of second line is

$$\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k}) = \vec{a_2} + \mu \vec{b_2}$$

Comparing $\vec{a_2} = 4\hat{i} + 5\hat{j} + 6\hat{k}$ and $\vec{b_2} = 2\hat{i} + 3\hat{j} + \hat{k}$

We know that length of S.D. between two (skew) lines is

$$\frac{|(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|} \quad \ldots(i)$$

Now $\vec{a_2} - \vec{a_1} = 4\hat{i} + 5\hat{j} + 6\hat{k} - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= 4\hat{i} + 5\hat{j} + 6\hat{k} - \hat{i} - 2\hat{j} - 3\hat{k} = 3\hat{i} + 3\hat{j} + 3\hat{k}$

Again $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

Expanding along first row,

$\vec{b_1} \times \vec{b_2} = \hat{i}(-3-6) - \hat{j}(1-4) + \hat{k}(3+6) = -9\hat{i} + 3\hat{j} + 9\hat{k}$

$\therefore \quad (\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 3(-9) + 3(3) + 3(9)$

$= -27 + 9 + 27 = 9$

and $\quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(-9)^2 + (3)^2 + (9)^2} = \sqrt{81 + 9 + 81}$

$= \sqrt{171} = \sqrt{9 \times 19} = 3\sqrt{19}$

Putting these values in (*i*),

length of shortest distance $= \dfrac{|9|}{3\sqrt{19}} = \dfrac{9}{3\sqrt{19}} = \dfrac{3}{\sqrt{19}}$.

17. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (1 - t)\hat{i} + (t - 2)\hat{j} + (3 - 2t)\hat{k} \text{ and}$$

$$\vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}.$$

Sol. The first line is $\vec{r} = (1 - t)\hat{i} + (t - 2)\hat{j} + (3 - 2t)\hat{k}$

$= \hat{i} - t\hat{i} + t\hat{j} - 2\hat{j} + 3\hat{k} - 2t\hat{k}$

$= (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}) = \vec{a}_1 + t\vec{b}_1$

Comparing $\vec{a}_1 = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b}_1 = -\hat{i} + \hat{j} - 2\hat{k}$

The second line is $\vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}$

$= s\hat{i} + \hat{i} + 2s\hat{j} - \hat{j} - 2s\hat{k} - \hat{k}$

$= (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}) = \vec{a}_2 + s\vec{b}_2$

Comparing $\vec{a}_2 = \hat{i} - \hat{j} - \hat{k}$, $\vec{b}_2 = \hat{i} + 2\hat{j} - 2\hat{k}$

We know that the S.D. between the two (skew) lines is given by

$$d = \frac{|(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|} \quad ...(i)$$

Now $\vec{a}_2 - \vec{a}_1 = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \hat{j} - 4\hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$

$= (-2 + 4)\hat{i} - (2 + 2)\hat{j} + (-2 - 1)\hat{k} = 2\hat{i} - 4\hat{j} - 3\hat{k}$

$\therefore |\vec{b}_1 \times \vec{b}_2| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{29}$

Again $(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2) = (\hat{j} - 4\hat{k}).(2\hat{i} - 4\hat{j} - 3\hat{k})$

$= (0)(2) + (1)(-4) + (-4)(-3) = 8$

Putting these values in eqn. (i),

$$\text{S.D. } (d) = \frac{|8|}{\sqrt{29}} = \frac{8}{\sqrt{29}}.$$