Exercise 7.7

I. Rule to evaluate $\int \sqrt{\text{Pure Quadratic}}\ dx$, *i.e.*,

$$\int \sqrt{ax^2 + b}\ dx.$$

Apply directly one of these formulae according to form of integrand:

1. $\int \sqrt{a^2 - x^2}\ dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

2. $\int \sqrt{x^2 + a^2}\ dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right|.$

3. $\int \sqrt{x^2 - a^2}\ dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right|.$

II. Rule to evaluate $\int \sqrt{\text{Quadratic}}\ dx$, *i.e.*, $\int \sqrt{ax^2 + bx + c}\ dx$

Step I. Make coefficient of x^2 unity by taking $|a|$ common. Now complete the squares by adding and subtracting $\left(\frac{1}{2}\text{Coefficient of } x\right)^2$.

Now applying one of the above three formulae (according to the form of the integrand) will give value of required integral.

Integrate the functions in Exercises 1 to 9:

1. $\sqrt{4 - x^2}$

Sol. $\int \sqrt{4 - x^2}\ dx = \int \sqrt{2^2 - x^2}\ dx$

$$= \frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2} + c$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2} + c$$

2. $\sqrt{1 - 4x^2}$

Sol. $\int \sqrt{1 - 4x^2}\, dx = \int \sqrt{1^2 - (2x)^2}\, dx$

$$= \frac{\frac{(2x)}{2}\sqrt{1^2 - (2x)^2} + \frac{1^2}{2}\sin^{-1}\left(\frac{2x}{1}\right)}{2 \to \text{Coefficient of } x \text{ in } 2x} + c$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{2}\left[x\sqrt{1 - 4x^2} + \frac{1}{2}\sin^{-1}\frac{2x}{1}\right] + c = \frac{x}{2}\sqrt{1 - 4x^2} + \frac{1}{4}\sin^{-1} 2x + c.$$

3. $\sqrt{x^2 + 4x + 6}$

Sol. $\int \sqrt{x^2 + 4x + 6}\, dx$

Coefficient of x^2 is unity. So let us complete squares by adding and subtracting $\left(\frac{1}{2}\text{Coefficient of } x\right)^2 = 2^2$

$$= \int \sqrt{x^2 + 4x + 4 + 6 - 4}\, dx = \int \sqrt{(x + 2)^2 + 2}\, dx$$

$$= \int \sqrt{(x + 2)^2 + (\sqrt{2})^2}\, dx = \left(\frac{x + 2}{2}\right)\sqrt{(x + 2)^2 + (\sqrt{2})^2}$$

$$+ \frac{(\sqrt{2})^2}{2}\log\left| x + 2 + \sqrt{(x + 2)^2 + (\sqrt{2})^2} \right| + c$$

$$\left[\because \int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log | x + \sqrt{x^2 + a^2}|\right]$$

$$= \frac{(x + 2)}{2}\sqrt{x^2 + 4 + 4x + 2}$$

$$+ \frac{2}{2}\log | x + 2 + \sqrt{x^2 + 4 + 4x + 2} | + c$$

$$= \frac{(x + 2)}{2}\sqrt{x^2 + 4x + 6} + \log | x + 2 + \sqrt{x^2 + 4x + 6} | + c.$$

4. $\sqrt{x^2 + 4x + 1}$

Sol. $\int \sqrt{x^2 + 4x + 1}\, dx = \int \sqrt{x^2 + 4x + 2^2 + 1 - 4}\, dx$

$$\left(\text{We have added and subtracted } \left(\frac{1}{2}\text{ coefficient of } x\right)^2 = 2^2\right)$$

$$= \int \sqrt{(x+2)^2 - 3}\; dx = \int \sqrt{(x+2)^2 - (\sqrt{3})^2}\; dx$$

$$= \left(\frac{x+2}{2}\right) \sqrt{(x+2)^2 - (\sqrt{3})^2} - \frac{(\sqrt{3})^2}{2} \log \left| x + 2 + \sqrt{(x+2)^2 - (\sqrt{3})^2} \right| + c$$

$$\left[\because \int \sqrt{x^2 - a^2}\; dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log | x + \sqrt{x^2 - a^2} |\right]$$

$$= \left(\frac{x+2}{2}\right) \sqrt{x^2 + 4x + 1} - \frac{3}{2} \log \left| x + 2 + \sqrt{x^2 + 4x + 1} \right| + c$$

$[\because\ (x + 2)^2 - (\sqrt{3})^2 = x^2 + 4x + 4 - 3 = x^2 + 4x + 1]$

5. $\int \sqrt{\mathbf{1 - 4x - x^2}}$

Sol. $\int \sqrt{1 - 4x - x^2}\; dx = \int \sqrt{-x^2 - 4x + 1}\; dx$

Making coefficient of x^2 unity

$$= \int \sqrt{-(x^2 + 4x - 1)}\; dx$$

(**Note.** You can't take this (–) sign out of this bracket because square root of – 1 is imaginary)

$$= \int \sqrt{-(x^2 + 4x + 2^2 - 4 - 1)}\; dx = \int \sqrt{-[(x+2)^2 - 5]}\; dx$$

$$= \int \sqrt{5 - (x+2)^2}\; dx = \int \sqrt{(\sqrt{5})^2 - (x+2)^2}\; dx$$

$$= \frac{x+2}{2} \sqrt{(\sqrt{5})^2 - (x+2)^2} + \frac{(\sqrt{5})^2}{2} \sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + c$$

$$\left[\because \int \sqrt{a^2 - x^2}\; dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a}\right]$$

$$= \frac{x+2}{2} \sqrt{1 - 4x - x^2} + \frac{5}{2} \sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + c$$

$[\because\ (\sqrt{5})^2 - (x + 2)^2 = 5 - (x^2 + 4 + 4x)$
$= 5 - x^2 - 4 - 4x = 1 - 4x - x^2]$

6. $\sqrt{\mathbf{x^2 + 4x - 5}}$

Sol. $\int \sqrt{x^2 + 4x - 5}\; dx = \int \sqrt{x^2 + 4x + 2^2 - 4 - 5}\; dx$

$$= \int \sqrt{(x+2)^2 - 9}\; dx = \int \sqrt{(x+2)^2 - 3^2}\; dx$$

$$= \left(\frac{x+2}{2}\right) \sqrt{(x+2)^2 - 3^2} - \frac{3^2}{2} \log \left| x + 2 + \sqrt{(x+2)^2 - 3^2} \right| + c$$

$$\left[\because \int \sqrt{x^2 - a^2}\; dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log | x + \sqrt{x^2 - a^2} |\right]$$

$$= \left(\frac{x+2}{2}\right)\sqrt{x^2+4x-5} - \frac{9}{2}\log\left|x+2+\sqrt{x^2+4x-5}\right| + c$$

$$[\because\ (x+2)^2 - 3^2 = x^2+4x+4-9 = x^2+4x-5]$$

7. $\sqrt{1+3x-x^2}$

Sol. $\int \sqrt{1+3x-x^2}\,dx = \int \sqrt{-x^2+3x+1}\,dx$

$$= \int \sqrt{-(x^2-3x-1)}\,dx$$

$$= \int \sqrt{-\left[x^2-3x+\left(\frac{3}{2}\right)^2 - \frac{9}{4} - 1\right]}\,dx = \int \sqrt{-\left[\left(x-\frac{3}{2}\right)^2 - \frac{13}{4}\right]}\,dx$$

$$= \int \sqrt{\frac{13}{4} - \left(x-\frac{3}{2}\right)^2}\,dx = \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2}\,dx$$

$$= \left(\frac{x-\frac{3}{2}}{2}\right)\sqrt{\left(\frac{\sqrt{13}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2} + \frac{\left(\frac{\sqrt{13}}{2}\right)^2}{2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right) + c$$

$$\left[\because \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \left(\frac{2x-3}{4}\right)\sqrt{1+3x-x^2} + \frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right) + c$$

$$\left[\because \left(\frac{\sqrt{13}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2 = \frac{13}{4} - \left(x^2+\frac{9}{4}-3x\right)\right.$$

$$\left. = \frac{13}{4} - x^2 - \frac{9}{4} + 3x = 1+3x-x^2\right]$$

8. $\sqrt{x^2+3x}$

Sol. $\int\sqrt{x^2+3x}\,dx = \int\sqrt{x^2+3x+\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\,dx = \int\sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\,dx$

$$= \frac{x+\frac{3}{2}}{2}\sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2} - \frac{\left(\frac{3}{2}\right)^2}{2}\log\left|x+\frac{3}{2}+\sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\right| + c$$

$$\left[\because \int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|\right]$$

$$= \frac{2x+3}{4}\sqrt{x^2+3x} - \frac{9}{8}\log\left|x+\frac{3}{2}+\sqrt{x^2+3x}\right| + c$$

$$\left[\because \left(x+\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 = x^2+3x+\frac{9}{4}-\frac{9}{4} = x^2+3x\right]$$

9. $\sqrt{1+\frac{x^2}{9}}$

Sol. $\int \sqrt{1+\frac{x^2}{9}}\,dx = \int \sqrt{\frac{9+x^2}{9}}\,dx = \int \frac{\sqrt{x^2+3^2}}{3}\,dx = \frac{1}{3}\int \sqrt{x^2+3^2}\,dx$

$$= \frac{1}{3}\left[\frac{x}{2}\sqrt{x^2+3^2} + \frac{3^2}{2}\log\left|x+\sqrt{x^2+3^2}\right|\right] + c$$

$$\left[\because \int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|\right]$$

$$= \frac{x}{6}\sqrt{x^2+9} + \frac{3}{2}\log|x+\sqrt{x^2+9}| + c.$$

Choose the correct answer in Exercises 10 to 11:

10. $\int \sqrt{1+x^2}\,dx$ **is equal to**

(A) $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|\left(x+\sqrt{1+x^2}\right)\right| + C$

(B) $\frac{2}{3}(1+x^2)^{3/2} + C$ (C) $\frac{2}{3}x(1+x^2)^{3/2} + C$

(D) $\frac{x^2}{2}\sqrt{1+x^2} + \frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right| + C.$

Sol. $\int \sqrt{1+x^2}\,dx = \int \sqrt{x^2+1^2}\,dx$

$$= \frac{x}{2}\sqrt{x^2+1^2} + \frac{1^2}{2}\log|x+\sqrt{x^2+1^2}| + C$$

$$\left[\because \int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|\right]$$

$$= \frac{x}{2}\sqrt{x^2+1} + \frac{1}{2}\log|x+\sqrt{x^2+1}| + C.$$

11. $\int \sqrt{x^2-8x+7}\,dx$ **is equal to**

(A) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} + 9\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

(B) $\frac{1}{2}(x+4)\sqrt{x^2-8x+7} + 9\log\left|x+4+\sqrt{x^2-8x+7}\right| + C$

(C) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - 3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

(D) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - \frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + C.$

Sol. $\int \sqrt{x^2 - 8x + 7}\ dx = \int \sqrt{x^2 - 8x + 4^2 - 16 + 7}\ dx$

$= \int \sqrt{(x-4)^2 - 9}\ dx = \int \sqrt{(x-4)^2 - 3^2}\ dx$

$= \left(\frac{x-4}{2}\right) \sqrt{(x-4)^2 - 3^2} - \frac{3^2}{2} \log |x - 4 + \sqrt{(x-4)^2 - 3^2}| + C$

$\left[\because \int \sqrt{x^2 - a^2}\ dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}|\right]$

$= \left(\frac{x-4}{2}\right) \sqrt{x^2 - 8x + 7} - \frac{9}{2} \log |x - 4 + \sqrt{x^2 - 8x + 7}| + C.$

$[\because (x-4)^2 - 3^2 = x^2 - 8x + 16 - 9 = x^2 - 8x + 7]$