Exercise 7.8

Definition of definite integral as the limit of a sum:

$$\int_a^b f(x)\ dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[f(a) + f(a + h) + f(a + 2h) + \ldots\ldots + f(a + (n - 1)\ h)]$$

where $nh = b - a$

Note. The series within brackets represents the sum of *n* terms.

Evaluate the following definite integrals as limit of sums:

1. $\int_a^b x\ dx$

Sol. Step I. Comparing $\int_a^b x\ dx$ with $\int_a^b f(x)\ dx$ we have

$$a = a,\ b = b \text{ and } f(x) = x \qquad \ldots(i)$$

$\therefore \quad nh = b - a = b - a$

Step II. Putting $x = a,\ a + h,\ a + 2h,\ \ldots\ldots,\ a + (n - 1)\ h$ in (i), we have $f(a) = a$, $f(a + h) = a + h$,

$f(a + 2h) = a + 2h,\ \ldots\ldots,\ f(a + (n - 1)h) = a + (n - 1)h$

Step III. Putting these values in

$$\int_a^b f(x)\, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h\ [f(a) + f(a + h) + f(a + 2h) + \ldots\ldots + f(a + (n - 1)h)]$$

where $nh = b - a$**, we have**

$$\int_a^b x\, dx = \underset{h \to 0}{\text{Lt}}\ h\ [a + (a + h) + (a + 2h) + \ldots\ldots + (a + (n - 1)h)]$$

where $nh = b - a$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[na + h(1 + 2 + 3 + \ldots\ldots + (n - 1)]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[anh + hh \frac{n(n-1)}{2} \right] \left[\because \quad 1 + 2 + 3 + \ldots\ldots + (n-1) = \frac{n(n-1)}{2} \right]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[anh + \frac{nh(nh-h)}{2} \right].$$

Step IV. Putting $nh = b - a$,

$$= \underset{h \to 0}{\text{Lt}} \left[a(b-a) + \frac{(b-a)(b-a-h)}{2} \right].$$

Step V. Taking Limits as $h \to 0$ (*i.e.*, putting $h = 0$ here)

$$= a(b-a) + \frac{(b-a)(b-a)}{2}$$

$$= (b-a)\left[a + \frac{b-a}{2} \right] = (b-a)\left[\frac{2a+b-a}{2} \right]$$

$$= \frac{(b-a)(b+a)}{2} = \frac{b^2 - a^2}{2}.$$

2. $\int_0^5 (x+1)\, dx$

Sol. Step I. Comparing $\int_0^5 (x+1)\, dx$ with $\int_a^b f(x)\, dx$, we have

$a = 0$, $b = 5$ and $f(x) = x + 1$ ...(*i*)

$\therefore \quad nh = b - a = 5 - 0 = 5.$

Step II. Putting $x = a, a + h, a + 2h, \ldots\ldots, a + (n-1)h$ in (*i*), we have

$f(a) = f(0) = 0 + 1 = 1$, $f(a+h) = f(h) = h + 1$,

$f(a+2h) = f(2h) = 2h + 1, \ldots\ldots,$

$f(a + (n-1)h) = f((n-1)h) = (n-1)h + 1.$

Step III. Putting these values in

$$\int_a^b f(x)\, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a+(n-1)h)],$$ **we have**

$$\int_0^5 (x+1)\, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[1 + (h+1) + (2h+1) + \ldots\ldots + [(n-1)h+1)]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[n + h(1 + 2 + \ldots\ldots + (n-1)] = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[nh + hh \frac{n(n-1)}{2} \right]$$

$$\left[\because \quad 1 + 2 + 3 + \ldots\ldots + (n-1) = \frac{n(n-1)}{2} \right]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[nh + \frac{(nh)(nh-h)}{2} \right].$$

Step IV. Putting $nh = 5$, $= \underset{h \to 0}{\text{Lt}} \left[5 + \frac{5(5-h)}{2}\right]$.

Step V. Taking limits as $h \to 0$ (*i.e.*, putting $h = 0$ here)

$$= 5 + \frac{5(5-0)}{2} = 5 + \frac{25}{2} = \frac{10+25}{2} = \frac{35}{2}.$$

3. $\int_2^3 x^2\, dx$

Sol. Step I. Comparing $\int_2^3 x^2 \; dx$ with $\int_a^b f(x)$, we have

$a = 2$, $b = 3$ and $f(x) = x^2$ ...(*i*)

$\therefore \quad nh = b - a = 3 - 2 = 1.$

Step II. Putting $x = a, a + h, a + 2h, ..., a + (n - 1)h$ in (*i*), we have

$f(a) = f(2) = 2^2 = 4$

$f(a + h) = f(2 + h) = (2 + h)^2 = 4 + 4h + h^2$

$f(a + 2h) = f(2 + 2h) = (2 + 2h)^2 = 4 + 8h + 2^2h^2$

|

$f(a + (n - 1)h) = f(2 + (n - 1)h) = (2 + (n - 1)h)^2$

$= 4 + 4(n - 1)h + (n - 1)^2h^2.$

Step III. Putting these values in

$$\int_a^b f(x)\, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[f(a) + f(a + h) + f(a + 2h) + ... + f(a + (n - 1)h)]$$

where $nh = 1$, we have

$$\int_2^3 x^2 \; dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[4 + (4 + 4h + h^2) + (4 + 8h + 2^2h^2) + ... + (4 + 4(n - 1)h + (n - 1)^2h^2)]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[4n + 4h(1 + 2 + ... + (n - 1)) + h^2 (1^2 + 2^2) + ... + (n - 1)^2)]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[4nh + 4hh\frac{n(n-1)}{2} + hhh\frac{n(n-1)(2n-1)}{6}\right]$$

$$\left[\because \quad 1 + 2 + ... + (n-1) = \frac{n(n-1)}{2} \text{ and } 1^2 + 2^2 + ... + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\right.$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[4nh + 4nh\frac{(nh-h)}{2} + \frac{nh(nh-h)(2nh-h)}{6}\right].$$

Step IV. Putting $nh = 1$;

$$= \underset{h \to 0}{\text{Lt}} \left[4 + 2(1-h) + 1\frac{(1-h)(2-h)}{6}\right].$$

Step V. Taking limits as $h \to 0$ (*i.e.*, putting $h = 0$ here)

$= 4 + 2(1 - 0) + \dfrac{1(2)}{6} = 6 + \dfrac{1}{3} = \dfrac{19}{3}$.

4. $\displaystyle\int_1^4 (x^2 - x)\, dx$

Sol. Step I. Comparing $\displaystyle\int_1^4 (x^2 - x)\, dx$ with $\displaystyle\int_a^b f(x)\, dx$,

we have

$$a = 1,\ b = 4,\ f(x) = x^2 - x \qquad ...(i)$$

$$\therefore \quad nh = b - a = 4 - 1 = 3.$$

Step II. Putting $x = a, a + h, a + 2h, ... a + (n - 1)h$ in (i),

$$f(a) = f(1) = 1^2 - 1 = 1 - 1 = 0$$

$$f(a + h) = f(1 + h) = (1 + h)^2 - (1 + h)$$
$$= 1 + h^2 + 2h - 1 - h = h + h^2$$

$$f(a + 2h) = f(1 + 2h) = (1 + 2h)^2 - (1 + 2h)$$
$$= 1 + 4h^2 + 4h - 1 - 2h$$
$$= 2h + 4h^2$$

$$\vdots$$

$$f(a + (n - 1)h) = (1 + (n - 1)h)^2 - (1 + (n - 1)h)$$
$$= 1 + (n - 1)^2 h^2 + 2(n - 1)h - 1 - (n - 1)h$$
$$= (n - 1)h + (n - 1)^2 h^2.$$

Step III. Putting these values in

$$\int_a^b f(x)\, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[f(a) + f(a + h) + f(a + 2h) + ... + f(a + (n - 1)h)]$$

we have

$$\int_1^4 (x^2 - x)\, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[0 + h + h^2 + 2h + 4h^2 + ... + (n - 1)h + (n - 1)^2h^2)$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[h(1 + 2 + ... + (n - 1)) + h^2(1^2 + 2^2 + ... + (n - 1)^2)]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[h \,.\, h \,.\, \frac{n(n - 1)}{2} + h \,.\, h \,.\, h\, \frac{n(n - 1)(2n - 1)}{6}\right]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[nh\, \frac{(nh - h)}{2} + \frac{(nh)(nh - h)(2nh - h)}{6}\right].$$

Step IV. Putting $nh = 3$

$$= \underset{h \to 0}{\text{Lt}} \left[\frac{3(3 - h)}{2} + \frac{3(3 - h)(6 - h)}{6}\right].$$

Step V. Taking limits as $h \to 0$ (Putting $h = 0$ here)

$$= \frac{3(3 - 0)}{2} + \frac{3(3 - 0)(6 - 0)}{6} = \frac{9}{2} + 9 = \frac{27}{2}.$$

5. $\int_{-1}^{1} e^x \, dx$

Sol. Step I. Comparing $\int_{-1}^{1} e^x \; dx$ with $\int_a^b f(x) \; dx$, we have

$a = -1, \; b = 1$ and $f(x) = e^x$ ...(*i*)

$\therefore \quad nh = b - a = 1 - (-1) = 2.$

Step II. Putting $x = a, a + h, a + 2h, \ldots\ldots, a + (n - 1)h$ in (*i*), we have

$f(a) = f(-1) = e^{-1}$

$f(a + h) = f(-1 + h) = e^{-1 + h} = e^{-1} \cdot e^h$

$f(a + 2h) = f(-1 + 2h) = e^{-1 + 2h} = e^{-1} \cdot e^{2h}$

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$f(a + (n - 1)h) = f(-1 + (n - 1)h) = e^{-1 + (n-1)h} = e^{-1} \, e^{(n-1)h}.$

Step III. Putting these values in

$$\int_a^b f(x) \, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \; h[f(a) + f(a + h) + f(a + 2h) + \ldots\ldots + f(a + (n - 1)h)],$$

we have

$$\int_{-1}^{1} e^x \; dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \; h[e^{-1} + e^{-1} e^h + e^{-1} e^{2h} + \ldots\ldots + e^{-1} e^{(n-1)h}]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \; h \, e^{-1} \frac{[(e^h)^n - 1]}{e^h - 1}$$ [$\because$ The series within brackets is a G.P. series with First term $A = e^{-1}$ and common ratio $R = e^h$, Number of terms is n and S_n of G.P. $= A \dfrac{(R^n - 1)}{R - 1}$].

$$= \int_{-1}^{1} e^x \; dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \; h \, e^{-1} \frac{(e^{nh} - 1)}{e^h - 1}.$$

Step IV. Putting $nh = 2$, $= \underset{h \to 0}{\text{Lt}} \; h \, e^{-1} \dfrac{(e^2 - 1)}{e^h - 1}$

$$= e^{-1} (e^2 - 1) \underset{h \to 0}{\text{Lt}} \frac{h}{e^h - 1} = e^{-1} (e^2 - 1) \times 1 \left[\because \underset{x \to 0}{\text{Lt}} \frac{x}{e^x - 1} = 1\right]$$

$$= e^{-1 + 2} - e^{-1} = e^1 - e^{-1} = e - e^{-1}.$$

6. $\int_0^4 (x + e^{2x}) \, dx$

Sol. Step I. Comparing $\int_0^4 (x + e^{2x}) \; dx$ with $\int_a^b f(x) \; dx$, we have

$a = 0, \; b = 4$ and $f(x) = x + e^{2x}$ ...(*i*)

$\therefore \quad nh = b - a = 4 - 0 = 4.$

Step II. Putting $x = a, a + h, a + 2h, \ldots\ldots, a + (n - 1)h$ in (*i*), we have

$f(a) = f(0) = 0 + e^0 = 1$

$$f(a+h) = f(h) = h + e^{2h}$$
$$f(a+2h) = f(2h) = 2h + e^{4h}$$
$$\cdot$$
$$\cdot$$
$$f(a+(n-1)h) = f((n-1)h) = (n-1)h + e^{2(n-1)h}.$$

Step III. Putting these values in

$$\int_a^b f(x)\,dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[f(a) + f(a+h) + f(a+2h) + \ldots.. + f(a+(n-1)h)],$$

we have

$$\int_0^4 (x + e^{2x})\,dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[1 + (h + e^{2h}) + (2h + e^{4h}) + \ldots\ldots ((n-1)h + e^{2(n-1)h})]$$

(G.P. series : A = 1, R = e^{2h}, $n = n$)

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[(h + 2h + .. + (n-1)h) + (1 + e^{2h} + e^{4h} + \ldots + e^{2(n-1)h})]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h\left[h(1 + 2 + \ldots\ldots + (n-1)) + A\left(\frac{R^n - 1}{R - 1}\right)\right]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h\left[h\,\frac{n(n-1)}{2} + \frac{1((e^{2h})^n - 1)}{e^{2h} - 1}\right]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[\frac{nh(nh - h)}{2} + \frac{h(e^{2nh} - 1)}{e^{2h} - 1}\right].$$

Step IV. Putting $nh = 4$, $= \underset{h \to 0}{\text{Lt}} \left[\frac{4(4-h)}{2} + \frac{h(e^8 - 1)}{e^{2h} - 1}\right].$

Step V. Taking limits as $h \to 0$

$$= \frac{4(4-0)}{2} + (e^8 - 1) \underset{h \to 0}{\text{Lt}} \frac{h}{e^{2h} - 1} = 8 + (e^8 - 1)\,\frac{1}{2} \underset{h \to 0}{\text{Lt}} \frac{2h}{e^{2h} - 1}$$

$$= 8 + \frac{(e^8 - 1)}{2}. \qquad \left[\because \underset{h \to 0}{\text{Lt}} \frac{e^{2h}}{e^{2h} - 1}\left(\Rightarrow \underset{x \to 0}{\text{Lt}} \frac{x}{e^x - 1}\right) = 1\right]$$