Exercise 7.3

Find the integrals of the following functions in Exercises 1 to 9:

1. $\sin^2(2x + 5)$

Sol. $\int \sin^2(2x+5)\ dx = \int \frac{1}{2}(1 - \cos 2(2x+5))\ dx$

$$\left[\because \ \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)\ ; \text{put } \theta = 2x+5\right]$$

$$= \frac{1}{2}\int (1-\cos(4x+10))\ dx \quad = \frac{1}{2}\left[\int 1\, dx - \int \cos(4x+10)\, dx\right]$$

$$= \frac{1}{2}\left[x - \frac{\sin(4x+10)}{4 \to \text{Coeff. of } x}\right] + c = \frac{1}{2}\ x - \frac{1}{8}\ \sin(4x+10) + c.$$

2. $\sin 3x \cos 4x$

Sol. $\int \sin 3x \cos 4x\ dx = \frac{1}{2}\int 2\sin 3x \cos 4x\ dx$

$$= \frac{1}{2}\int (\sin(3x+4x) + \sin(3x-4x))\ dx$$

$$[\because \ 2 \sin A \cos B = \sin(A+B) + \sin(A-B)]$$

$$= \frac{1}{2}\int (\sin 7x + \sin(-x))\ dx \quad = \frac{1}{2}\int (\sin 7x - \sin x)\ dx$$

$$= \frac{1}{2}\left[\int \sin 7x\, dx - \int \sin x\, dx\right] = \frac{1}{2}\left[\frac{-\cos 7x}{7} - (-\cos x)\right] + c$$

$$= \frac{-1}{14}\cos 7x + \frac{1}{2}\cos x + c.$$

3. $\cos 2x \cos 4x \cos 6x$

Sol. $\cos 2x \cos 4x \cos 6x = \frac{1}{2}(2\cos 6x \cos 4x)\cos 2x$

$$= \frac{1}{2}[\cos(6x+4x) + \cos(6x-4x)]\cos 2x$$

$$[\because \ 2\cos x \,.\, \cos y = \cos(x+y) + \cos(x-y)]$$

$= \frac{1}{2}$ (cos 10x + cos 2x) cos 2x $= \frac{1}{4}$ (2 cos 10x cos 2x + 2 $\cos^2$ 2x)

$= \frac{1}{4}$ [cos (10x + 2x) + cos (10x − 2x) + 1 + cos 4x]

$= \frac{1}{4}$ (cos 12x + cos 8x + cos 4x + 1)

$\therefore \int \cos 2x \cos 4x \cos 6x \; dx = \frac{1}{4} \int (\cos 12x + \cos 8x + \cos 4x + 1)\, dx$

$= \frac{1}{4} \left[\int \cos 12x\, dx + \int \cos 8x\, dx + \int \cos 4x\, dx + \int 1\, dx\right]$

$= \frac{1}{4} \left(\frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4} + x\right) + c.$

Note. We know that $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$

$\therefore \quad 4 \sin^3 \theta = 3 \sin \theta - \sin 3\theta$

Dividing by 4, $\sin^3 \theta = \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta$ ...(*i*)

Similarly, $\cos^3 \theta = \frac{3}{4} \cos \theta + \frac{1}{4} \cos 3\theta$ ...(*ii*)

[$\because \quad \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$]

4. $\sin^3 (2x + 1)$

Sol. To evaluate $\int \sin^3 (2x + 1)\; dx$

We know by Eqn. (*i*) of above note that $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4} \sin 3\theta$

Putting $\theta = 2x + 1$, we have

$\sin^3 (2x + 1) = \frac{3}{4} \sin (2x + 1) - \frac{1}{4} \sin 3 (2x + 1)$

$= \frac{3}{4} \sin (2x + 1) - \frac{1}{4} \sin (6x + 3)$

$\therefore \int \sin^3 (2x + 1)\; dx = \frac{3}{4} \int \sin (2x + 1)\; dx - \frac{1}{4} \int \sin (6x + 3)\; dx$

$= \frac{3}{4} \left(\frac{-\cos (2x + 1)}{2}\right) - \frac{1}{4} \left(\frac{-\cos (6x + 3)}{6 \to \text{Coeff. of } x}\right) + c$

$= \frac{-3}{8} \cos (2x + 1) + \frac{1}{24} \cos (6x + 3) + c.$

OR

To integrate $\sin^n x$ where n is odd, put $\cos x = t$.

$\therefore \quad \int \sin^3 (2x + 1)\; dx = \int \sin^2 (2x + 1)\; \sin (2x + 1)\, dx$

$= \frac{-1}{2} \int [1 - \cos^2 (2x + 1)]\; (-2 \sin (2x + 1))\, dx$ ...(*i*)

Put cos (2x + 1) = t

$\therefore \quad -\sin (2x + 1) \frac{d}{dx} (2x + 1) = \frac{dt}{dx} \quad \therefore -2 \sin (2x + 1)\, dx = dt$

$\therefore \quad$ From (*i*), the given integral $= \frac{-1}{2} \int (1 - t^2)\; dt$

$$= \frac{-1}{2}\left(t - \frac{t^3}{3}\right) + c = \frac{-1}{2}\ t + \frac{1}{6}\ t^3 + c$$

$$= \frac{-1}{2}\ \cos\ (2x + 1) + \frac{1}{6}\ \cos^3\ (2x + 1) + c.$$

5. $\sin^3 x \cos^3 x$

Sol. $\int \sin^3 x \cos^3 x \ dx = \int (\sin x \cos x)^3 \ dx$

$$= \int \left(\frac{1}{2} 2 \sin x \cos x\right)^3 \ dx = \int \left(\frac{1}{2} \sin 2x\right)^3 \ dx$$

$$= \frac{1}{8} \int \sin^3 2x \ dx = \frac{1}{8} \int \left(\frac{3}{4} \sin 2x - \frac{1}{4} \sin 6x\right) dx$$

$$\left(\text{Putting } \theta = 2x \text{ in } \sin^3 \theta = \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta\right)$$

$$= \frac{3}{32} \int \sin 2x \ dx - \frac{1}{32} \int \sin 6x \ dx$$

$$= \frac{-3}{32} \frac{\cos 2x}{2} - \frac{1}{32}\left(\frac{-\cos 6x}{6}\right) + c = \frac{-3}{64} \cos 2x + \frac{1}{192} \cos 6x + c.$$

OR

To evaluate $\int \sin^3 x \cos^3 x \, dx$, Put either $\sin x = t$ or $\cos x = t$. (The form of answer given in N.C.E.R.T. book II can be obtained by putting $\cos x = t$)

6. $\sin x \sin 2x \sin 3x$

Sol. $\sin x \sin 2x \sin 3x = \frac{1}{2} (2 \sin 3x \sin 2x) \sin x$

$$= \frac{1}{2} [\cos (3x - 2x) - \cos (3x + 2x)] \sin x$$

$[\because \ 2 \sin x \sin y = \cos (x - y) - \cos (x + y)]$

$$= \frac{1}{2} (\cos x - \cos 5x) \sin x = \frac{1}{4} (2 \cos x \sin x - 2 \cos 5x \sin x)$$

$$= \frac{1}{4} [\sin 2x - \{\sin (5x + x) - \sin (5x - x)\}]$$

$[\because \ 2 \cos x \sin y = \sin (x + y) - \sin (x - y)]$

$$= \frac{1}{4} (\sin 2x - \sin 6x + \sin 4x)$$

$$\therefore \int \sin x \sin 2x \sin 3x \ dx = \frac{1}{4} \int (\sin 2x + \sin 4x - \sin 6x) \ dx$$

$$= \frac{1}{4} \left[\int \sin 2x \, dx + \int \sin 4x \, dx - \int \sin 6x \, dx\right]$$

$$= \frac{1}{4} \left(-\frac{\cos 2x}{2} - \frac{\cos 4x}{4} + \frac{\cos 6x}{6}\right) + c.$$

7. $\sin 4x \sin 8x$

Sol. $\int \sin 4x \sin 8x \ dx = \frac{1}{2} \int 2 \sin 4x \sin 8x \ dx$

$$= \frac{1}{2} \int [\cos (4x - 8x) - \cos (4x + 8x)] \ dx$$

$[\because \ 2 \sin A \sin B = \cos (A - B) - \cos (A + B)]$

$$= \frac{1}{2} \int (\cos(-4x) - \cos 12x)\ dx = \frac{1}{2} \int (\cos 4x - \cos 12x)\ dx$$

$$[\because \quad \cos(-\theta) = \cos\theta]$$

$$= \frac{1}{2}\left[\int \cos 4x\, dx - \int \cos 12x\, dx\right] = \frac{1}{2}\left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12}\right] + c.$$

8. $\dfrac{\mathbf{1 - \cos x}}{\mathbf{1 + \cos x}}$

Sol. $$\int \frac{1 - \cos x}{1 + \cos x}\ dx = \int \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}}\ dx = \int \tan^2 \frac{x}{2}\ dx$$

$$\left(\because \quad 1 - \cos\theta = 2\sin^2\frac{\theta}{2} \text{ and } 1 + \cos\theta = 2\cos^2\frac{\theta}{2}\right)$$

$$= \int \left(\sec^2 \frac{x}{2} - 1\right) dx \qquad (\because \quad \tan^2\theta = \sec^2\theta - 1)$$

$$= \int \sec^2 \frac{x}{2}\ dx - \int 1\,dx = \frac{\tan\frac{x}{2}}{\frac{1}{2} \to \text{Coeff. of } x} - x + c = 2\tan\frac{x}{2} - x + c.$$

9. $\dfrac{\mathbf{\cos x}}{\mathbf{1 + \cos x}}$

Sol. $$\int \frac{\cos x}{1 + \cos x}\ dx$$

Adding and subtracting 1 in the numerator of integrand,

$$= \int \frac{1 + \cos x - 1}{1 + \cos x}\,dx = \int \left(\frac{1 + \cos x}{1 + \cos x} - \frac{1}{1 + \cos x}\right) dx \quad \left(\because \frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}\right)$$

$$= \int \left(1 - \frac{1}{2\cos^2 \frac{x}{2}}\right) dx = \int 1\ dx - \frac{1}{2} \int \sec^2 \frac{x}{2}\ dx$$

$$= x - \frac{1}{2}\,\frac{\tan\frac{x}{2}}{\frac{1}{2}} + c = x - \tan\frac{x}{2} + c.$$

Find the integrals of the functions in Exercises 10 to 18:

10. $\mathbf{\sin^4 x}$

Sol. $$\int \sin^4 x\ dx = \int (\sin^2 x)^2\ dx = \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx$$

$$= \int \frac{(1 - \cos 2x)^2}{4}\ dx = \frac{1}{4} \int (1 + \cos^2 2x - 2\cos 2x)\ dx$$

$$= \frac{1}{4} \int \left(1 + \left(\frac{1 + \cos 4x}{2}\right) - 2\cos 2x\right) dx \quad \left[\because \quad \cos^2\theta = \frac{1 + \cos 2\theta}{2}\right]$$

$$= \frac{1}{4}\int\left(\frac{2+1+\cos 4x-4\cos 2x}{2}\right)dx = \frac{1}{8}\int (3+\cos 4x - 4\cos 2x)\,dx$$

$$= \frac{1}{8}\left[3\int 1\,dx + \int \cos 4x\,dx - 4\int \cos 2x\,dx\right]$$

$$= \frac{1}{8}\left[3x + \frac{\sin 4x}{4} - \frac{4\sin 2x}{2}\right] + c = \frac{3}{8}x + \frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + c$$

11. $\cos^4 2x$

Sol. $\int \cos^4 2x \; dx = \int (\cos^2 2x)^2 \; dx$

$$= \int\left(\frac{1+\cos 4x}{2}\right)^2 dx = \int \frac{1}{4}(1+\cos 4x)^2 \; dx$$

$$= \frac{1}{4}\int (1+\cos^2 4x + 2\cos 4x)\,dx$$

$$= \frac{1}{4}\int\left(1 + \frac{1+\cos 8x}{2} + 2\cos 4x\right) dx \qquad \left[\because \; \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$= \frac{1}{4}\int\left(\frac{2+1+\cos 8x+4\cos 4x}{2}\right)dx = \frac{1}{8}\int (3+\cos 8x + 4\cos 4x)\,dx$$

$$= \frac{1}{8}\left[3\int 1\,dx + \int \cos 8x\,dx + 4\int \cos 4x\,dx\right]$$

$$= \frac{1}{8}\left[3x + \frac{\sin 8x}{8} + \frac{4\sin 4x}{4}\right] + c = \frac{3}{8}x + \frac{1}{64}\sin 8x + \frac{1}{8}\sin 4x + c$$

12. $\dfrac{\sin^2 x}{1+\cos x}$

Sol. $\displaystyle\int \frac{\sin^2 x}{1+\cos x}\,dx = \int \frac{1-\cos^2 x}{1+\cos x}\,dx = \int \frac{(1-\cos x)(1+\cos x)}{1+\cos x}\,dx$

$$= \int (1-\cos x)\; dx = \int 1\; dx - \int \cos x \; dx \quad = x - \sin x + c.$$

Note. It may be noted that letters $a, b, c, d, \ldots, q$ of English Alphabet and letters $\alpha, \beta, \gamma, \delta$ of Greek Alphabet are generally treated as constants.

13. $\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$

Sol. $\displaystyle\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\,dx = \int \frac{(2\cos^2 x - 1) - (2\cos^2\alpha - 1)}{\cos x - \cos\alpha}\,dx$

$$= \int \frac{2\cos^2 x - 1 - 2\cos^2\alpha + 1}{\cos x - \cos\alpha}\,dx \qquad = \int \frac{2\cos^2 x - 2\cos^2\alpha}{\cos x - \cos\alpha}\,dx$$

$$= 2\int \frac{\cos^2 x - \cos^2\alpha}{\cos x - \cos\alpha}\,dx \qquad = 2\int \frac{(\cos x - \cos\alpha)(\cos x + \cos\alpha)}{(\cos x - \cos\alpha)}\,dx$$

$$= 2\int (\cos x + \cos\alpha)\; dx \qquad = 2\left[\int \cos x\,dx + \int \cos\alpha\,dx\right]$$

$$= 2\,[\sin x + \boldsymbol{\cos\alpha} \int \mathbf{1} \;\; \boldsymbol{dx}] = 2\,[\sin x + (\cos\alpha)\,x] + c$$

$$= 2\sin x + 2x\cos\alpha + c.$$

Remark. $\int \sin a \; dx = \sin a \int 1 \; dx = x \sin a$.

Please note that $\int \sin a \, dx \neq -\cos a$.

14. $\dfrac{\cos x - \sin x}{1 + \sin 2x}$

Sol. Let $I = \int \dfrac{\cos x - \sin x}{1 + \sin 2x} dx = \int \dfrac{\cos x - \sin x}{\cos^2 x + \sin^2 x + 2 \sin x \cos x} \; dx$

$$= \int \frac{\cos x - \sin x}{(\cos x + \sin x)^2} \; dx \qquad ...(i)$$

Put $\cos x + \sin x = t$.

$\therefore -\sin x + \cos x = \dfrac{dt}{dx}$. Therefore $(\cos x - \sin x)\, dx = dt$.

$\therefore$ From (*i*), $I = \int \dfrac{dt}{t^2} = \int t^{-2} \; dt = \dfrac{t^{-1}}{-1} + c$

$\Rightarrow \; I = \dfrac{-1}{t} + c = \dfrac{-1}{\cos x + \sin x} + c.$

15. $\tan^3 2x \sec 2x$

Sol. Let $I = \int \tan^3 2x \; \sec 2x \; dx = \int \tan^2 2x \tan 2x \sec 2x \; dx$

$= \int (\sec^2 2x - 1) \sec 2x \tan 2x \; dx \qquad [\because \; \tan^2 \theta = \sec^2 \theta - 1]$

$= \dfrac{1}{2} \int (\sec^2 2x - 1)(2 \sec 2x \tan 2x) \; dx \qquad ...(i)$

Put $\sec 2x = t$. Therefore $\sec 2x \tan 2x \dfrac{d}{dx}(2x) = \dfrac{dt}{dx}$

$\therefore \; 2 \sec 2x \tan 2x \; dx = dt$

$\therefore$ From (*i*), $I = \dfrac{1}{2} \int (t^2 - 1) \; dt = \dfrac{1}{2} \left(\int t^2 \, dt - \int 1 \, dt \right)$

$$= \frac{1}{2}\left(\frac{t^3}{3} - t\right) + c = \frac{1}{6} t^3 - \frac{1}{2} t + c$$

Putting $t = \sec 2x$, $= \dfrac{1}{6} \sec^3 2x - \dfrac{1}{2} \sec 2x + c$.

16. $\tan^4 x$

Sol. $\int \tan^4 x \, dx = \int \tan^2 x \tan^2 x \; dx = \int \tan^2 x (\sec^2 x - 1) \; dx$

$= \int (\tan^2 x \sec^2 x - \tan^2 x) \; dx = \int \tan^2 x \sec^2 x \; dx - \int \tan^2 x \; dx$

$= \int \tan^2 x \sec^2 x \; dx - \int (\sec^2 x - 1) \; dx$

$= \int \tan^2 x \sec^2 x \; dx - \int \sec^2 x \; dx + \int 1 \; dx$

$\downarrow$

For this integral, put **$\tan x = t$.**

$\therefore \; \sec^2 x = \dfrac{dt}{dx}$ or $\sec^2 x \; dx = dt$

$$= \int t^2 \, dt - \tan x + x + c = \frac{t^3}{3} - \tan x + x + c$$

Put $t = \tan x$, $= \frac{1}{3} \tan^3 x - \tan x + x + c.$

17. $\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$

Sol. $\displaystyle\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} \, dx = \int \left(\frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x} \right) dx$

$$\left(\because \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \right)$$

$$= \int \left(\frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x} \right) dx = \int \left(\frac{\sin x}{\cos x \cos x} + \frac{\cos x}{\sin x \sin x} \right) dx$$

$$= \int (\tan x \sec x + \cot x \operatorname{cosec} x) \, dx$$

$$= \int \sec x \tan x \, dx + \int \operatorname{cosec} x \cot x \, dx = \sec x - \operatorname{cosec} x + c.$$

18. $\dfrac{\cos 2x + 2 \sin^2 x}{\cos^2 x}$

Sol. $\displaystyle\int \frac{\cos 2x + 2 \sin^2 x}{\cos^2 x} \, dx = \int \frac{(1 - 2 \sin^2 x) + 2 \sin^2 x}{\cos^2 x} \, dx$

$$= \int \frac{1}{\cos^2 x} \, dx = \int \sec^2 x \, dx = \tan x + c.$$

Integrate the functions in Exercises 19 to 22:

Note. Method to evaluate $\displaystyle\int \frac{1}{\sin^p x \cos^q x} \, dx$ if $(p + q)$ is a negative even integer $(= -n$ (say)); then multiply Numerator and Denominator of integrand by $\sec^n x$.

19. $\dfrac{1}{\sin x \cos^3 x}$

Sol. Let $\mathrm{I} = \displaystyle\int \frac{1}{\sin x \cos^3 x} \, dx$ ...(*i*)

Here $p + q = -1 - 3 = -4$ is a negative even integer.
So multiplying both Numerator and Denominator of integrand of (*i*) by $\sec^4 x$,

$$\mathrm{I} = \int \frac{\sec^4 x}{\sin x \cos^3 x \sec^4 x} \, dx = \int \frac{\sec^4 x}{\tan x} \, dx$$

$$\left(\because \sin x \cos^3 x \sec^4 x = \sin x \cos^3 x \cdot \frac{1}{\cos^4 x} = \frac{\sin x}{\cos x} = \tan x \right)$$

or $\mathrm{I} = \displaystyle\int \frac{\sec^2 x \sec^2 x}{\tan x} \, dx = \int \frac{(1 + \tan^2 x) \sec^2 x}{\tan x} \, dx$ ...(*ii*)

Put $\tan x = t$

$\therefore \quad \sec^2 x = \frac{dt}{dx} \quad \Rightarrow \quad \sec^2 x\, dx = dt$

$\therefore \quad$ From (*ii*), $I = \int \frac{(1+t^2)}{t}\, dt = \int \left(\frac{1}{t} + \frac{t^2}{t}\right) dt$

$= \int \left(\frac{1}{t} + t\right) dt = \int \frac{1}{t}\, dt + \int t\, dt = \log |t| + \frac{t^2}{2} + c$

Putting $t = \tan x$, $= \log |\tan x| + \frac{1}{2} \tan^2 x + c.$

20. $\frac{\cos 2x}{(\cos x + \sin x)^2}$

Sol. Let $I = \int \frac{\cos 2x}{(\cos x + \sin x)^2}\, dx = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2}\, dx$

$= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)(\cos x + \sin x)}\, dx = \int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx \quad ...(i)$

Put DENOMINATOR $\cos x + \sin x = t$

$\therefore \quad -\sin x + \cos x = \frac{dt}{dx} \quad \Rightarrow \quad (\cos x - \sin x)\, dx = dt$

$\therefore \quad$ From (*i*), $I = \int \frac{dt}{t} = \log |t| + c = \log |\cos x + \sin x| + c$

Note. Another method to evaluate integral (*i*) is, apply

$$\int \frac{f'(x)}{f(x)}\, dx = \log |f(x)|.$$

21. $\sin^{-1} (\cos x)$

Sol. $\int \sin^{-1} (\cos x)\, dx = \int \sin^{-1} \sin \left(\frac{\pi}{2} - x\right) dx$

$= \int \left(\frac{\pi}{2} - x\right) dx = \int \frac{\pi}{2}\, dx - \int x\, dx$

$= \frac{\pi}{2} \int 1\, dx - \int x^1\, dx = \frac{\pi}{2} x - \frac{x^2}{2} + c.$

22. $\frac{1}{\cos (x-a) \cos (x-b)}$

Sol. Let $I = \int \frac{1}{\cos (x-a) \cos (x-b)}\, dx \quad ...(i)$

Here $(x - a) - (x - b) = x - a - x + b = b - a \quad ...(ii)$

By looking at Eqn. (*ii*), dividing and multiplying the integrand in (*i*) by $\sin (b - a)$,

$I = \frac{1}{\sin (b-a)} \int \frac{\sin (b-a)}{\cos (x-a) \cos (x-b)}\, dx$

$= \frac{1}{\sin (b-a)} \int \frac{\sin [(x-a) - (x-b)]}{\cos (x-a) \cos (x-b)}\, dx \quad$ [By (*ii*)]

$= \frac{1}{\sin (b-a)} \int \frac{\sin (x-a) \cos (x-b) - \cos (x-a) \sin (x-b)}{\cos (x-a) \cos (x-b)}\, dx$

[$\because \sin (A - B) = \sin A \cos B - \cos A \sin B$]

$$= \frac{1}{\sin(b-a)} \int \left[\frac{\sin(x-a)\cos(x-b)}{\cos(x-a)\cos(x-b)} - \frac{\cos(x-a)\sin(x-b)}{\cos(x-a)\cos(x-b)} \right] dx$$

$$\left(\because \frac{A-B}{C} = \frac{A}{C} - \frac{B}{C} \right)$$

$$= \frac{1}{\sin(b-a)} \int [\tan(x-a) - \tan(x-b)]\, dx$$

$$= \frac{1}{\sin(b-a)} [-\log|\cos(x-a)| + \log|\cos(x-b)|] + c$$

$$\left(\because \int \tan x\, dx = -\log|\cos x|\right)$$

$$= \frac{1}{\sin(b-a)} \log\left|\frac{\cos(x-b)}{\cos(x-a)}\right| + c. \quad \left(\because \log m - \log n = \log\frac{m}{n}\right)$$

Choose the correct answer in Exercises 23 and 24:

23. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\, dx$ **is equal to**

(A) $\tan x + \cot x + C$ **(B)** $\tan x + \operatorname{cosec} x + C$

(C) $-\tan x + \cot x + C$ **(D)** $\tan x + \sec x + C$

Sol. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\, dx$

$$= \int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} - \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) dx \qquad \left[\because \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}\right]$$

$$= \int \left(\frac{1}{\cos^2 x} - \frac{1}{\sin^2 x} \right) dx = \int (\sec^2 x - \operatorname{cosec}^2 x)\, dx$$

$$= \int \sec^2 x\, dx - \int \operatorname{cosec}^2 x\, dx = \tan x - (-\cot x) + C$$

$= \tan x + \cot x + C$ $\quad \therefore$ Option (A) is the correct answer.

24. $\int \frac{e^x(1+x)}{\cos^2(e^x x)}\, dx$ **equals**

(A) $-\cot(ex^x) + C$ **(B)** $\tan(xe^x) + C$

(C) $\tan(e^x) + C$ **(D)** $\cot(e^x) + C$

Sol. Let $I = \int \frac{e^x(1+x)}{\cos^2(e^x x)}\, dx$ ...(*i*)

Put $e^x \cdot x = t$

[To evaluate $\int$ (T-function or Inverse T-function $f(x)$) $f'(x)\, dx$, put $f(x) = t$]

Applying Product Rule, $e^x \cdot 1 + xe^x = \frac{dt}{dx}$

or $e^x(1+x)\, dx = dt$

$\therefore$ From (*i*), $I = \int \frac{dt}{\cos^2 t} = \int \sec^2 t\, dt$

$= \tan t + C = \tan(x e^x) + C$ $\therefore$ Option (B) is the correct answer.