Exercise 7.5

Integrate the (rational) functions in Exercises 1 to 6:

1. $\dfrac{x}{(x+1)(x+2)}$

Sol. To integrate the (rational) function $\dfrac{x}{(x+1)(x+2)}$.

Let integrand $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$ ...(*i*)

(Partial Fractions)

Multiplying by L.C.M. = $(x + 1)(x + 2)$,

$x = A(x + 2) + B(x + 1) \quad = Ax + 2A + Bx + B$

Comparing coefficients of x on both sides, $A + B = 1$ ...(*ii*)

Comparing constants, $2A + B = 0$ ...(*iii*)

Let us solve Eqns. (*ii*) and (*iii*) for A and B.

Eqn. (*iii*) – Eqn. (*ii*) gives, $A = -1$

Putting $A = -1$ in (*ii*), $-1 + B = 1 \Rightarrow B = 2$

Putting values of A and B in (*i*), $\dfrac{x}{(x+1)(x+2)} = \dfrac{-1}{x+1} + \dfrac{2}{x+2}$

$$\therefore \int \frac{x}{(x+1)(x+2)}\, dx = -\int \frac{1}{x+1}\, dx + 2\int \frac{1}{x+2}\, dx$$

$$= -\log|x+1| + 2\log|x+2| + c$$

$$= \log|x+2|^2 - \log|x+1| + c = \log\frac{(x+2)^2}{|x+1|} + c.$$

$(\because \ |t|^2 = t^2)$

2. $\dfrac{1}{x^2 - 9}$

Sol. To integrate the (rational) function $\dfrac{1}{x^2 - 9}$

$$\int \frac{1}{x^2 - 9}\, dx = \int \frac{1}{x^2 - 3^2}\, dx$$

$$= \frac{1}{2\times 3} \log \left| \frac{x-3}{x+3} \right| + c \quad \left[\because \int \frac{1}{x^2-a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| \right]$$

$$= \frac{1}{6} \log \left| \frac{x-3}{x+3} \right| + c.$$

OR

Integrand $\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$

Now proceed as in the solution of Q.No.1.

3. $\mathbf{\frac{3x-1}{(x-1)(x-2)(x-3)}}$.

Sol. To integrate the (rational) function $\frac{3x-1}{(x-1)(x-2)(x-3)}$

Let integrand $\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$ ...(*i*)

Multiplying by L.C.M. = $(x-1)(x-2)(x-3)$, we have

$3x - 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

$= A(x^2 - 5x + 6) + B(x^2 - 4x + 3) + C(x^2 - 3x + 2)$

$= Ax^2 - 5Ax + 6A + Bx^2 - 4Bx + 3B + Cx^2 - 3Cx + 2C$

Comparing coefficients of x^2, x and constant terms on both sides, we have

Coefficients of x^2: $A + B + C = 0$ ...(*ii*)

Coefficient of x: $-5A - 4B - 3C = 3$ or $5A + 4B + 3C = -3$ ...(*iii*)

Constants: $6A + 3B + 2C = -1$ ...(*iv*)

Let us solve (*ii*), (*iii*) and (*iv*) for A, B, C.

Let us first form two Eqns. in two unknowns say A and B.

Eqn. (*iii*) – 3 Eqn. (*i*) gives (to eliminate C),

$5A + 4B + 3C - 3A - 3B - 3C = -3$

or $2A + B = -3$ ...(*v*)

Eqn. (*iv*) – 2 Eqn. (*i*) gives (to eliminate C),

$6A + 3B + 2C - 2A - 2B - 2C = -1$

or $4A + B = -1$ ...(*vi*)

Eqn. (*vi*) – Eqn. (*v*) gives (to eliminate B),

$2A = -1 + 3 = 2 \Rightarrow A = \frac{2}{2} = 1.$

Putting A = 1 in (*v*), $2 + B = -3 \Rightarrow B = -5$

Putting A = 1 and B = – 5 in (*ii*), $1 - 5 + C = 0$

or $C - 4 = 0$ or $C = 4$

Putting values of A, B, C in (*i*),

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{4}{x-3}$$

$$\therefore \int \frac{3x-1}{(x-1)(x-2)(x-3)}$$

$$= \int \frac{1}{x-1}\,dx - 5\int \frac{1}{x-2}\,dx + 4\int \frac{1}{x-3}\,dx$$

$$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + c.$$

4. $\boldsymbol{\dfrac{x}{(x-1)(x-2)(x-3)}}$

Sol. To integrate the (rational) function $\dfrac{x}{(x-1)(x-2)(x-3)}$.

Let integrand $\dfrac{x}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$ ...(*i*)

(Partial fractions)

Multiplying by L.C.M. = $(x-1)(x-2)(x-3)$,

$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

$= A(x^2 - 5x + 6) + B(x^2 - 4x + 3) + C(x^2 - 3x + 2)$

$= Ax^2 - 5Ax + 6A + Bx^2 - 4Bx + 3B + Cx^2 - 3Cx + 2C$

Comparing coefficients of x^2, x and constant terms on both sides, we have

$\boldsymbol{x^2}$**:** $A + B + C = 0$ ...(*ii*)

$\boldsymbol{x}$**:** $-5A - 4B - 3C = 1$ or $5A + 4B + 3C = -1$ ...(*iii*)

Constants: $6A + 3B + 2C = 0$ ...(*iv*)

Let us solve Eqns. (*ii*), (*iii*) and (*iv*) for A, B, C.

Let us first form two Eqns. in two unknowns say A and B.

Eqn. (*iii*) – 3 × Eqn. (*ii*) gives | To eliminate C

$5A + 4B + 3C - 3A - 3B - 3C = -1$ or $2A + B = -1$ ...(*v*)

Eqn. (*iv*) – 2 × Eqn. (*ii*) gives | To eliminate C

$4A + B = 0$ ...(*vi*)

Eqn. (*vi*) – Eqn. (*v*) gives (To eliminate B)

$2A = 1 \quad \therefore \quad A = \dfrac{1}{2}$

Putting $A = \dfrac{1}{2}$ in (*v*), $1 + B = -1 \Rightarrow B = -2$

Putting $A = \dfrac{1}{2}$ and $B = -2$ in (*ii*),

$\dfrac{1}{2} - 2 + C = 0 \Rightarrow C = \dfrac{-1}{2} + 2 = \dfrac{-1+4}{2} = \dfrac{3}{2}$

Putting these values of A, B, C in (*i*), we have

$$\frac{x}{(x-1)(x-2)(x-3)} = \frac{\frac{1}{2}}{x-1} - \frac{2}{x-2} + \frac{\frac{3}{2}}{x-3}$$

$$\therefore \int \frac{x}{(x-1)(x-2)(x-3)}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x-1}\,dx - 2\int \frac{1}{x-2}\,dx + \frac{3}{2}\int \frac{1}{x-3}\,dx$$

$$= \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + c.$$

5. $\dfrac{2x}{x^2+3x+2}$

Sol. To integrate the (rational) function $\dfrac{2x}{x^2+3x+2}$.

Now $x^2 + 3x + 2 = x^2 + 2x + x + 2 = x(x + 2) + 1(x + 2)$
$= (x + 1)(x + 2)$

$\therefore$ Integrand $\dfrac{2x}{x^2+3x+2} = \dfrac{2x}{(x+1)(x+2)}$

$= \dfrac{A}{x+1} + \dfrac{B}{x+2}$ ...(*i*)

(Partial Fractions)

Multiplying both sides by L.C.M. $= (x + 1)(x + 2)$,

$2x = A(x + 2) + B(x + 1) \quad = Ax + 2A + Bx + B$

Comparing coefficients of x and constant terms on both sides, we have

Coefficients of x: $A + B = 2$ ...(*ii*)

Constant terms: $2A + B = 0$ ...(*iii*)

Let us solve (*ii*) and (*iii*) for A and B.

(*iii*) – (*ii*) gives $A = -2$.

Putting $A = -2$ in (*ii*), $-2 + B = 2$. $\therefore$ $B = 4$

Putting values of A and B in (*i*), $\dfrac{2x}{x^2+3x+2} = \dfrac{-2}{x+1} + \dfrac{4}{x+2}$

$\therefore \displaystyle\int \frac{2x}{x^2+3x+2}\,dx = -2\int \frac{1}{x+1}\,dx + 4\int \frac{1}{x+2}\,dx$

$= -2\log|x + 1| + 4\log|x + 2| + c$

$= 4\log|x + 2| - 2\log|x + 1| + c$

Remark: Alternative method to evaluate $\displaystyle\int \frac{2x}{x^2+3x+2}\,dx$

is $\displaystyle\int \frac{\text{Linear}}{\text{Quadratic}}\,dx$ as explained in solutions in Exercise 7.4 (Exercise 18 and Exercise 22.

6. $\dfrac{1-x^2}{x(1-2x)}$

Sol. To integrate (rational) function $\dfrac{1-x^2}{x(1-2x)} = \dfrac{1-x^2}{x-2x^2} = \dfrac{-x^2+1}{-2x^2+x}$

[Here Degree of numerator = Degree of Denominator = 2

$\therefore$ We must divide numerator by denominator to make the degree of numerator smaller than degree of denominator so that we can form partial fractions.]

$$\begin{array}{r|l} -2x^2 + x & -x^2 + 1 \; \left(\frac{1}{2} \right. \\ & -x^2 + \frac{x}{2} \\ & \;\;+ \quad - \\ \hline & -\frac{x}{2} + 1 \end{array}$$

$$\therefore \quad \frac{1-x^2}{x(1-2x)} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}} = \frac{1}{2} + \frac{\left(-\frac{x}{2}+1\right)}{x(1-2x)}$$

$$\therefore \quad \int \frac{1-x^2}{x(1-2x)}\,dx = \int \left(\frac{1}{2} + \frac{\left(-\frac{x}{2}+1\right)}{x(1-2x)} \right) dx$$

$$= \frac{1}{2}\int 1\,dx + \int \frac{-\frac{x}{2}+1}{x(1-2x)}\,dx \qquad ...(i)$$

Let integrand $\frac{-\frac{x}{2}+1}{x(1-2x)} = \frac{A}{x} + \frac{B}{1-2x}$ ...(*ii*)

Multiplying by L.C.M. $= x(1-2x)$,

$$-\frac{x}{2} + 1 = A(1-2x) + Bx \quad = A - 2Ax + Bx$$

Comparing coefficients of x, $-2A + B = \frac{-1}{2}$ ...(*iii*)

Comparing constants, $A = 1$ ...(*iv*)

Putting $A = 1$ from (*iv*) in (*iii*),

$$-2 + B = \frac{-1}{2} \Rightarrow B = \frac{-1}{2} + 2 = \frac{-1+4}{2} \quad \text{or} \quad B = \frac{3}{2}$$

Putting values of A and B in (*ii*),

$$\frac{-\frac{x}{2}+1}{x(1-2x)} = \frac{1}{x} + \frac{\frac{3}{2}}{1-2x}$$

$$\therefore \quad \int \frac{-\frac{x}{2}+1}{x(1-2x)}\,dx = \int \frac{1}{x}\,dx + \frac{3}{2}\int \frac{1}{1-2x}\,dx$$

$$= \log|x| + \frac{3}{2}\log\frac{|1-2x|}{-2 \to \text{Coefficient of } x} + c$$

$$= \log|x| - \frac{3}{4}\log|1-2x| + c$$

Putting this value in (*i*),

$$\int \frac{1-x^2}{x(1-2x)}\,dx = \frac{1}{2}x + \log|x| - \frac{3}{4}\log|1-2x| + c.$$

Integrate the following functions in Exercises 7 to 12:

7. $\dfrac{x}{(x^2+1)(x-1)}$

Sol. To integrate the (rational) function $\dfrac{x}{(x^2+1)(x-1)}$.

Let integrand $\dfrac{x}{(x^2+1)(x-1)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-1}$ ...(*i*)

(Partial Fractions)

Multiplying by L.C.M. = $(x^2 + 1)(x - 1)$ on both sides,

$$x = (Ax + B)(x - 1) + C(x^2 + 1)$$

$$\Rightarrow \quad x = Ax^2 - Ax + Bx - B + Cx^2 + C,$$

Comparing coefficients of x^2, x and constant terms on both sides, we have

x^2 $\quad A + C = 0$ ...(*ii*)

x $\quad - A + B = 1$ ...(*iii*)

Constants $\quad - B + C = 0$ ...(*iv*)

Let us solve Eqns. (*ii*), (*iii*) and (*iv*) for A, B, C

Adding (*ii*) and (*iii*) to eliminate A, B + C = 1 ...(*v*)

Adding (*iv*) and (*v*), $2C = 1 \Rightarrow C = \dfrac{1}{2}$

From (*iv*), $-B = -C \Rightarrow B = C = \dfrac{1}{2}$

From (*ii*), $A = -C = \dfrac{-1}{2}$

Putting these values of A, B, C in (*i*),

$$\frac{x}{(x^2+1)(x-1)} = \frac{\frac{-1}{2}x + \frac{1}{2}}{x^2+1} + \frac{\frac{1}{2}}{x-1}$$

$$= \frac{-1}{2}\frac{x}{x^2+1} + \frac{1}{2}\cdot\frac{1}{x^2+1} + \frac{1}{2}\frac{1}{x-1}$$

$$= \frac{-1}{4}\frac{2x}{x^2+1} + \frac{1}{2}\frac{1}{x^2+1} + \frac{1}{2}\frac{1}{x-1}$$

$$\therefore \int \frac{x}{(x^2+1)(x-1)}\,dx$$

$$= \frac{-1}{4}\int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$$

$$\Rightarrow \int \frac{x}{(x^2+1)(x-1)}\,dx = \frac{-1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x$$

$$+ \frac{1}{2}\log|x-1| + c \qquad \left(\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right)$$

$$= \frac{-1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + c$$

$[\because \; x^2 + 1 > 0 \Rightarrow |x^2+1| = x^2 + 1]$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + c.$$

8. $\dfrac{x}{(x-1)^2(x+2)}$

Sol. To integrate the (rational) function $\dfrac{x}{(x-1)^2(x+2)}$.

Let integrand $\dfrac{x}{(x-1)^2(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2}$ ...(*i*)

(Partial fractions)

Multiplying both sides of (*i*) by L.C.M. = $(x - 1)^2 (x + 2)$

$x = A(x - 1)(x + 2) + B(x + 2) + C(x - 1)^2$

or $x = A(x^2 + \mathbf{2x - x} - 2) + B(x + 2) + C(x^2 + 1 - 2x)$

or $x = Ax^2 + Ax - 2A + Bx + 2B + Cx^2 + C - 2Cx$

Comparing coefficients of x^2, x and constant terms on both sides

$\mathbf{x^2}$ $\quad A + C = 0$ ...(*ii*)

$\mathbf{x}$ $\quad A + B - 2C = 1$ ...(*iii*)

Constants $\quad - 2A + 2B + C = 0$ ...(*iv*)

Let us solve (*ii*), (*iii*) and (*iv*) for A, B, C

From (*ii*), $A = - C$

Putting $A = - C$ in (*iv*), $2C + 2B + C = 0$

$\Rightarrow \quad 2B = - 3C \quad \Rightarrow \quad B = \dfrac{-3C}{2}$

Putting values of A and B in (*iii*),

$$- C - \frac{-3C}{2} - 2C = 1 \Rightarrow - 2C - 3C - 4C = 2$$

$\Rightarrow \quad - 9C = 2 \quad \Rightarrow \quad C = \dfrac{-2}{9}$

Putting $C = \dfrac{-2}{9}$, $B = \dfrac{-3C}{2} = \dfrac{-3}{2}\left(\dfrac{-2}{9}\right) = \dfrac{1}{3}$ $\quad \therefore A = - C = \dfrac{2}{9}$

Putting these values of A, B, C in (*i*),

$$\frac{x}{(x-1)^2(x+2)} = \frac{\frac{2}{9}}{x-1} + \frac{\frac{1}{3}}{(x-1)^2} - \frac{\frac{2}{9}}{x+2}$$

$$\therefore \quad \int \frac{x}{(x-1)^2(x+2)}\, dx$$

$$= \frac{2}{9} \int \frac{1}{x-1}\, dx + \frac{1}{3} \int (x-1)^{-2}\, dx - \frac{2}{9} \int \frac{1}{x+2}\, dx$$

$$= \frac{2}{9} \log | x - 1 | + \frac{1}{3} \frac{(x-1)^{-1}}{(-1)(1)} - \frac{2}{9} \log | x + 2 | + c$$

$$= \frac{2}{9} (\log | x - 1 | - \log | x + 2 |) - \frac{1}{3(x-1)} + c$$

$$= \frac{2}{9} \log \left| \frac{x-1}{x+2} \right| - \frac{1}{3(x-1)} + c.$$

9. $\dfrac{\mathbf{3x + 5}}{\mathbf{x^3 - x^2 - x + 1}}$

Sol. To integrate the (rational) function $\dfrac{3x+5}{x^3 - x^2 - x + 1}$.

Now denominator $= x^3 - x^2 - x + 1$
$= x^2 (x - 1) - 1(x - 1) = (x - 1)(x^2 - 1)$
$= (x - 1)(x - 1)(x + 1) = (x - 1)^2 (x + 1)$

$\therefore$ Integrand $\dfrac{3x+5}{x^3 - x^2 - x + 1} = \dfrac{3x+5}{(x-1)^2 (x+1)}$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \quad ...(i) \quad \text{(Partial fractions)}$$

Multiplying by L.C.M. $= (x - 1)^2 (x + 1)$,
$3x + 5 = A(x - 1)(x + 1) + B(x + 1) + C(x - 1)^2$
$= A(x^2 - 1) + B(x + 1) + C(x^2 + 1 - 2x)$
$= Ax^2 - A + Bx + B + Cx^2 + C - 2Cx$

Comparing coefficients of x^2, x and constant terms on both sides,

x^2 $\quad A + C = 0 \quad ...(ii)$

x $\quad B - 2C = 3 \quad ...(iii)$

Constants $\quad - A + B + C = 5 \quad ...(iv)$

Let us solve Eqns. (*ii*), (*iii*) and (*iv*) for A, B, C.
From (*ii*), $A = - C$ and from (*iii*), $B = 2C + 3$
Putting these values of A and B in (*iv*),

$$C + 2C + 3 + C = 5 \Rightarrow 4C = 2 \Rightarrow C = \frac{2}{4} = \frac{1}{2}$$

$$\therefore \quad A = - C = -\frac{1}{2}$$

and $\quad B = 2C + 3 = 2\left(\dfrac{1}{2}\right) + 3 = 1 + 3 = 4.$

Putting these values of A, B, C in (*i*)

$$\frac{3x+5}{x^3 - x^2 - x + 1} = \frac{\frac{-1}{2}}{x-1} + \frac{4}{(x-1)^2} + \frac{\frac{1}{2}}{x+1}$$

$$\therefore \int \frac{3x+5}{x^3 - x^2 - x + 1}\, dx$$

$$= \frac{-1}{2} \int \frac{1}{x-1}\, dx + 4 \int (x-1)^{-2}\, dx + \frac{1}{2} \int \frac{1}{x+1}\, dx$$

$$= \frac{-1}{2} \log |x - 1| + 4 \frac{(x-1)^{-1}}{\underset{\text{Coeff. of } x}{\underbrace{(-1)(1)}}} + \frac{1}{2} \log |x + 1| + c$$

$$= \frac{1}{2} (\log |x + 1| - \log |x - 1|) - \frac{4}{x-1} + c$$

$$= \frac{1}{2} \log \left| \frac{x+1}{x-1} \right| - \frac{4}{x-1} + c.$$

10. $\boldsymbol{\dfrac{2x-3}{(x^2-1)(2x+3)}}$

Sol. To integrate the rational function $\dfrac{2x-3}{(x^2-1)(2x+3)}$.

Let integrand $\dfrac{2x-3}{(x^2-1)(2x+3)} = \dfrac{2x-3}{(x-1)(x+1)(2x+3)}$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+3} \quad ...(i)$$

Multiplying both sides by L.C.M. = $(x-1)(x+1)(2x+3)$,

$2x - 3 = A(x+1)(2x+3) + B(x-1)(2x+3) + C(x-1)(x+1)$

or $2x - 3 = A(2x^2 + \mathbf{3x + 2x} + 3) + B(2x^2 + \mathbf{3x - 2x} - 3) + C(x^2 - 1)$

Comparing coefficients of x^2, x and constant terms on both sides,

$\boldsymbol{x^2}$ $\quad 2A + 2B + C = 0 \quad ...(ii)$

$\boldsymbol{x}$ $\quad 5A + B = 2 \quad ...(iii)$

Constants $\quad 3A - 3B - C = -3 \quad ...(iv)$

Let us solve Eqns. (*ii*), (*iii*) and (*iv*) for A, B, C.

Eqn. (*ii*) + Eqn. (*iv*) gives (to eliminate C)

$$5A - B = -3 \quad ...(v)$$

Adding Eqns. (*iii*) and (*v*), $10A = -1 \Rightarrow A = \dfrac{-1}{10}$

Putting $A = \dfrac{-1}{10}$ in (*iii*), $\dfrac{-5}{10} + B = 2 \Rightarrow B = 2 + \dfrac{1}{2} = \dfrac{5}{2}$

Putting values of A and B in (*ii*),

$$\frac{-1}{5} + 5 + C = 0 \qquad \therefore \ C = \frac{1}{5} - 5 = \frac{1-25}{25} = \frac{-24}{5}$$

Putting values of A, B, C in (*i*),

$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{\frac{-1}{10}}{x-1} + \frac{\frac{5}{2}}{x+1} - \frac{\frac{24}{5}}{2x+3}$$

$$\therefore \int \frac{2x-3}{(x^2-1)(2x+3)}\,dx$$

$$= \frac{-1}{10}\int \frac{1}{x-1}\,dx + \frac{5}{2}\int \frac{1}{x+1}\,dx - \frac{24}{5}\int \frac{1}{2x+3}\,dx$$

$$= \frac{-1}{10}\,\frac{\log|x-1|}{1 \to \text{Coeff. of } x} + \frac{5}{2}\,\frac{\log|x+1|}{1} - \frac{24}{5}\,\frac{\log|2x+3|}{2 \to \text{Coeff. of } x} + c$$

$$= \frac{-1}{10}\log|x-1| + \frac{5}{2}\log|x+1| - \frac{12}{5}\log|2x+3| + c$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + c.$$

11. $\boldsymbol{\dfrac{5x}{(x+1)(x^2-4)}}$

Sol. To integrate the rational function $\dfrac{5x}{(x+1)(x^2-4)}$.

Let integrand $\dfrac{5x}{(x+1)(x^2-4)} = \dfrac{5x}{(x+1)(x+2)(x-2)}$

$$= \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-2} \quad ...(i) \text{ (Partial fractions)}$$

Multiplying both sides of (*i*) by L.C.M.

$= (x + 1)(x + 2)(x - 2),$

$$5x = A(x + 2)(x - 2) + B(x + 1)(x - 2) + C(x + 1)(x + 2)$$
$$= A(x^2 - 4) + B(x^2 - x - 2) + C(x^2 + 3x + 2)$$
$$= Ax^2 - 4A + Bx^2 - Bx - 2B + Cx^2 + 3Cx + 2C.$$

Comparing coefficients of x^2, x and constant terms on both sides,

$\boldsymbol{x^2}$ $\quad A + B + C = 0$ ...(*ii*)

$\boldsymbol{x}$ $\quad - B + 3C = 5$ ...(*iii*)

Constants $\quad - 4A - 2B + 2C = 0$

Dividing by $- 2$, $2A + B - C = 0$ ...(*iv*)

Let us solve (*ii*), (*iii*) and (*iv*) for A, B, C

Eqn. (*ii*) × 2 – Eqn. (*iv*) gives (To eliminate A) because Eqn. (*iii*) does not involve A.

$$2A + 2B + 2C - (2A + B - C) = 0,$$

i.e., $\quad 2A + 2B + 2C - 2A - B + C = 0$

$\Rightarrow \quad B + 3C = 0$ ...(*v*)

Adding Eqns. (*iii*) and (*v*),

$$6C = 5 \qquad \Rightarrow \qquad C = \frac{5}{6}$$

Putting $C = \frac{5}{6}$ in (*iii*), $- B + \frac{15}{6} = 5 \quad \Rightarrow \quad - B = 5 - \frac{15}{6}$

$$\Rightarrow \quad - B = \frac{30 - 15}{6} = \frac{15}{6} = \frac{5}{2} \quad \Rightarrow \quad B = \frac{-5}{2}$$

Putting $B = \frac{-5}{2}$ and $C = \frac{5}{6}$ in (*ii*), $A - \frac{5}{2} + \frac{5}{6} = 0$

$$\Rightarrow \quad A = \frac{5}{2} - \frac{5}{6} = \frac{15 - 5}{6} = \frac{10}{6} = \frac{5}{3}$$

Putting values of A, B, C in (*i*),

$$\frac{5x}{(x + 1)(x^2 - 4)} = \frac{\frac{5}{3}}{x + 1} - \frac{\frac{5}{2}}{x + 2} + \frac{\frac{5}{6}}{x - 2}$$

$$\therefore \quad \int \frac{5x}{(x + 1)(x^2 - 4} \, dx = \frac{5}{3} \int \frac{1}{x + 1} \, dx - \frac{5}{2} \int \frac{1}{x + 2} \, dx + \frac{5}{6} \int \frac{1}{x - 2} \, dx$$

$$= \frac{5}{3} \log | x + 1 | - \frac{5}{2} \log | x + 2 | + \frac{5}{6} \log | x - 2 | + c.$$

12. $\boldsymbol{\dfrac{x^3 + x + 1}{x^2 - 1}}$

Sol. Here degree of numerator is greater than degree of denominator. Therefore, dividing the numerator by the denominator,

$$
\begin{array}{r|l}
x^2 - 1 & x^3 + x + 1 \;(\; x \\
 & x^3 - x \\
 & \underline{- \quad +} \\
 & 2x + 1
\end{array}
$$

$$\therefore \quad \frac{x^3 + x + 1}{x^2 - 1} = x + \frac{2x + 1}{x^2 - 1} \qquad ...(i)$$

$$\left[\textbf{Rational function} = \textbf{Quotient} + \frac{\textbf{Remainder}}{\textbf{Divisor}}\right]$$

Let $\frac{2x+1}{x^2-1} = \frac{2x+1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$ ...(*ii*)

Multiplying by L.C.M. = $(x + 1)(x - 1)$, we have

$2x + 1 = A(x - 1) + B(x + 1)$

or $2x + 1 = Ax - A + Bx + B$

By equating the coefficients of x and constant terms, we get

$A + B = 2$ ...(*iii*)

and $-A + B = 1$ ...(*iv*)

(*iii*) + (*iv*) gives $2B = 3 \Rightarrow B = \frac{3}{2}$

Putting $B = \frac{3}{2}$ in (*iii*), we get $A + \frac{3}{2} = 2$ or $A = \frac{1}{2}$

Putting values of A and B in eqn. (*ii*), we have

$$\frac{2x+1}{x^2-1} = \frac{\frac{1}{2}}{x+1} + \frac{\frac{3}{2}}{x-1}$$

Putting this value of $\frac{2x+1}{x^2-1}$ in (*i*),

$$\frac{x^3+x+1}{x^2-1} = x + \frac{\frac{1}{2}}{x+1} + \frac{\frac{3}{2}}{x-1}$$

$$\therefore \int \frac{x^3+x+1}{x^2-1}\,dx = \int x\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{3}{2}\int \frac{1}{x-1}\,dx$$

$$= \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + c.$$

Integrate the following functions in Exercises 13 to 17:

13. $\frac{2}{(1-x)(1+x^2)}$

Sol. To find integral of the Rational function $\frac{2}{(1-x)(1+x^2)}$.

Let integrand $\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}$ ...(*i*)

(Partial Fractions)

Multiplying by L.C.M. = $(1 - x)(1 + x^2)$

$2 = A(1 + x^2) + (Bx + C)(1 - x)$

or $2 = A + Ax^2 + Bx - Bx^2 + C - Cx$

Comparing coefficients of x^2, x and constant terms, we have

$\boldsymbol{x^2}$ $A - B = 0$ ...(*ii*)

$\boldsymbol{x}$ $B - C = 0$ ...(*iii*)

Constant terms $A + C = 2$ ...(*iv*)

Let us solve (*ii*), (*iii*), (*iv*) for A, B, C

From (*ii*), $A = B$ and from (*iii*), $B = C$

$\therefore \quad A = B = C$

Putting A = C in (*iv*), C + C = 2 or 2C = 2 or C = 1

$\therefore \quad A = C = 1 \qquad \therefore \quad B = A = 1$

Putting these values of A, B, C in (*i*),

$$\frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2} = \frac{1}{1-x} + \frac{x}{1+x^2} + \frac{1}{1+x^2}$$

$$= \frac{1}{1-x} + \frac{1}{2}\frac{2x}{1+x^2} + \frac{1}{1+x^2}$$

$$\therefore \int \frac{2}{(1-x)(1+x^2)}\,dx = \int \frac{1}{1-x}\,dx + \frac{1}{2}\int \frac{2x}{1+x^2}\,dx + \int \frac{1}{1+x^2}\,dx$$

$$= \frac{\log|1-x|}{-1 \to \text{Coefficient of } x} + \frac{1}{2}\log|1+x^2| + \tan^{-1}x + c$$

$$\left[\because \int \frac{2x}{1+x^2}\,dx = \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$$

$$= -\log|1-x| + \frac{1}{2}\log(1+x^2) + \tan^{-1}x + c$$

($\because$ $1 + x^2 > 0$, therefore $|1 + x^2| = 1 + x^2$)

Note. $\log|1 - x| = \log|-(x-1)|$

$= \log|x - 1|$ because $|-t| = |t|$.

14. $\mathbf{\dfrac{3x-1}{(x+2)^2}}$

Sol. To find integral of rational function $\dfrac{3x-1}{(x+2)^2}$.

Let $I = \int \dfrac{3x-1}{(x+2)^2}\,dx$ ...(*i*)

Form $\int \dfrac{\text{Polynomial function}}{(\text{Linear})^k}\,dx$ where k is a positive integer,

put Linear = t.

Here put $x + 2 = t \quad \Rightarrow \quad x = t - 2$

$\therefore \quad \dfrac{dx}{dt} = 1 \quad \Rightarrow \quad dx = dt$

Putting these values in (*i*),

$$I = \int \frac{3(t-2)-1}{t^2}\,dt = \int \frac{3t-6-1}{t^2}\,dt = \int \frac{3t-7}{t^2}\,dt$$

$$= \int \left(\frac{3t}{t^2} - \frac{7}{t^2}\right) dt = \int \left(\frac{3}{t} - \frac{7}{t^2}\right) dt$$

$$= 3\int \frac{1}{t}\,dt - 7\int t^{-2}\,dt = 3\log|t| - 7\frac{t^{-1}}{-1} + c$$

$$= 3\log|t| + \frac{7}{t} + c$$

Putting $t = x + 2$, $= 3 \log |x + 2| + \dfrac{7}{x+2} + c$.

Remark. Alternative solution is Let $\dfrac{3x - 1}{(x+2)^2} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2}$.

15. $\dfrac{1}{x^4 - 1}$

Sol. To find integral of $\dfrac{1}{x^4 - 1}$.

Let integrand $\dfrac{1}{x^4 - 1} = \dfrac{1}{(x^2 - 1)(x^2 + 1)}$.

Put $x^2 = y$ **only** to form partial fractions.

$$= \frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1} \quad ...(i)$$

Multiplying by L.C.M. $= (y - 1)(y + 1)$

$1 = A(y + 1) + B(y - 1)$ or $1 = Ay + A + By - B$

Comparing coeffs. of y and constant terms, we have

Coefficients of y: $A + B = 0$ ...(ii)

Constant terms $A - B = 1$ ...(iii)

Adding (ii) and (iii), $2A = 1 \Rightarrow A = \dfrac{1}{2}$

Putting $A = \dfrac{1}{2}$ in (ii), $\dfrac{1}{2} + B = 0 \Rightarrow B = \dfrac{-1}{2}$

Putting values of A, B **and y** in (i),

$$\frac{1}{x^4 - 1} = \frac{\frac{1}{2}}{x^2 - 1} - \frac{\frac{1}{2}}{x^2 + 1}$$

$$\therefore \int \frac{1}{x^4 - 1}\, dx = \frac{1}{2}\int \frac{1}{x^2 - 1^2}\, dx - \frac{1}{2}\int \frac{1}{x^2 + 1}\, dx$$

$$= \frac{1}{2}\,\frac{1}{2.1} \log \left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1} x + c$$

$$\left[\because \int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right|\right]$$

Note. Must put $y = x^2$ in (i) along with values of A and B before writing values of integrals.

Remark. Alternative solution is:

$$\frac{1}{x^4 - 1} = \frac{1}{(x^2 - 1)(x^2 + 1)} = \frac{1}{(x-1)(x+1)(x^2+1)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx + D}{x^2 + 1}$$

But the above given solution is better.

16. $\dfrac{\mathbf{1}}{\boldsymbol{x(x^n+1)}}$

Sol. Let I = $\displaystyle\int \frac{1}{x(x^n+1)}\, dx$

Multiplying both numerator and denominator of integrand by nx^{n-1}.

$$\left[\because \frac{d}{dx}(x^n+1) = nx^{n-1}\right]$$

$$I = \int \frac{nx^{n-1}}{n\,x^{n-1}\,x(x^n+1)}\, dx = \frac{1}{n}\int \frac{n\,x^{n-1}}{x^n\,(x^n+1)}\, dx \qquad ...(i)$$

$$(\because \quad n-1+1 = n)$$

Put $x^n = t$. Therefore $n\,x^{n-1} = \dfrac{dt}{dx}$. $\quad\therefore\quad n\,x^{n-1}\,dx = dt$.

$\therefore$ From (i), I = $\displaystyle\frac{1}{n}\int \frac{dt}{t(t+1)} = \frac{1}{n}\int \frac{1}{t(t+1)}\, dt$

Adding and subtracting t in the numerator of integrand,

$$= \frac{1}{n}\int \frac{t+1-t}{t(t+1)}\, dt = \frac{1}{n}\int \left(\frac{t+1}{t(t+1)} - \frac{t}{t(t+1)}\right) dt \left[\because \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}\right]$$

$$= \frac{1}{n}\left[\int \frac{1}{t}\, dt - \int \frac{1}{t+1}\, dt\right] = \frac{1}{n}\,[\log|\,t\,| - \log|\,t+1\,| + c]$$

$$= \frac{1}{n}\log\left|\frac{t}{t+1}\right| + c$$

Putting $t = x^n$, $= \dfrac{1}{n}\log\left|\dfrac{x^n}{x^n+1}\right| + c$

Remark: Alternative solution for $\displaystyle\int \frac{1}{t(t+1)}\, dt$ is:

Let $\dfrac{1}{t(t+1)} = \dfrac{A}{t} + \dfrac{B}{t+1}$.

But the above given solution is better.

17. $\dfrac{\boldsymbol{\cos x}}{\boldsymbol{(1-\sin x)(2-\sin x)}}$

Sol. Let I = $\displaystyle\int \frac{\cos x}{(1-\sin x)(2-\sin x)}\, dx \qquad ...(i)$

Put $\sin x = t$. Therefore $\cos x = \dfrac{dt}{dx} \Rightarrow \cos x\, dx = dt$,

$\therefore$ From (i), $\displaystyle\int \frac{1}{(1-t)(2-t)}\, dt = \int \frac{(2-t)-(1-t)}{(1-t)(2-t)}\, dt$

[$\because$ Difference of two factors in the denominator namely $1-t$ and $2-t$ is $(2-t)-(1-t) = 2-t-1+t = 1$]

$$= \int \left(\frac{2-t}{(1-t)(2-t)} - \frac{(1-t)}{(1-t)(2-t)}\right) dt \left[\because \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}\right]$$

$$= \int \left(\frac{1}{1-t} - \frac{1}{2-t} \right) dt = \int \frac{1}{1-t}\, dt - \int \frac{1}{2-t}\, dt$$

$$= \frac{\log|1-t|}{-1 \to \text{Coefficient of } t} - \frac{\log|2-t|}{-1} + c$$

$$= -\log|1-t| + \log|2-t| + c$$

$$= \log|2-t| - \log|1-t| + c \qquad = \log\left|\frac{2-t}{1-t}\right| + c$$

Putting $t = \sin x$, $= \log\left|\frac{2-\sin x}{1-\sin x}\right| + c$

Remark: Alternative solution for $\int \frac{1}{(1-t)(2-t)}\, dt$ is

Let $\frac{1}{(1-t)(2-t)} = \frac{A}{1-t} + \frac{B}{2-t}$

Integrate the following functions for Exercises 18 to 21:

18. $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

Sol. To integrate the rational function $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$. ...(*i*)

Put $x^2 = y$ in the integrand to get

$$= \frac{(y+1)(y+2)}{(y+3)(y+4)} = \frac{y^2+3y+2}{y^2+7y+12} \qquad ...(ii)$$

Here degree of numerator = degree of denominator (= 2)
So have to perform long division to make the degree of numerator smaller than degree of denominator so that the concept of forming partial fractions becomes valid.

$$y^2 + 7y + 12 \;\big)\; y^2 + 3y + 2 \;\big(\; 1$$
$$y^2 + 7y + 12$$
$$- \quad - \quad -$$
$$\overline{\quad - 4y - 10 \quad}$$

∴ From (*i*) and (*ii*),

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \frac{(y+1)(y+2)}{(y+3)(y+4)} = 1 + \frac{(-4y-10)}{(y+3)(y+4)} \qquad ...(iii)$$

Let us form partial fractions of $\frac{(-4y-10)}{(y+3)(y+4)}$.

$$\text{Let } \frac{-4y-10}{(y+3)(y+4)} = \frac{A}{y+3} + \frac{B}{y+4} \qquad ...(iv)$$

Multiplying by L.C.M. = $(y+3)(y+4)$

$$-4y - 10 = A(y+4) + B(y+3) = Ay + 4A + By + 3B$$

Comparing coefficients of y, $A + B = -4$ ...(*v*)

Comparing constants, $4A + 3B = -10$ ...(*vi*)

Let us solve Eqns. (*v*) and (*vi*) for A and B.

Eqn. (*v*) × 4 gives, $4A + 4B = -16$ ...(*vii*)

Eqn. (*vi*) – Eqn. (*vii*) gives, $-B = 6$ or $B = -6$.
Putting $B = -6$ in (*v*), $A - 6 = -4 \Rightarrow A = -4 + 6 = 2$
Putting these values of A and B in (*iv*),

$$\frac{-4y-10}{(y+3)(y+4)} = \frac{2}{y+3} - \frac{6}{y+4}$$

Putting this value in (*iii*),

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 + \frac{2}{y+3} - \frac{6}{y+4}$$

In R.H.S., Putting $y = x^2$ (before integration)

$$= 1 + \frac{2}{x^2+3} - \frac{6}{x^2+4}$$

$$\therefore \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\,dx$$

$$= \int 1\,dx + 2\int \frac{1}{x^2+(\sqrt{3})^2}\,dx - 6\int \frac{1}{x^2+2^2}\,dx$$

$$= x + 2\,.\,\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 6\,.\,\frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + c.$$

19. $\dfrac{\mathbf{2x}}{\mathbf{(x^2+1)(x^2+3)}}$

Sol. Let $I = \displaystyle\int \frac{2x}{(x^2+1)(x^2+3)}\,dx$

Put $x^2 = t$. Differentiating both sides $2x\,dx = dt$

$$\therefore \quad I = \int \frac{dt}{(t+1)(t+3)}$$

Dividing and multiplying by 2,
$(\because\ (t+3)-(t+1) = t+3-t-1 = 2)$

$$= \frac{1}{2}\int \frac{2}{(t+1)(t+3)}\,dt = \frac{1}{2}\int \frac{(t+3)-(t+1)}{(t+1)(t+3)}\,dt$$

$$= \frac{1}{2}\int \left(\frac{1}{t+1} - \frac{1}{t+3}\right)dt = \frac{1}{2}\,[\log|t+1| - \log|t+3|] + c$$

$$= \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + c = \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + c = \frac{1}{2}\log\left(\frac{x^2+1}{x^2+3}\right) + c.$$

20. $\dfrac{\mathbf{1}}{\mathbf{x(x^4-1)}}$

Sol. Let $I = \displaystyle\int \frac{1}{x(x^4-1)}\,dx$

Multiplying both numerator and denominator of integrand by $4x^3$.

$$\left(\because \frac{d}{dx}(x^4-1) = 4x^3\right)$$

$$I = \int \frac{4x^3}{4x^4 (x^4 - 1)} dx = \frac{1}{4} \int \frac{4x^3}{x^4 (x^4 - 1)} dx \quad ...(i)$$

Put $x^4 = t$. Therefore $4x^3 = \frac{dt}{dx} \Rightarrow 4x^3 \, dx = dt.$

$\therefore$ From (i), $I = \frac{1}{4} \int \frac{dt}{t(t-1)} = \frac{1}{4} \int \frac{t-(t-1)}{t(t-1)} dt$

[$\because$ $t - (t - 1) = t - t + 1 = 1$]

$$= \frac{1}{4} \int \left(\frac{t}{t(t-1)} - \frac{(t-1)}{t(t-1)} \right) dt = \frac{1}{4} \int \left(\frac{1}{t-1} - \frac{1}{t} \right) dt$$

$$= \frac{1}{4} \left[\int \frac{1}{t-1} dt - \int \frac{1}{t} dt \right] = \frac{1}{4} [\log | t - 1 | - \log | t |] + c$$

$$= \frac{1}{4} \log \left| \frac{t-1}{t} \right| + c$$

Putting $t = x^4$, $= \frac{1}{4} \log \left| \frac{x^4 - 1}{x^4} \right| + c.$

Remark: Alternative solution is:

$$\frac{1}{x(x^4-1)} = \frac{1}{x(x^2-1)(x^2+1)} = \frac{1}{x(x-1)(x+1)(x^2+1)}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{Dx+E}{x^2+1}$$

But the solution given above is much better.

21. $\mathbf{\dfrac{1}{(e^x - 1)}}$

Sol. Let $I = \int \frac{1}{e^x - 1} dx \quad ...(i)$

Put $e^x = t$. Therefore $e^x = \frac{dt}{dx} \Rightarrow e^x \, dx = dt \Rightarrow dx = \frac{dt}{e^x}$

(Rule to evaluate $\int f(e^x) \, dx$, put $e^x = t$)

$\therefore$ From (i), $I = \int \frac{1}{t-1} \frac{dt}{e^x} = \int \frac{1}{t-1} \frac{dt}{t} = \int \frac{1}{t(t-1)} dt$

$$= \int \frac{t-(t-1)}{t(t-1)} dt = \int \left(\frac{t}{t(t-1)} - \frac{(t-1)}{t(t-1)} \right) dt = \int \frac{1}{t-1} dt - \int \frac{1}{t} dt$$

$$= \log | t - 1 | - \log | t | + c = \log \left| \frac{t-1}{t} \right| + c.$$

Putting $t = e^x$, $= \log \left| \frac{e^x - 1}{e^x} \right| + c$.

Choose the correct answer in each of the Exercises 22 and 23:

22. $\int \frac{x\,dx}{(x-1)(x-2)}$ **equals**

(A) $\log \left|\frac{(x-1)^2}{x-2}\right| + C$ **(B)** $\log \left|\frac{(x-2)^2}{x-1}\right| + C$

(C) $\log \left|\left(\frac{x-1}{x-2}\right)^2\right| + C$ **(D)** $\log |(x-1)(x-2)| + C.$

Sol. Let integrand $\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$ ...(*i*)

(Partial fractions)

Multiplying by L.C.M. $= (x-1)(x-2)$,

$$x = A(x-2) + B(x-1)$$
$$= Ax - 2A + Bx - B$$

Comparing coefficients of x and constant terms on both sides,

Coefficients of x: $A + B = 1$ ...(*ii*)

Constant terms: $-2A - B = 0$ ...(*iii*)

Let us solve (*ii*) and (*iii*) for A and B

Adding (*ii*) and (*iii*), $-A = 1$ or $A = -1$

Putting $A = -1$ in (*ii*) $-1 + B = 1$ or $B = 2$

Putting values of A and B in (*i*),

$$\frac{x}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{2}{x-2}$$

$$\therefore \int \frac{x}{(x-1)(x-2)}\,dx = -\int \frac{1}{x-1}\,dx + 2\int \frac{1}{x-2}\,dx$$
$$= -\log|x-1| + 2\log|x-2| + c$$
$$= \log|(x-2)^2| - \log|x-1| + c$$

$(\because\ n \log m = \log m^n)$

$$= \log\left|\frac{(x-2)^2}{x-1}\right| + c$$

$\therefore$ Option (B) is the correct answer.

23. $\int \frac{dx}{x(x^2+1)}$ **equals**

(A) $\log|x| - \frac{1}{2}\log(x^2+1) + C$

(B) $\log|x| + \frac{1}{2}\log(x^2+1) + C$

(C) $-\log|x| + \frac{1}{2}\log(x^2+1) + C$

(D) $\frac{1}{2}\log|x| + \log(x^2+1) + C.$

Sol. Let $I = \int \frac{1}{x(x^2+1)}\,dx$

Multiplying both numerator and denominator of integrand by $2x$.

$\left(\because \frac{d}{dx}(x^2+1) = 2x\right)$

$$\Rightarrow \quad I = \int \frac{2x}{2x^2(x^2+1)}\, dx \qquad ...(i)$$

Put $x^2 = t$. $\quad \therefore \quad 2x = \frac{dt}{dx} \Rightarrow 2x\, dx = dt$

$\therefore$ From (i), $I = \int \frac{dt}{2t(t+1)} = \frac{1}{2}\int \frac{1}{t(t+1)}\, dt$

Adding and subtracting t in the numerator of integrand,

$$= \frac{1}{2}\int \frac{(t+1)-t}{t(t+1)}\, dt = \frac{1}{2}\int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt$$

$$= \frac{1}{2}(\log|t| - \log|t+1|) + c$$

Putting $t = x^2$, $I = \frac{1}{2}(\log|x^2| - \log|x^2+1|) + c$

$$= \frac{1}{2}(2\log|x| - \log(x^2+1)) + c$$

$(\because \; x^2 + 1 \geq 1 > 0$ and hence $|x^2+1| = x^2+1)$

$$= \log|x| - \frac{1}{2}\log(x^2+1) + c$$

$\therefore$ Option (A) is the correct answer.