Exercise 7.6

Integrate the functions in Exercises 1 to 8:

1. $\boldsymbol{x \sin x}$

Sol. $\displaystyle\int \underset{\text{I}}{x} \underset{\text{II}}{\sin x}\, dx$

Applying Product Rule $\displaystyle \text{I} \int \text{II}\, dx - \int \left(\frac{d}{dx}(\text{I}) \int \text{II}\, dx \right) dx$

$$= x \int \sin x \, dx - \int \left(\frac{d}{dx}(x) \int \sin x\, dx \right) dx$$

$$= x\,(-\cos x) - \int 1\,(-\cos x)\, dx = -x \cos x - \int -\cos x \, dx$$

$$= -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + c$$

Note. $\displaystyle\int \sin x \, dx = -\cos x.$

2. $\boldsymbol{x \sin 3x}$

Sol. $\displaystyle\int \underset{\text{I}}{x} \underset{\text{II}}{\sin 3x}\, dx$

Applying Product Rule $\displaystyle \text{I} \int \text{II}\, dx - \int \left(\frac{d}{dx}(\text{I}) \int \text{II}\, dx \right) dx$

$$= x \int \sin 3x \, dx - \int \left(\frac{d}{dx}(x) \int \sin 3x\, dx \right) dx$$

$$= x \left(\frac{-\cos 3x}{3} \right) - \int \left[1 \left(\frac{-\cos 3x}{3} \right) \right] dx + c$$

$$= \frac{-1}{3}\, x \cos 3x + \frac{1}{3} \int \cos 3x \, dx + c$$

$$= \frac{-1}{3} x \cos 3x + \frac{1}{3} \frac{\sin 3x}{3} + c = \frac{-1}{3} x \cos 3x + \frac{1}{9} \sin 3x + c.$$

3. $\boldsymbol{x^2 e^x}$

Sol. $\int \underset{\text{I}}{x^2} \underset{\text{II}}{e^x} \, dx$

Applying Product Rule $\text{I} \int \text{II} \, dx - \int \left(\frac{d}{dx} (\text{I}) \int \text{II} \, dx \right) dx$

$$= x^2 \int e^x \, dx - \int \left[\left(\frac{d}{dx} x^2 \right) \int e^x \, dx \right] dx = x^2 e^x - \int 2x \, e^x \, dx$$

$$= x^2 e^x - 2 \int \underset{\text{I}}{x} \, \underset{\text{II}}{e^x} \, dx$$

Again Applying Product Rule

$$= x^2 e^x - 2 \left[x \int e^x \, dx - \int \left[\frac{d}{dx} (x) \int e^x \, dx \right] dx \right]$$

$$= x^2 e^x - 2 \left(x e^x - \int 1.e^x \, dx \right) = x^2 e^x - 2 \left(x e^x - \int e^x \, dx \right)$$

$$= x^2 e^x - 2x e^x + 2 \int e^x \, dx + c \quad = x^2 e^x - 2x e^x + 2e^x + c$$

$$= e^x (x^2 - 2x + 2) + c.$$

4. $\boldsymbol{x \log x}$

Sol. $\int x \log x \, dx = \int \underset{\text{I}}{(\log x)} \, . \, \underset{\text{II}}{x} \, dx$

Applying Product Rule $\text{I} \int \text{II} \, dx - \int \left[\frac{d}{dx} (\text{I}) \int \text{II} \, dx \right] dx$

$$= (\log x) \int x \, dx - \int \left[\frac{d}{dx} (\log x) \int x \, dx \right] dx$$

$$= (\log x) \frac{x^2}{2} - \int \frac{1}{x} \frac{x^2}{2} dx = \frac{1}{2} x^2 \log x - \frac{1}{2} \int x \, dx$$

$$\left(\because \frac{x^2}{x} = \frac{x \, . \, x}{x} = x \right)$$

$$= \frac{1}{2} x^2 \log x - \frac{1}{2} \frac{x^2}{2} + c = \frac{x^2}{2} \log x - \frac{x^2}{4} + c.$$

5. $\boldsymbol{x \log 2x}$

Sol. $\int x \log 2x \, dx = \int \underset{\text{I}}{(\log 2x)} \, . \, \underset{\text{II}}{x} \, dx$

Applying Product Rule $\text{I} \int \text{II} \, dx - \int \left(\frac{d}{dx} (\text{I}) \int \text{II} \, dx \right) dx$

$$= (\log 2x) \int x \, dx - \int \left(\frac{d}{dx} (\log 2x) \int x \, dx \right) dx$$

$$= (\log 2x) \frac{x^2}{2} - \int \frac{1}{2x} \, . \, 2 \, . \, \frac{x^2}{2} \, dx$$

$$= \frac{1}{2} x^2 \log 2x - \frac{1}{2} \int x \, dx \qquad \left[\because \frac{x^2}{x} = \frac{x \cdot x}{x} = x\right]$$

$$= \frac{1}{2} x^2 \log 2x - \frac{1}{2} \frac{x^2}{2} + c = \frac{x^2}{2} \log 2x - \frac{x^2}{4} + c.$$

6. $x^2 \log x$

Sol. $\int x^2 \log x \, dx = \int \underset{\text{I}}{(\log x)} \underset{\text{II}}{x^2} \, dx$

Applying Product Rule: $\text{I} \int \text{II} \, dx - \int \left(\frac{d}{dx}(\text{I}) \int \text{II} \, dx\right) dx$

$$= \log x \int x^2 \, dx - \int \left(\frac{d}{dx}(\log x) \int x^2 \, dx\right) dx$$

$$= (\log x) \frac{x^3}{3} - \int \frac{1}{x} \frac{x^3}{3} \, dx = \frac{x^3}{3} \log x - \frac{1}{3} \int x^2 \, dx \quad \left[\because \frac{x^3}{x} = x^2\right]$$

$$= \frac{x^3}{3} \log x - \frac{1}{3} \frac{x^3}{3} + c = \frac{x^3}{3} \log x - \frac{x^3}{9} + c.$$

7. $x \sin^{-1} x$

Sol. Let $\text{I} = \int x \sin^{-1} x \, dx.$

Put $x = \sin \theta$. Differentiating both sides $dx = \cos \theta \, d\theta$

$$\therefore \quad \text{I} = \int \sin \theta \, . \, \theta \, . \cos \theta \, d\theta = \frac{1}{2} \int \theta \, . \, 2 \sin \theta \cos \theta \, d\theta$$

$$= \frac{1}{2} \int \underset{\text{I}}{\theta} \, \underset{\text{II}}{\sin 2\theta} \, d\theta$$

Integrating by parts

$$= \frac{1}{2} \left[\theta\left(-\frac{\cos 2\theta}{2}\right) - \int 1 . \left(-\frac{\cos 2\theta}{2}\right) d\theta\right]$$

$$= \frac{1}{4} \left[-\theta \cos 2\theta + \int \cos 2\theta \, d\theta\right] = \frac{1}{4} \left[-\theta \cos 2\theta + \frac{\sin 2\theta}{2}\right] + c$$

$$= \frac{1}{4} [-\theta (1 - 2 \sin^2 \theta) + \sin \theta \cos \theta] + c$$

$(\because \sin 2\theta = 2 \sin \theta \cos \theta)$

$$= \frac{1}{4} [-\sin^{-1} x \, . \, (1 - 2x^2) + x \sqrt{1 - x^2}\,] + c$$

$$\left[\because \cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - x^2}\right]$$

$$= \frac{1}{4} (2x^2 - 1) \sin^{-1} x + \frac{x\sqrt{1 - x^2}}{4} + c.$$

8. $x \tan^{-1} x$

Sol. Let $\text{I} = \int x \tan^{-1} x \, dx = \int (\tan^{-1} x) \, . \, x \, dx$

$$= (\tan^{-1} x) \, . \, \frac{x^2}{2} - \int \frac{1}{1 + x^2} \, . \, \frac{x^2}{2} \, dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2}\, dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$\left[\because \frac{x^2}{1+x^2} = \frac{1+x^2-1}{1+x^2} = 1 - \frac{1}{1+x^2}\right]$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} (x - \tan^{-1} x) + c$$

$$= \frac{1}{2} [x^2 \tan^{-1} x - x + \tan^{-1} x] + c \;=\; \frac{1}{2} [(x^2 + 1) \tan^{-1} x - x] + c.$$

Integrate the functions in Exercises 9 to 15:

9. $\boldsymbol{x \cos^{-1} x}$

Sol. Let $\quad I = \int x \cos^{-1} x \; dx \qquad \ldots(i)$

Put $\cos^{-1} x = \theta$. Therefore $x = \cos \theta$.

$$\therefore \quad \frac{dx}{d\theta} = -\sin\theta \;\Rightarrow\; dx = -\sin\theta \, d\theta$$

$$\therefore \text{ From}(i),\ I = \int (\cos\theta)\,\theta\,(-\sin\theta\, d\theta) = \frac{-1}{2}\int \theta\,(2\sin\theta\cos\theta)\, d\theta$$

$$= \frac{-1}{2} \int \underset{\text{I}}{\theta}\ \underset{\text{II}}{\sin 2\theta}\, d\theta$$

Applying Product Rule: $I \int II \; d\theta - \int \left[\frac{d}{d\theta}(I) \int II \, d\theta\right] d\theta$

$$= \frac{-1}{2}\left[\theta\left(\frac{-\cos 2\theta}{2}\right) - \int 1\left(\frac{-\cos 2\theta}{2}\right) d\theta\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{2}\theta\cos 2\theta + \frac{1}{2}\int \cos 2\theta\, d\theta\right] = \frac{1}{4}\theta\cos 2\theta - \frac{1}{4}\left(\frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{1}{4}\theta\cos 2\theta - \frac{1}{8}(2\sin\theta\cos\theta) + c$$

$$= \frac{1}{4}\theta\,(2\cos^2\theta - 1) - \frac{1}{4}\sqrt{1-\cos^2\theta}\,.\cos\theta + c$$

Putting $\cos\theta = x$ and $\theta = \cos^{-1} x$;

$$= \frac{1}{4}(\cos^{-1} x)(2x^2 - 1) - \frac{1}{4}\sqrt{1-x^2}\,.\,x + c$$

$$= (2x^2 - 1)\frac{\cos^{-1} x}{4} - \frac{x}{4}\sqrt{1-x^2} + c.$$

10. $\boldsymbol{(\sin^{-1} x)^2}$

Sol. **Put $x = \sin\theta$.** Differentiating both sides, $dx = \cos\theta\, d\theta$

$$\therefore \int (\sin^{-1} x)^2\, dx = \int \underset{\text{I}}{\theta^2}\ \underset{\text{II}}{\cos\theta}\, d\theta = \theta^2 \sin\theta - \int 2\theta\sin\theta\, d\theta$$

$$= \theta^2\sin\theta - 2\int \underset{\text{I}}{\theta}\ \underset{\text{II}}{\sin\theta}\, d\theta$$

$= \theta^2 \sin\theta - 2\left[\theta(-\cos\theta) - \int 1.(-\cos\theta)\,d\theta\right]$

$= \theta^2 \sin\theta + 2\theta\cos\theta - 2\int \cos\theta\,d\theta = \theta^2\sin\theta + 2\theta\cos\theta - 2\sin\theta + c$

$= x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\,\sin^{-1}x - 2x + c.$

$\left(\because \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}\right)$

11. $\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}$

Sol. Let $I = \int \dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}\,dx$ ...(*i*)

Put $\cos^{-1}x = \theta$. $\Rightarrow x = \cos\theta$

Therefore $\dfrac{dx}{d\theta} = -\sin\theta \Rightarrow dx = -\sin\theta\,d\theta$

$\therefore$ From (*i*), $I = \int \dfrac{(\cos\theta)\,\theta}{\sqrt{1-\cos^2\theta}}(-\sin\theta\,d\theta)$

$= -\int \dfrac{\theta\cos\theta\sin\theta}{\sin\theta}\,d\theta \quad \left(\because \sqrt{1-\cos^2\theta} = \sqrt{\sin^2\theta} = \sin\theta\right)$

$= -\int \underset{\text{I}}{\theta}\,\underset{\text{II}}{\cos\theta}\,d\theta$

Applying Product Rule: $\text{I}\int \text{II}\,d\theta - \int\left[\dfrac{d}{d\theta}(\text{I})\int \text{II}\,d\theta\right]d\theta$

$= -\left[\theta.\sin\theta - \int 1.\sin\theta\,d\theta\right] = -\theta\sin\theta + \int \sin\theta\,d\theta$

$= -\theta\sin\theta - \cos\theta + c = -\theta\sqrt{1-\cos^2\theta} - \cos\theta + c$

Putting $\theta = \cos^{-1}x$ and $\cos\theta = x$,

$= -(\cos^{-1}x)\sqrt{1-x^2} - x + c = -[\sqrt{1-x^2}\,\cos^{-1}x + x] + c.$

12. $x\sec^2 x$

Sol. $\int \underset{\text{I}}{x}\,\underset{\text{II}}{\sec^2 x}\,dx$

Applying Product Rule: $\text{I}\int \text{II}\,dx - \int\left[\dfrac{d}{dx}(\text{I})\int \text{II}\,dx\right]dx$

$= x\int \sec^2 x\,dx - \int\left[\dfrac{d}{dx}(x)\int \sec^2 x\,dx\right]dx$

$= x\tan x - \int 1.\tan x\,dx = x\tan x - \int \tan x\,dx$

$= x\tan x - (-\log|\cos x|) + c = x\tan x + \log|\cos x| + c.$

13. $\tan^{-1}x$

Sol. Let $I = \int \tan^{-1}x\,dx = \int (\tan^{-1}x).1\,dx$

$= \tan^{-1}x.x - \int \dfrac{1}{1+x^2}.x\,dx = x\tan^{-1}x - \dfrac{1}{2}\int \dfrac{2x}{1+x^2}\,dx$

$= x\tan^{-1}x - \dfrac{1}{2}\log|(1+x^2)| + c.\left[\because \int \dfrac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$

$= x \tan^{-1} x - \frac{1}{2} \log(1 + x^2) + c$

$[\because \; 1 + x^2 \geq 1 > 0$ and hence $|1 + x^2| = 1 + x^2]$

14. $x (\log x)^2$

Sol. $\int x (\log x)^2 \, dx = \int \underset{\text{I}}{(\log x)^2} \, . \underset{\text{II}}{x} \, dx$

Applying Product Rule: $\text{I} \int \text{II} \, dx - \int \left[\frac{d}{dx} (\text{I}) \int \text{II} \, dx \right] dx$

$= (\log x)^2 \int x \, dx - \int \left[\frac{d}{dx} (\log x)^2 \int x \, dx \right] dx$

$= (\log x)^2 \frac{x^2}{2} - \int \frac{2 (\log x)}{x} \frac{x^2}{2} \, dx$

$\left[\because \frac{d}{dx} (\log x)^2 = 2(\log x)^1 \frac{d}{dx} (\log x) = 2 \log x \, . \frac{1}{x} = \frac{2 \log x}{x} \right]$

$= \frac{x^2}{2} (\log x)^2 - \int \underset{\text{I}}{(\log x)} \, \underset{\text{II}}{x} \, dx \qquad \left[\because \frac{x^2}{x} = \frac{x \, . \, x}{x} = x \right]$

Again applying Product Rule: $\text{I} \int \text{II} \, dx - \int \left[\frac{d}{dx} (\text{I}) \int \text{II} \, dx \right] dx$

$= \frac{x^2}{2} (\log x)^2 - \left[(\log x) \frac{x^2}{2} - \int \left(\frac{1}{x} \frac{x^2}{2} \right) dx \right] + c$

$= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{1}{2} \int x \, dx + c$

$= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{x^2}{4} + c.$

15. $(x^2 + 1) \log x$

Sol. $\int (x^2 + 1) \log x \, dx = \int \underset{\text{I}}{(\log x)} \, \underset{\text{II}}{(x^2 + 1)} \, dx$

Applying Product Rule: $\text{I} \int \text{II} \, dx - \int \left[\frac{d}{dx} (\text{I}) \int \text{II} \, dx \right] dx$

$= \log x \left(\frac{x^3}{3} + x \right) - \int \frac{1}{x} \left(\frac{x^3}{3} + x \right) dx$

$= \left(\frac{x^3}{3} + x \right) \log x - \int \left(\frac{x^2}{3} + 1 \right) dx$

$= \left(\frac{x^3}{3} + x \right) \log x - \frac{1}{3} \int x^2 \, dx - \int 1 \, dx$

$= \left(\frac{x^3}{3} + x \right) \log x - \frac{1}{3} \frac{x^3}{3} - x + c = \left(\frac{x^3}{3} + x \right) \log x - \frac{x^3}{9} - x + c.$

Integrate the functions in Exercises 16 to 22:

16. $e^x (\sin x + \cos x)$

Sol. Here $I = \int e^x (\sin x + \cos x)\, dx$

It is of the form $\int e^x [f(x) + f'(x)]\, dx$

Let us take $f(x) = \sin x$ so that $f'(x) = \cos x$

$$I = e^x f(x) + c = e^x \sin x + c.$$

$$\left[\because \int e^x (f(x) + f'(x))\, dx = e^x f(x) + c\right]$$

17. $\dfrac{x e^x}{(1+x)^2}$

Sol. Here $I = \int \dfrac{xe^x}{(x+1)^2}\, dx = \int \dfrac{(x+1)-1}{(x+1)^2} e^x\, dx$

$$= \int e^x \left[\frac{x+1}{(x+1)^2} - \frac{1}{(x+1)^2}\right] dx = \int e^x \left[\frac{1}{x+1} + \frac{-1}{(x+1)^2}\right] dx$$

It is of the form $\int e^x [f(x) + f'(x)]\, dx$

Let us take $f(x) = \dfrac{1}{x+1}$ so that $f'(x) = \dfrac{d}{dx}[(x+1)^{-1}]$

$$= -(x+1)^{-2} = \frac{-1}{(x+1)^2}$$

$\therefore$ $I = e^x f(x) + c = \dfrac{e^x}{x+1} + c.$ $\left[\because \int e^x (f(x) + f'(x))\, dx = e^x f(x) + c\right]$

18. $e^x \left(\dfrac{1+\sin x}{1+\cos x}\right)$

Sol. Here $I = \int e^x \cdot \dfrac{1+\sin x}{1+\cos x}\, dx = \int e^x \cdot \dfrac{1 + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\, dx$

$$= \int e^x \cdot \left[\frac{1}{2\cos^2\frac{x}{2}} + \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right] dx = \int e^x \left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right) dx$$

$$= \int e^x \left(\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2}\right) dx$$

It is of the form $\int e^x [f(x) + f'(x)]\, dx$

Let us take $f(x) = \tan\dfrac{x}{2}$ so that $f'(x) = \dfrac{1}{2}\sec^2\dfrac{x}{2}$

$\therefore$ $I = e^x f(x) + c = e^x \tan\dfrac{x}{2} + c.$

$$\left[\because \int e^x (f(x) + f'(x))\, dx = e^x f(x) + c\right]$$

19. $e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$

Sol. Let $I = \int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)\,dx$

It is of the form $\int e^x\,(f(x)+f'(x))\,dx$

Here $f(x) = \frac{1}{x} = x^{-1}$ and so $f'(x) = (-1)\,x^{-2} = \frac{-1}{x^2}$

$\therefore \qquad I = e^x f(x) + c \qquad [\because \int e^x\,(f(x)+f'(x)]\,dx = e^x f(x) + c$

$= e^x\,\frac{1}{x} + c = \frac{e^x}{x} + c.$

20. $\frac{(x-3)\,e^x}{(x-1)^3}$

Sol. Here $I = \int \frac{(x-3)\,e^x}{(x-1)^3}\,dx = \int \frac{(x-1)-2}{(x-1)^3}\,e^x\,dx$

$= \int e^x\left[\frac{x-1}{(x-1)^3}-\frac{2}{(x-1)^3}\right]dx = \int e^x\left[\frac{1}{(x-1)^2}+\frac{-2}{(x-1)^3}\right]dx$

It is of the form $\int e^x\,[f(x)+f'(x)]\,dx$

Let us take $f(x) = \frac{1}{(x-1)^2}$ so that $f'(x) = \frac{d}{dx}\,[(x-1)^{-2}]$

$= -2(x-1)^{-3} = \frac{-2}{(x-1)^3}$

$\therefore \qquad I = e^x f(x) + c = \frac{e^x}{(x-1)^2} + c.$

$\left[\because \int e^x\,(f(x)+f'(x))\,dx = e^x\,f(x)\right]$

Note. Rule to evaluate $\int e^{ax}\sin bx\,dx$ or $\int e^{ax}\cos bx\,dx$

Let $I = \int \underset{\text{I}}{e^{ax}}\,\underset{\text{II}}{\sin bx}\,dx$ or $\int \underset{\text{I}}{e^{ax}}\,\underset{\text{II}}{\cos bx}\,dx$

Integrate twice by product Rule and transpose term containing I from R.H.S. to L.H.S.

21. $e^{2x}\sin x$

Sol. Let $I = \int \underset{\text{I}}{e^{2x}}\,\underset{\text{II}}{\sin x}\,dx$ ...(*i*)

Applying Product Rule: $\text{I}\int \text{II}\,dx - \int\left[\frac{d}{dx}(\text{I})\int \text{II}\,dx\right]dx$

$\Rightarrow \quad I = e^{2x}\,(-\cos x) - \int e^{2x}\,.\,2\,.\,(-\cos x)\,dx$

$\left[\because \frac{d}{dx}e^{2x} = e^{2x}\,\frac{d}{dx}(2x) = 2e^{2x}\right]$

$\Rightarrow \quad I = -e^{2x}\cos x + 2\int \underset{\text{I}}{e^{2x}}\ \underset{\text{II}}{\cos x}\ dx$

Again Applying Product Rule:

$$I = -e^{2x}\cos x + 2\left[e^{2x}\sin x - \int 2e^{2x}\sin x\,dx\right]$$

$\Rightarrow \quad I = -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx$

$\Rightarrow \quad I = e^{2x}(-\cos x + 2\sin x) - 4I$ [By (*i*)]

Transposing $-4I$ to L.H.S.; $5I = e^{2x}(2\sin x - \cos x)$

$\therefore \quad I\left(= \int e^{2x}\sin x\,dx\right) = \dfrac{e^{2x}}{5}(2\sin x - \cos x) + c$

Remark: The above question can also be done as:

Applying Product Rule: taking $\sin x$ as first function and e^{2x} as second function.

22. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Sol. **Put $x = \tan\theta$.** Differentiating both sides $dx = \sec^2\theta\,d\theta$.

$\therefore \quad \int \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)dx = \int \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right).\sec^2\theta\,d\theta$

$= \int \sin^{-1}(\sin 2\theta).\sec^2\theta\,d\theta = \int 2\theta\sec^2\theta\,d\theta$

$= 2\int \underset{\text{I}}{\theta}\ \underset{\text{II}}{\sec^2\theta}\,d\theta$

Applying product rule

$= 2\left[\theta.\tan\theta - \int 1.\tan\theta\,d\theta\right] = 2\left[\theta\tan\theta - \int \tan\theta\,d\theta\right]$

$= 2[\theta\tan\theta - \log\sec\theta] + c$

$= 2[\tan^{-1}x.x - \log\sqrt{1+x^2}] + c$

$\left[\because \sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+x^2}\right]$

$= 2\left[x\tan^{-1}x - \dfrac{1}{2}\log(1+x^2)\right] + c$

$= 2x\tan^{-1}x - \log(1+x^2) + c.$

Choose the correct answer in Exercises 23 and 24.

23. $\int x^2 e^{x^3}\,dx$ **equals**

(A) $\dfrac{1}{3}e^{x^3} + C$ (B) $\dfrac{1}{3}e^{x^2} + C$

(C) $\dfrac{1}{2}e^{x^3} + C$ (D) $\dfrac{1}{2}e^{x^2} + C$

Sol. Let $I = \int x^2 e^{x^3}\,dx = \dfrac{1}{3}\int e^{(x^3)}(3x^2)\,dx \quad \left[\because \dfrac{d}{dx}x^3 = 3x^2\right]$ ...(*i*)

Put $x^3 = t$. Therefore $3x^2 = \dfrac{dt}{dx}$. Therefore $3x^2\,dx = dt$

$\therefore$ From (i), $I = \frac{1}{3} \int e^t \, dt = \frac{1}{3} e^t + C$

Putting $t = x^3$, $= \frac{1}{3} e^{x^3} + C$

$\therefore$ Option (B) is the correct answer.

24. $\int e^x \sec x (1 + \tan x) \, dx$ **equals**

(A) $e^x \cos x + C$ **(B)** $e^x \sec x + C$

(C) $e^x \sin x + C$ **(D)** $e^x \tan x + C$

Sol. Let $I = \int e^x \sec x (1 + \tan x) \, dx = \int e^x (\sec x + \sec x \tan x) \, dx$

It is of the form $\int e^x (f(x) + f'(x)) \, dx$

Here $f(x) = \sec x$ and so $f'(x) = \sec x \tan x$

$\therefore$ $I = e^x f(x) + C$ $\left[\because \int e^x (f(x) + f'(x)) \, dx = e^x f(x) + C\right]$

$= e^x \sec x + C$

$\therefore$ Option (B) is the correct answer.