NCERT Class 12 Maths Solutions

Chapter - 6

Exercise 6.5

1. Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = (2x - 1)^2 + 3$ **(ii)** $f(x) = 9x^2 + 12x + 2$

(iii) $f(x) = -(x-1)^2 + 10$ **(iv)** $g(x) = x^3 + 1$.

Sol. (i) **Given:** $f(x) = (2x - 1)^2 + 3$

We know that for all $x \in R$, $(2x - 1)^2 \geq 0$

$\Rightarrow$ Adding 3 to both sides, $(2x - 1)^2 + 3 \geq 0 + 3$

$\Rightarrow$ $f(x) \geq 3$

The minimum value of $f(x)$ is 3 and is obtained when $2x - 1 = 0$, [$\because$ Minimum value of $(2x - 1)^2$ is 0] *i.e.*, when $x = \frac{1}{2}$. There is no maximum value of $f(x)$.

[$\because$ Maximum value of $f(x) = (2x - 1)^2 + 3 \to \infty$ as $x \to \infty$ and hence does not exist].

(ii) **Given:** $f(x) = 9x^2 + 12x + 2$

Making coefficient of x^2 unity,

$$= 9\left[x^2 + \frac{12x}{9} + \frac{2}{9}\right] = 9\left[x^2 + \frac{4x}{3} + \frac{2}{9}\right]$$

Add and subtract $\left(\frac{1}{2}\text{ coeff. of } x\right)^2$

$$= \left(\frac{1}{2} \times \frac{4}{3}\right)^2 = \left(\frac{2}{3}\right)^2,$$

$$= 9\left[x^2 + \frac{4x}{3} + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2 + \frac{2}{9}\right] = 9\left[\left(x + \frac{2}{3}\right)^2 - \frac{4}{9} + \frac{2}{9}\right]$$

or $f(x) = 9\left(x + \frac{2}{3}\right)^2 - 4 + 2 = 9\left(x + \frac{2}{3}\right)^2 - 2$ ...(*i*)

We know that for all $x \in$ R, $9\left(x + \frac{2}{3}\right)^2 \geq 0$

Adding – 2 to both sides, $9\left(x + \frac{2}{3}\right)^2 - 2 \geq -2$

$\Rightarrow$ Using (*i*), $f(x) \geq -2$

$\therefore$ Minimum value of $f(x)$ is – 2 and is obtained when

$$x + \frac{2}{3} = 0 \text{ i.e., when } x = -\frac{2}{3}.$$

From (*i*), maximum value of $f(x) \to \infty$ as $x \to \infty$ (or as $x \to -\infty$) and hence does not exist.

(*iii*) **Given:** $f(x) = -(x - 1)^2 + 10$ ...(*i*)

We know that for all $x \in$ R, $(x - 1)^2 \geq 0$

Multiplying by – 1, $-(x - 1)^2 \leq 0$

Adding 10 to both sides, $-(x - 1)^2 + 10 \leq 10$

$\Rightarrow$ Using (*i*), $f(x) \leq 10$

$\therefore$ Maximum value of $f(x)$ is 10 and is obtained when $x - 1 = 0$ *i.e.,* when $x = 1$.

From (*i*), minimum value of $f(x) \to -\infty$ as $x \to \infty$ (or as $x \to -\infty$) and hence does not exist.

(*iv*) **Given:** $g(x) = x^3 + 1$ ...(*i*)

As $x \to \infty$, $g(x) \to \infty$

As $x \to -\infty$, $g(x) \to -\infty$

$\therefore$ Maximum value of $g(x)$ does not exist and also minimum value of $g(x)$ does not exist.

2. Find the maximum and minimum values, if any, of the following functions given by

(*i*) $f(x) = |x + 2| - 1$ (*ii*) $g(x) = -|x + 1| + 3$

(*iii*) $h(x) = \sin(2x) + 5$ (*iv*) $f(x) = |\sin 4x + 3|$

(*v*) $h(x) = x + 1, x \in (-1, 1)$

Sol. (*i*) **Given:** $f(x) = |x + 2| - 1$ ...(*i*)

We know that for all $x \in$ R, $|x + 2| \geq 0$

Adding – 1 to both sides, $|x + 2| - 1 \geq -1$

$\Rightarrow$ Using (*i*), $f(x) \geq -1$

$\therefore$ Minimum value of $f(x)$ is -1 and is obtained when $x + 2 = 0$ *i.e.,* when $x = -2$.

From (*i*), maximum value of $f(x) \to \infty$ as $x \to \infty$ (or as $x \to -\infty$) and hence does not exist.

(*ii*) **Given:** $g(x) = -|x+1| + 3$

We know that for all $x \in R$, $|x+1| \geq 0$

Multiplying by -1 to both sides

$\Rightarrow \quad -|x+1| \leq 0$

Adding 3 to both sides,

$\Rightarrow -|x+1| + 3 \leq 3 \quad \Rightarrow g(x) \leq 3$

$\therefore$ The maximum value of $g(x)$ is 3 and is obtained when $|x+1| = 0$, *i.e.,* when $x + 1 = 0$ *i.e.,* when $x = -1$. There is no minimum value of $g(x)$. [Because minimum value of $g(x) = -|x+1| + 3 \to -\infty$ as $x \to \pm\infty$ and hence does not exist].

(*iii*) **Given:** $h(x) = \sin(2x) + 5$ ...(*i*)

We know that for all $x \in R$, $-1 \leq \sin 2x \leq 1$

Adding 5 to all sides, $-1 + 5 \leq \sin 2x + 5 \leq 1 + 5$

$\Rightarrow \quad 4 \leq h(x) \leq 6$ (By (*i*))

$\therefore$ Minimum value of $h(x)$ is 4 and maximum value is 6.

(*iv*) **Given:** $f(x) = |\sin 4x + 3|$

We know that for all $x \in R$, $-1 \leq \sin 4x \leq 1$

Adding 3 throughout,

$2 \leq \sin 4x + 3 \leq 4 \quad \Rightarrow 2 \leq |\sin 4x + 3| \leq 4$

[$\because$ $\sin 4x + 3 \geq 2$ and hence > 0, $\therefore$ $|\sin 4x + 3| = \sin 4x + 3$]

$\therefore$ The minimum value of $f(x)$ is 2 and the maximum value of $f(x)$ is 4.

(*v*) **Given:** $h(x) = x + 1$, $x \in (-1, 1)$ ...(*i*)

Given: $x \in (-1, 1) \quad \Rightarrow \quad -1 < x < 1$

Adding 1 to all sides, $1 - 1 < x + 1 < 1 + 1$

i.e., $0 < h(x)$ [By (*i*)] < 2 ...(*ii*)

$\therefore$ Neither minimum value nor maximum value of $h(x)$ exists.

($\because$ Equality sign is absent at both ends of inequality (*ii*). We know from (*ii*) that minimum value of $h(x)$ is > 0 and maximum value is < 2 but what exactly they are can't be said).

3. Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

(*i*) $f(x) = x^2$ **(*ii*)** $g(x) = x^3 - 3x$

(*iii*) $h(x) = \sin x + \cos x,\ 0 < x < \dfrac{\pi}{2}$

(*iv*) $f(x) = \sin x - \cos x,\ 0 < x < 2\pi$

(*v*) $f(x) = x^3 - 6x^2 + 9x + 15$ **(*vi*)** $g(x) = \dfrac{x}{2} + \dfrac{2}{x},\ x > 0.$

(vii) $g(x) = \dfrac{1}{x^2 + 2}$ **(viii)** $f(x) = x\sqrt{1 - x}$, $x > 0$.

Sol. (*i*) **Given:** $f(x) = x^2$ ...(*i*)

$\therefore \quad f'(x) = 2x$ and $f''(x) = 2$

Putting $f'(x) = 0$ to get turning points, we have $2x = 0$

or $x = \dfrac{0}{2} = 0$

(Turning point)

Let us apply second derivative test.

When $x = 0$, $f''(x) = 2$ (positive)

$\therefore$ $x = 0$ is a point of local minima and local minimum value $= f(0) = 0^2$ [From (*i*)]

$= 0$

Therefore, local minima at $x = 0$ and local minimum value $= 0$.

(*ii*) **Given:** $g(x) = x^3 - 3x$ ...(*i*)

$\therefore \quad g'(x) = 3x^2 - 3$ and $g''(x) = 6x$

Putting $g'(x) = 0$ to get turning points, we have

$3x^2 - 3 = 0$ or $3(x^2 - 1) = 0$ or $3(x + 1)(x - 1) = 0$

But $3 \neq 0$. Therefore, either $x + 1 = 0$ or $x - 1 = 0$

i.e., $x = -1$ or $x = 1$. (Turning points)

Let us apply second derivative test.

At $x = -1$, $g''(x) = 6x = 6(-1) = -6$ (Negative)

$\therefore$ $x = -1$ is a point of local maxima and local maximum value $= g(-1) = (-1)^3 - 3(-1)$ [From (*i*)] $= -1 + 3 = 2$.

At $x = 1$, $g''(x) = 6x = 6(1) = 6$ (positive)

$\therefore$ $x = 1$ is a point of local minima and local minimum value

$= g(1) = 1^3 - 3(1) = 1 - 3 = -2$

Therefore, Local maximum at $x = -1$ and local maximum value $= 2$. Local minimum at $x = 1$ and local minimum value $= -2$.

(*iii*) **Given:** $h(x) = \sin x + \cos x$ $\left(0 < x < \dfrac{\pi}{2}\right)$ ...(*i*)

$\therefore$ $h'(x) = \cos x - \sin x$ and $h''(x) = -\sin x - \cos x$

Putting $h'(x) = 0$ to get turning points,

we have $\cos x - \sin x = 0 \Rightarrow -\sin x = -\cos x$

Dividing by $-\cos x$, $\dfrac{\sin x}{\cos x} = 1$ or $\tan x = 1$

which is positive. Therefore, x can have values in both Ist and IIIrd quadrants.

Here, x is only in Ist quadrant

$\left[\because \quad 0 < x < \dfrac{\pi}{2} \text{ (given)}\right]$

$\therefore \quad \tan x = 1 = \tan \frac{\pi}{4}$

$\therefore \quad x = \frac{\pi}{4}$ (only turning point)

At $\quad x = \frac{\pi}{4}$, $h''(x) = -\sin x - \cos x$

$$= -\sin \frac{\pi}{4} - \cos \frac{\pi}{4} = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}} = -\sqrt{2} \text{ is negative.}$$

$\therefore x = \frac{\pi}{4}$ is a point of local maxima and local maximum value

$$= h\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} + \cos \frac{\pi}{4} \qquad \text{(From } (i))$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$\therefore$ Local maximum at $x = \frac{\pi}{4}$, and local maximum value $= \sqrt{2}$.

(*iv*) **Given:** $f(x) = \sin x - \cos x$ ...(*i*) $(0 < x < 2\pi)$

$\therefore \quad f'(x) = \cos x + \sin x$

and $\quad f''(x) = -\sin x + \cos x$

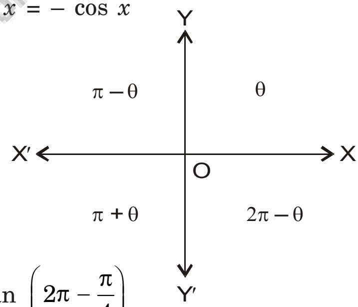
Putting $f'(x) = 0$ to get turning points, we have

$$\cos x + \sin x = 0 \Rightarrow \sin x = -\cos x$$

Dividing by $\cos x$, $\frac{\sin x}{\cos x} = -1$

$\Rightarrow \quad \tan x = -1$ is negative.

Therefore, x is in both second and fourth quadrants.



$\therefore \quad \tan x = -1 = -\tan \frac{\pi}{4}$

$$= \tan\left(\pi - \frac{\pi}{4}\right) \text{ or } \tan\left(2\pi - \frac{\pi}{4}\right)$$

$\Rightarrow \quad \tan x = \tan \frac{3\pi}{4}$ or $\tan \frac{7\pi}{4}$

$\therefore \quad x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$ (Turning points)

Let us apply second derivative test.

At $x = \frac{3\pi}{4}$, $f''(x) = -\sin x + \cos x = -\sin \frac{3\pi}{4} + \cos \frac{3\pi}{4}$

$$= -\sin \frac{4\pi - \pi}{4} + \cos \frac{4\pi - \pi}{4} = -\sin\left(\pi - \frac{\pi}{4}\right) + \cos\left(\pi - \frac{\pi}{4}\right)$$

$$= -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}} = -\sqrt{2} \text{ (negative)}$$

$\therefore\ x = \frac{3\pi}{4}$ is a point of local maxima and local maximum value

$$= f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} \qquad \text{(From } (i))$$

$$= \sin\left(\pi - \frac{\pi}{4}\right) - \cos\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

At $x = \frac{7\pi}{4}$,

$$f''(x) = -\sin x + \cos x = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$$

$$= -\sin\left(\frac{8\pi - \pi}{4}\right) + \cos\left(\frac{8\pi - \pi}{4}\right)$$

$$= -\sin\left(2\pi - \frac{\pi}{4}\right) + \cos\left(2\pi - \frac{\pi}{4}\right)$$

$$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \quad \text{(Positive)}$$

$\therefore\ x = \frac{7\pi}{4}$ is a point of local minima and local minimum value

$$= f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} \quad \text{(From } (i))$$

$$= \sin\left(2\pi - \frac{\pi}{4}\right) - \cos\left(2\pi - \frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$$

$$= \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}.$$

Therefore, Local maxima at $x = \frac{3\pi}{4}$ and local maximum value $= \sqrt{2}$. Local minima at $x = \frac{7\pi}{4}$ and local minimum value $= -\sqrt{2}$.

(*v*) **Given:** $f(x) = x^3 - 6x^2 + 9x + 15$ ...(*i*)

$\therefore\quad f'(x) = 3x^2 - 12x + 9 \quad$ and $\quad f''(x) = 6x - 12$

Putting $f'(x) = 0$ to get turning points, we have

$$3x^2 - 12x + 9 = 0$$

Dividing by 3, $x^2 - 4x + 3 = 0$

or $\quad x^2 - x - 3x + 3 = 0 \quad$ or $\quad (x - 1)(x - 3) = 0$

$\therefore$ Either $x - 1 = 0$ or $x - 3 = 0$

i.e., $x = 1$ or $x = 3$. (Turning points)

Let us apply second derivative test.

When $x = 1$, $f''(x) = 6x - 12$

$= 6 - 12 = -6$ (negative)

$\therefore$ $x = 1$ is a point of local maxima and local maximum value

$= f(1) = (1)^3 - 6(1)^2 + 9(1) + 15 = 1 - 6 + 9 + 15 = 19$

When $x = 3$ $f''(x) = 6x - 12 = 6(3) - 12 = 6$ (positive)

$\therefore$ $x = 3$ is a point of local minima and local minimum value

$= f(3) = (3)^3 - 6(3)^2 + 9(3) + 15$

$= 27 - 54 + 27 + 15 = 15$

Therefore, Local maxima at $x = 1$ and local maximum value = 19. Local minima at $x = 3$ and local minimum value = 15.

(*vi*) **Given:** $g(x) = \dfrac{x}{2} + \dfrac{2}{x}, x > 0$

$$\therefore \quad g'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2} = \frac{(x+2)(x-2)}{2x^2} \quad ...(i)$$

For turning points, putting $g'(x) = 0$

$$\Rightarrow \quad \frac{(x+2)(x-2)}{2x^2} = 0$$

$\Rightarrow (x + 2)(x - 2) = 0 \quad \Rightarrow x = -2, 2$

But $x > 0$ (given) $\therefore$ $x = -2$ is rejected.

Hence $x = 2$ is the only turning point.

Let us apply first derivative test

When x is slightly < 2, let $x = 1.9$

From (*i*), $$g'(1.9) = \frac{(1.9+2)(1.9-2)}{2(1.9)^2} = \frac{(+\text{ ve})(-\text{ ve})}{(+\text{ ve})} = -\text{ ve}$$

When x is slightly > 2, let $x = 2.1$

$$g'(2.1) = \frac{(2.1+2)(2.1-2)}{2(2.1)^2} = \frac{(+\text{ ve})(+\text{ ve})}{(+\text{ ve})} = +\text{ ve}$$

Thus, $g'(x)$ changes sign from negative to positive as x increases through 2.

$\therefore$ $x = 2$ is a point of local minima and local minimum value

$$= g(2) = \frac{2}{2} + \frac{2}{2} = 1 + 1 = 2.$$

Note. Second derivative test, $g''(x)$

$$= \frac{d}{dx}\left(\frac{1}{2} - \frac{2}{x^2}\right) = \frac{4}{x^3} \qquad \therefore \quad g''(2) = \frac{4}{8} = \frac{1}{2} > 0$$

$\Rightarrow g(x)$ has local minimum value at $x = 2$ and local minimum value $= g(2) = \dfrac{2}{2} + \dfrac{2}{2} = 1 + 1 = 2.$

(*vii*) **Given:** $h(x) = \dfrac{1}{x^2 + 2} = (x^2 + 2)^{-1}$ ...(*i*)

$\therefore \quad h'(x) = (-1)(x^2+2)^{-2}(2x) = -\dfrac{2x}{(x^2+2)^2}$

and $\quad h''(x) = -\left[\dfrac{(x^2+2)^2 \,.\, 2 - 2x \,.\, 2(x^2+2)\,2x}{(x^2+2)^4}\right]$

$$= \frac{-2(x^2+2)\,[x^2+2-4x^2)}{(x^2+2)^4} = \frac{-2(2-3x^2)}{(x^2+2)^3}$$

Putting $h'(x) = 0$ to get turning points, we have

$$\frac{-2x}{(x^2+2)^2} = 0 \Rightarrow -2x = 0 \Rightarrow x = \frac{0}{-2} = 0$$

Let us apply second derivative test.

At $x = 0$, $\quad h''(x) = \dfrac{-2(2-3x^2)}{(x^2+2)^3} = \dfrac{-2(2-0)}{(0+2)^3} = \dfrac{-4}{8}$

$$= \frac{-1}{2} \quad \text{(Negative)}$$

$\therefore \quad x = 0$ is a point of local maxima and local maximum value

$$= h(0) = \frac{1}{0+2} = \frac{1}{2} \quad (\text{From } (i))$$

Therefore, Local maxima at $x = 0$ and local maximum value $= \dfrac{1}{2}$.

(*viii*) **Given:** $f(x) = x\sqrt{1-x}$, $x \le 1$

$\therefore \quad f'(x) = x\dfrac{1}{2}(1-x)^{-1/2}\dfrac{d}{dx}(1-x) + \sqrt{1-x} \,.\, 1$

$\therefore \quad f'(x) = x \,.\, \dfrac{1}{2\sqrt{1-x}}(-1) + \sqrt{1-x} \,.\, 1$

$$= \frac{-x}{2\sqrt{1-x}} + \sqrt{1-x} = \frac{-x+2(1-x)}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}} \quad ...(i)$$

For turning points, putting $f'(x) = 0$

$$\Rightarrow \frac{2-3x}{2\sqrt{1-x}} = 0 \Rightarrow 2 - 3x = 0 \quad \therefore x = \frac{2}{3}$$

Let us apply first derivative test.

When x is slightly $< \dfrac{2}{3}$, let $x = 0.6$

From (*i*), $\quad f'(0.6) = \dfrac{2-3(0.6)}{2\sqrt{1-0.6}} = \dfrac{2-1.8}{2\sqrt{0.4}} = \dfrac{0.2}{2\sqrt{0.4}} > 0$

When x is slightly $> \dfrac{2}{3}$, let $x = 0.7$

From (*i*), $\quad f'(0.7) = \dfrac{2-3\,.\,(0.7)}{2\sqrt{1-0.7}} = \dfrac{2-2.1}{2\sqrt{0.3}} = \dfrac{-0.1}{2\sqrt{0.3}} < 0$

Thus, $f'(x)$ changes sign from positive to negative as x increases through $\frac{2}{3}$.

$\therefore$ $x = \frac{2}{3}$ is a point of local maxima and local maximum value

$$= f\left(\frac{2}{3}\right) = x\sqrt{1-x} = \frac{2}{3}\sqrt{1-\frac{2}{3}} = \frac{2}{3} \times \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{9}.$$

Note. Apply second derivative test.

Differentiating both sides of (i) w.r.t. x,

$$f''(x) = \frac{1}{2} \cdot \frac{\sqrt{1-x}\,.\,(-3) - (2-3x)\,.\,\frac{1}{2\sqrt{1-x}}(-1)}{1-x}$$

$$f''\left(\frac{2}{3}\right) = \frac{1}{2} \cdot \frac{\left(\frac{1}{\sqrt{3}}\right)(-3) - 0}{\frac{1}{3}} = \frac{-9}{2\sqrt{3}} = -\frac{3\sqrt{3}}{2} < 0$$

$\therefore$ $f(x)$ has local maximum value at $x = \frac{2}{3}$.

4. Prove that the following functions do not have maxima or minima:

(i) $f(x) = e^x$ **(ii) $g(x) = \log x$**

(iii) $h(x) = x^3 + x^2 + x + 1$.

Sol. (i) **Given:** $f(x) = e^x$

$\therefore$ $f'(x) = e^x$

Putting $f'(x) = 0$ to get turning points, we have $e^x = 0$. But this gives no real value of x. [$\because$ $e^x > 0$ for all real x]

$\therefore$ No turning point.

Hence $f(x)$ does not have maxima or minima.

(ii) **Given:** $g(x) = \log x$

$\therefore$ $g'(x) = \frac{1}{x}$

Putting $g'(x) = 0$ to get turning points, we have

$$\frac{1}{x} = 0 \Rightarrow 1 = 0.$$

But this is impossible.

$\therefore$ No turning point.

Hence $f(x)$ does not have maxima or minima.

(iii) $h(x) = x^3 + x^2 + x + 1$

$h'(x) = 3x^2 + 2x + 1$

Putting $h'(x) = 0$, we have $3x^2 + 2x + 1 = 0$

$$\therefore \quad x = \frac{-2 \pm \sqrt{4-12}}{6} = \frac{-2 \pm \sqrt{-8}}{6}$$

$$= \frac{-2 \pm 2\sqrt{2}\,i}{6} = \frac{-1 \pm \sqrt{2}\,i}{3}$$

These values of x are imaginary.

$\therefore$ $h(x)$ does not have maxima or minima.

5. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(*i*) $f(x) = x^3,\ x \in [-2, 2]$

(*ii*) $f(x) = \sin x + \cos x,\ x \in [0, \pi]$

(*iii*) $f(x) = 4x - \frac{1}{2}x^2,\ x \in \left[-2, \frac{9}{2}\right]$

(*iv*) $f(x) = (x - 1)^2 + 3,\ x \in [-3, 1]$

Sol. (*i*) **Given:** $f(x) = x^3,\ x \in [-2, 2]$ ...(*i*)

$\therefore$ $f'(x) = 3x^2$

Putting $f'(x) = 0$ to get turning points, we have

$$3x^2 = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0 \in [-2, 2]$$

To find absolute maximum and absolute minimum value of the function, we are to find values of $f(x)$ at (*a*) turning point(s) and (*b*) at end points of the given closed interval [– 2, 2].

Putting $x = 0$ in (*i*), $f(0) = 0$

Putting $x = -2$ in (*i*), $f(-2) = (-2)^3 = -8$

Putting $x = 2$ in (*i*), $f(2) = (2)^3 = 8$

Out of these three values of $f(x)$; absolute minimum value $= -8$ and absolute maximum value is 8.

Remark. Absolute maximum and absolute minimum values of $f(x)$ are also called maximum and minimum values of $f(x)$.

(*ii*) **Given:** $f(x) = \sin x + \cos x,\ x \in [0, \pi]$ ...(*i*)

$\therefore$ $f'(x) = \cos x - \sin x$

Putting $f'(x) = 0$ to get turning points, we have

$\cos x - \sin x = 0 \Rightarrow -\sin x = -\cos x.$

Dividing by $-\cos x$, $\tan x = 1$ is positive.

$\therefore$ x is in I and III quadrants.

But $x \in [0, \pi]$ (given) can't be in third quadrant.

$\therefore$ x is in Ist quadrant.

Therefore, $\tan x = 1 = \tan \frac{\pi}{4}$

$\therefore$ $x = \frac{\pi}{4}$

Now let us find values of $f(x)$ at turning point $x = \frac{\pi}{4}$ and at end points $x = 0$ and $x = \pi$ of given closed interval [0, π].

Putting $x = \frac{\pi}{4}$ in (*i*), $f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} + \cos \frac{\pi}{4}$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Putting $x = 0$ in (i), $f(0) = \sin 0 + \cos 0$
$= 0 + 1 = 1$

Putting $x = \pi$ in (i), $f(\pi) = \sin \pi + \cos \pi$
$= 0 - 1 = -1$

$[\because \sin \pi = \sin 180° = \sin (180° - 0°) = \sin 0° = 0$
and $\cos \pi = \cos 180° = \cos (180° - 0°) = -\cos 0° = -1]$

$\therefore$ Absolute minimum value $= -1$ and absolute maximum value $= \sqrt{2}$.

(iii) **Given:** $f(x) = 4x - \frac{1}{2}x^2,\ x \in \left[-2, \frac{9}{2}\right]$ ...(i)

$\therefore \quad f'(x) = 4 - \frac{1}{2}(2x) = 4 - x.$

Putting $f'(x) = 0$ to find turning points,
we have $4 - x = 0$

i.e., $-x = -4$ *i.e.,* $x = 4 \in \left[-2, \frac{9}{2}\right]$

Now let us find values of $f(x)$ at turning point $x = 4$ and at end points $x = -2$ and $x = \frac{9}{2}$ of the given closed interval $\left[-2, \frac{9}{2}\right]$.

Putting $x = 4$ in (i), $f(4) = 16 - \frac{1}{2}(16) = 16 - 8 = 8$

Putting $x = -2$ in (i), $f(-2) = 4(-2) - \frac{1}{2}(4) = -8 - 2 = -10$

Putting $x = \frac{9}{2}$ in (i), $f\left(\frac{9}{2}\right) = 4\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right)^2$

$$= 18 - \frac{81}{8} = \frac{144 - 81}{8} = \frac{63}{8}$$

$\therefore$ Absolute minimum value is -10 and absolute maximum value is 8.

(iv) **Given:** $f(x) = (x - 1)^2 + 3,\ x \in [-3, 1]$ ...(i)

$\therefore \quad f'(x) = 2(x - 1)\frac{d}{dx}(x - 1) + 0 = 2(x - 1)$

Putting $f'(x) = 0$ to find turning points,
we have $2(x - 1) = 0$

$\Rightarrow \quad x - 1 = \frac{0}{2} = 0 \quad \Rightarrow \quad x = 1 \in [-3, 1]$

Now let us find values of $f(x)$ at turning point $x = 1$ and at the end point $x = -3$ of the given closed

interval [– 3, 1] ($\because$ the other end point $x = 1$ has already come out to be a turning point)

Putting $x = -3$ in (*i*), $f(-3) = (-3-1)^2 + 3$

$= (-4)^2 + 3 = 16 + 3 = 19$

Putting $x = 1$ in (*i*), $f(1) = (1-1)^2 + 3 = 0 + 3 = 3$

$\therefore$ Absolute minimum value is 3 and absolute maximum value is 19.

Note. To find absolute maximum or absolute minimum value of a function $f(x)$ when only one turning point comes out to be there and no closed interval is given, then we get only one value of $f(x)$ at such points and out of one value of $f(x)$ we can't make a decision about maximum value or minimum value. In such problems, we have to depend upon local minimum value and local maximum value.

6. Find the maximum profit that a company can make, if the profit function is given by

$$p(x) = 41 + 24x - 18x^2.$$

Sol. Given: Profit function is $p(x) = 41 + 24x - 18x^2$ ...(*i*)

$\therefore \quad p'(x) = 24 - 36x \quad$ and $\quad p''(x) = -36$

(The logic of finding $p''(x)$ is given in the note above)

Putting $p'(x) = 0$ to get turning points, we have

$$24 - 36x = 0 \quad \Rightarrow \quad -36x = -24$$

$$\Rightarrow \qquad x = \frac{24}{36} = \frac{2}{3}$$

At $x = \frac{2}{3}$, $p''(x) = -36$ (Negative)

$\therefore \quad p(x)$ has a local maximum value and hence maximum value at $x = \frac{2}{3}$.

Putting $x = \frac{2}{3}$ in (*i*),

$$\text{Maximum profit} = 41 + 24\left(\frac{2}{3}\right) - 18\left(\frac{4}{9}\right) = 41 + 16 - 8 = 49$$

Remark. The original statement in N.C.E.R.T. book $p(x) = 41 - 24x - 18x^2$ is wrong, because with this $p(x)$, turning point comes out to be $x = -\frac{2}{3}$ which being the number of units produced or sold can't be negative.

7. Find both the maximum value and the minimum value of $3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval [0, 3].

Sol. Let $\quad f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$ on [0, 3] ...(*i*)

$\therefore \qquad f'(x) = 12x^3 - 24x^2 + 24x - 48$

Putting $f'(x) = 0$ to find turning points, we have

$$12x^3 - 24x^2 + 24x - 48 = 0$$

Dividing every term by 12, $x^3 - 2x^2 + 2x - 4 = 0$

or $x^2(x - 2) + 2(x - 2) = 0$ or $(x - 2)(x^2 + 2) = 0$

$\therefore$ Either $x - 2 = 0$ or $x^2 + 2 = 0$

$\Rightarrow$ $x = 2$ $\Rightarrow$ $x^2 = -2$

$\Rightarrow$ $x = \pm\sqrt{-2}$

These values of x are imaginary and hence rejected.

Turning point $x = 2 \in [0, 3]$.

Now let us find values of $f(x)$ at turning point $x = 2$ and end points $x = 0$ and $x = 3$ of closed interval [0, 3].

Putting $x = 2$ in (i), $f(2) = 3(16) - 8(8) + 12(4) - 48(2) + 25$

$= 48 - 64 + 48 - 96 + 25 = -39$

Putting $x = 0$ in (i), $f(0) = 25$

Putting $x = 3$ in (i), $f(3) = 3(81) - 8(27) + 12(9) - 48(3) + 25$

$= 243 - 216 + 108 - 144 + 25$

$= 27 - 36 + 25 = 16.$

$\therefore$ Minimum (absolute) value is -39 (at $x = 2$) and maximum (absolute) value is 25 (at $x = 0$).

8. At what points on the interval $[0, 2\pi]$ does the function $\sin 2x$ attain its maximum value?

Sol. Let $f(x) = \sin 2x$, then $f'(x) = 2\cos 2x$

For maxima or minima, $f'(x) = 0$

$\Rightarrow \cos 2x = 0 \therefore 2x = (2n + 1)\dfrac{\pi}{2}$ or $x = (2n + 1)\dfrac{\pi}{4}$

Putting $n = 0, 1, 2, 3$; $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4} \in [0, 2\pi]$

Now let us find values of $f(x)$ at these turning points.

Now $f(x) = \sin 2x$

$\therefore \left[f(2n+1)\dfrac{\pi}{4}\right] = \sin(2n + 1)\dfrac{\pi}{2} = \sin\left(n\pi + \dfrac{\pi}{2}\right) = (-1)^n \sin\dfrac{\pi}{2}$

$[\because \sin(n\pi + \alpha) = (-1)^n \sin\alpha;\ n \in \text{I}]$

$= (-1)^n$

Putting $n = 0, 1, 2, 3$,

$f\left(\dfrac{\pi}{4}\right) = (-1)^0 = 1,$ $f\left(\dfrac{3\pi}{4}\right) = (-1)^1 = -1$

$f\left(\dfrac{5\pi}{4}\right) = (-1)^2 = 1,$ $f\left(\dfrac{7\pi}{4}\right) = (-1)^3 = -1$

Also let us find $f(x)$ at the end-points $x = 0$ and $x = 2\pi$ of $[0, 2\pi]$.

$f(0) = \sin 0 = 0$, $f(2\pi) = \sin 4\pi = 0$ $[\because \sin n\pi = 0$ where $n \in \text{I}]$

$\therefore f(x)$ attains its maximum value 1 at $x = \dfrac{\pi}{4}$ and $x = \dfrac{5\pi}{4}$.

Hence, the required points are $\left(\dfrac{\pi}{4}, 1\right)$ and $\left(\dfrac{5\pi}{4}, 1\right)$.

9. What is the maximum value of the function sin x + cos x?

Sol. Let $f(x) = \sin x + \cos x$ ...(i)

$\therefore \quad f'(x) = \cos x - \sin x$

Putting $f'(x) = 0$ to find turning points, we have

$\cos x - \sin x = 0 \quad \Rightarrow \quad -\sin x = -\cos x$

Dividing by $-\cos x$, $\frac{\sin x}{\cos x} = 1 \quad \Rightarrow \quad \tan x = 1 = \tan \frac{\pi}{4}$

$\therefore \quad x = n\pi + \frac{\pi}{4}$ where $n \in$ Z (turning points)

($\because$ If $\tan \theta = \tan \alpha$, then $\theta = n\pi + \alpha$ where $n \in$ Z)

Putting $x = n\pi + \frac{\pi}{4}$ in (i),

$$f\left(n\pi + \frac{\pi}{4}\right) = \sin\left(n\pi + \frac{\pi}{4}\right) + \cos\left(n\pi + \frac{\pi}{4}\right)$$

$$= (-1)^n \sin \frac{\pi}{4} + (-1)^n \cos \frac{\pi}{4}$$

[$\because \sin(n\pi + \alpha) = (-1)^n \sin \alpha$ and $\cos(n\pi + \alpha) = (-1)^n \cos \alpha$]

$= (-1)^n \frac{1}{\sqrt{2}} + (-1)^n \frac{1}{\sqrt{2}} = 2(-1)^n \frac{1}{\sqrt{2}}$ ($\because \quad t + t = 2t$)

$= (-1)^n \sqrt{2}$

If n is even; then $(-1)^n = 1$ and then $f\left(n\pi + \frac{\pi}{4}\right) = \sqrt{2}$

If n is odd, then $(-1)^n = -1$; and then $f\left(n\pi + \frac{\pi}{4}\right) = -\sqrt{2}$

$\therefore$ Maximum value of $f(x)$ is $\sqrt{2}$.

Note. Minimum value of $f(x)$ is $-\sqrt{2}$.

10. Find the maximum value of $2x^3 - 24x + 107$ in the interval [1, 3]. Find the maximum value of the same function in [– 3, – 1].

Sol. Let $f(x) = 2x^3 - 24x + 107$ ...(i)

$\therefore \quad f'(x) = 6x^2 - 24$

Let us put $f'(x) = 0$ to find turning points.

i.e., $6x^2 - 24 = 0 \quad \Rightarrow \quad 6x^2 = 24$

Dividing by 6, $x^2 = 24 \quad \Rightarrow \quad x = \pm 2$

$\therefore \quad x = -2$ and $x = 2$ are two turning points. ...(ii)

(a) Let us find maximum value of $f(x)$ given by (i) in the interval [1, 3].

From (ii), turning point $x = -2 \notin [1, 3]$.

So let us find values of $f(x)$ at turning point $x = 2$ and at end points $x = 1$ and $x = 3$ of closed interval [1, 3].

Putting $x = 2$ in (i), $f(2) = 2(8) - 24(2) + 107$

$= 16 - 48 + 107 = 123 - 48 = 75$

Putting $x = 1$ in (*i*), $f(1) = 2 - 24 + 107 = 109 - 24 = 85$

Putting $x = 3$ in (*i*), $f(3) = 2(27) - 24(3) + 107$

$= 54 - 72 + 107 = 161 - 72 = 89$

$\therefore$ Maximum value of $f(x)$ given by (*i*) in [1, 3] is 89 (at $x = 3$).

(*b*) Let us find maximum value of $f(x)$ given by (*i*) in the interval $[-3, -1]$.

From (*ii*), turning point $x = 2 \notin [-3, -1]$

So let us find values of $f(x)$ at turning point $x = -2$ and at end points $x = -3$ and $x = -1$ of closed interval $[-3, -1]$

Putting $x = -2$ in (*i*), $f(-2) = 2(-8) - 24(-2) + 107$

$= -16 + 48 + 107 = 139$

Putting $x = -3$ in (*i*), $f(-3) = 2(-27) - 24(-3) + 107$

$= -54 + 72 + 107 = 125$

Putting $x = -1$ in (*i*), $f(-1) = 2(-1) - 24(-1) + 107$

$= -2 + 24 + 107 = 129$

$\therefore$ Maximum value of $f(x)$ is 139 (at $x = -2$).

11. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval [0, 2]. Find the value of a.

Sol. Let $f(x) = x^4 - 62x^2 + ax + 9$

$\therefore \quad f'(x) = 4x^3 - 124x + a$ ...(*i*)

Because $f(x)$ attains its maximum value at $x = 1$ in the interval [0, 2], therefore, $f'(1) = 0$.

Putting $x = 1$ in (*i*), $f'(1) = 4 - 124 + a = 0$

or $a - 120 = 0$ or $a = 120$.

12. Find the maximum and minimum value of $x + \sin 2x$ on $[0, 2\pi]$.

Sol. Let $f(x) = x + \sin 2x$ ...(*i*)

$\therefore \quad f'(x) = 1 + 2 \cos 2x$

Putting $f'(x) = 0$ to find turning points, we have

$1 + 2\cos 2x = 0 \Rightarrow 2\cos 2x = -1$

$$\Rightarrow \qquad \cos 2x = \frac{-1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$\therefore \qquad 2x = 2n\pi \pm \frac{2\pi}{3} \text{ where } n \in Z$$

[$\because$ If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$ where $n \in Z$]

Dividing by 2, $x = n\pi \pm \dfrac{\pi}{3}$ where $n \in Z$

For $n = 0$, $x = \pm\dfrac{\pi}{3}$. But $x = -\dfrac{\pi}{3} \notin [0, 2\pi]$ $\therefore$ $x = \dfrac{\pi}{3}$

For $n = 1$, $x = \pi \pm \dfrac{\pi}{3} = \pi + \dfrac{\pi}{3}$ and $\pi - \dfrac{\pi}{3}$

i.e., $\dfrac{4\pi}{3}$ and $\dfrac{2\pi}{3}$ and both belong to $[0, 2\pi]$

For $n = 2$, $x = 2\pi \pm \frac{\pi}{3}$. But $x = 2\pi + \frac{\pi}{3} > 2\pi$ and hence $\notin [0, 2\pi]$

$\therefore \quad x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

It can be easily observed that for all other $n \in Z$,

$$x = n\pi \pm \frac{\pi}{3} \notin [0, 2\pi].$$

$\therefore$ The only turning points of $f(x)$ given by (i) which belong to given closed interval $[0, 2\pi]$ are

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

Now let us find values of $f(x)$ at these four turning points and at the end points $x = 0$ and $x = 2\pi$ of $[0, 2\pi]$.

Putting $x = \frac{\pi}{3}$ in (i), $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3}$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2} = 1.05 + 0.87$$

$$= 1.92 \text{ nearly}$$

$$\left(\because \frac{\pi}{3} = \frac{\frac{22}{7}}{3} = \frac{22}{21} = 1.05 \text{ and } \frac{\sqrt{3}}{2} = \frac{1.732}{2} = 0.866 = 0.87\right)$$

and $\left(\because \sin\frac{2\pi}{3} = \sin\frac{3\pi - \pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\right)$

Putting $x = \frac{2\pi}{3}$ in (i), $f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3}$

$$= 2\pi - \frac{\sqrt{3}}{2} = 2.10 - 0.87 = 1.23 \text{ nearly}$$

$$\left(\because \sin\frac{4\pi}{3} = \sin\frac{3\pi + \pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = \frac{-\sqrt{3}}{2}\right)$$

Putting $x = \frac{4\pi}{3}$ in (i), $f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin\frac{8\pi}{3}$

$$= \frac{4\pi}{3} + \frac{\sqrt{3}}{2} = 4(1.05) + 0.87 = 4.20 + 0.87 = 5.07$$

$$\left(\because \sin\frac{8\pi}{3} = \sin\left(\frac{6\pi + 2\pi}{3}\right) = \sin\left(2\pi + \frac{2\pi}{3}\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}\right)$$

Putting $x = \frac{5\pi}{3}$ in (i), $f\left(\frac{5\pi}{3}\right)$

$$= \frac{5\pi}{3} + \sin\frac{10\pi}{3} = \frac{5\pi}{3} - \frac{\sqrt{3}}{2}$$

$= 5(1.05) - 0.87 = 5.25 - 0.87 = 4.38$ nearly

$$\left[\because \sin \frac{10\pi}{3} = \sin \frac{6\pi + 4\pi}{3} = \sin\left(2\pi + \frac{4\pi}{3}\right) = \sin \frac{4\pi}{3} = \frac{-\sqrt{3}}{2}\right]$$

Putting $x = 0$ in (i), $f(0) = 0 + \sin 0 = 0$

Putting $x = 2\pi$ in (i), $f(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi$

$= 2(3.14) = 6.28$ nearly $(\because \sin n\pi = 0$ for every integer $n)$

$\therefore$ Maximum value $= 2\pi$ (at $x = 2\pi$)

and minimum value $= 0$ (at $x = 0$).

13. Find two numbers whose sum is 24 and whose product is as large as possible.

Sol. Let the two numbers be x and y.

Their sum = 24 (given) $\Rightarrow$ $x + y = 24$

$\therefore$ $y = 24 - x$ ...(i)

Let z denote their product *i.e.,* product of x and y

i.e., $z = xy$

Putting $y = 24 - x$ from (i),

$z = x(24 - x) = 24x - x^2$

Now z is a function of x only.

$\therefore$ $\frac{dz}{dx} = 24 - 2x$ and $\frac{d^2z}{dx^2} = -2$

Putting $\frac{dz}{dx} = 0$ to find turning points, we have

$24 - 2x = 0$ *i.e.,* $-2x = -24$. Therefore, $x = 12$.

At $x = 12$, $\frac{d^2z}{dx^2} = -2$ (negative)

$\therefore$ $x = 12$ is a point of (local) maxima.

(See Note at the end of solution of Q. No. 5)

$\therefore$ z is maximum at $x = 12$.

Putting $x = 12$ in (i), $y = 24 - 12 = 12$

$\therefore$ The two required numbers are 12 and 12.

14. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

Sol. Here $x + y = 60$, $x > 0$, $y > 0$ ...(i) (Given condition)

Let $P = xy^3$ ...(ii) (To be maximised)

To express P in terms of one independent variable, (here better y, because power of y is larger in the value of P), we have from (i)

$x = 60 - y$,

Putting $x = 60 - y$ in (ii), $P = (60 - y)y^3 = 60y^3 - y^4$

$\therefore$ $\frac{dP}{dy} = 180y^2 - 4y^3 = 4y^2(45 - y)$ ...(iii)

For max. or min., Putting $\frac{dP}{dy} = 0$

$\Rightarrow$ $4y^2(45 - y) = 0$ $\Rightarrow y = 0, 45$

Rejecting $y = 0$ because $y > 0$ $\therefore y = 45$

When y is slightly < 45, from (iii), $\frac{dP}{dy} = (+\text{ ve})(+\text{ ve}) = +\text{ ve}$

When y is slightly > 45, from (iii), $\frac{dP}{dy} = (+\text{ ve})(-\text{ ve}) = -\text{ ve}$

Thus, $\frac{dP}{dy}$ changes sign from + ve to – ve as y increases through 45.

$\therefore$ P is maximum when $y = 45$.

Hence, xy^3 is maximum when $x = 60 - 45 = 15$

and $y = 45$.

15. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.

Sol. Given: $x + y = 35 \Rightarrow y = 35 - x$ ...(i)

Let $z = x^2 y^5$

Putting $y = 35 - x$ from (i), $z = x^2 (35 - x)^5$

Now z is a function of x alone.

$$\therefore \quad \frac{dz}{dx} = x^2 \,.\, 5(35 - x)^4 (-1) + (35 - x)^5\, 2x$$

$$\text{or} \quad \frac{dz}{dx} = x(35 - x)^4 [-5x + (35 - x)\, 2]$$

$$\text{or} \quad \frac{dz}{dx} = x(35 - x)^4 (-5x + 70 - 2x) = x(35 - x)^4 (70 - 7x)$$

$$\text{or} \quad \frac{dz}{dx} = 7x\, (35 - x)^4 (10 - x) \qquad ...(ii)$$

Putting $\frac{dz}{dx} = 0$ to find turning points, we have

$$7x(35 - x)^4 (10 - x) = 0$$

But $7 \neq 0$ $\therefore$ Either $x = 0$ or $35 - x = 0$ or $10 - x = 0$

i.e., $x = 0$ or $x = 35$ or $x = 10$.

Now $x = 0$ is rejected because x is positive number (given).

Also, $x = 35$ is rejected because for $x = 35$, from (i)

$y = 35 - 35 = 0$; but y is given to be positive.

$\therefore$ $x = 10$ is the only admissible turning point.

Let us apply first derivative test because finding $\frac{d^2z}{dx^2}$ is tedious as you and we think so.

We know that $(35 - x)^4$ is never negative because index 4 is even.

When x is slightly < 10 (say $x = 9.8$); from (ii),

$$\frac{dz}{dx} = (+)\,(+)\,(+) = (+)$$

When x is slightly > 10 (say $x = 10.1$); from (ii),

$$\frac{dz}{dx} = (+)\,(+)\,(-) = (-)$$

$\therefore \quad \frac{dz}{dx}$ changes sign from (+) to (–) as x increases while passing through 10.

$\therefore \quad x = 10$ gives a point of (local) maxima.

(See Note at the end of solution of Q. No. 5)

i.e., z is maximum when $x = 10$.

Putting $x = 10$ in (i), $y = 35 - 10 = 25$.

$\therefore$ The two required numbers are 10 and 25.

16. Find two positive numbers whose sum is 16 and sum of whose cubes is minimum.

Sol. Let the two positive numbers be x and y.

Given: $x + y = 16 \quad \Rightarrow \quad y = 16 - x$ ...(i)

Let z denote the sum of their cubes *i.e.,* $z = x^3 + y^3$.

Putting $y = 16 - x$ from (i), $z = x^3 + (16 - x)^3$

$\Rightarrow \quad z = x^3 + (16)^3 - x^3 - 48x\,(16 - x)$

$[\because \ (a - b)^3 = a^3 - b^3 - 3ab(a - b)]$

$\Rightarrow \quad z = (16)^3 - 768x + 48x^2$

Now z is a function of x alone.

$$\therefore \quad \frac{dz}{dx} = -768 + 96x \text{ and } \frac{d^2z}{dx^2} = 96$$

Putting $\frac{dz}{dx} = 0$ to find turning points, we have

$$-768 + 96x = 0 \Rightarrow 96x = 768 \Rightarrow x = \frac{768}{96} = 8$$

At $x = 8$, $\frac{d^2z}{dx^2} = 96 \quad (+)$

$\therefore \quad x = 8$ is a point of (local) minima.

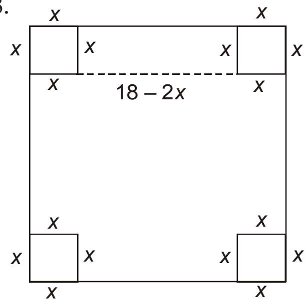
(See **Note** at the end of solution of Q. No. 5)

$\therefore \quad z$ is minimum when $x = 8$.

Putting $x = 8$ in (i), $y = 16 - 8 = 8$

$\therefore$ The required numbers are 8 and 8.

17. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?



Sol. Given: Each side of square piece of tin is 18 cm.

Let x cm be the side of each of the four squares cut off from each corner.

Then dimensions of the open box formed by folding the flaps after cutting off squares are $(18 - 2x)$, $(18 - 2x)$, x cm.

Let z denote the volume of the open box.

$\therefore \quad z$ = length × breadth × height $= (18 - 2x)(18 - 2x)x$

or $\quad z = (18 - 2x)^2\, x = (324 + 4x^2 - 72x)x$

or $\quad z = 4x^3 - 72x^2 + 324x$

$$\therefore \quad \frac{dz}{dx} = 12x^2 - 144x + 324 \quad \text{and} \quad \frac{d^2z}{dx^2} = 24x - 144$$

Putting $\frac{dz}{dx} = 0$ to find turning points, we have

$$12x^2 - 144x + 324 = 0$$

Dividing by 12, $x^2 - 12x + 27 = 0$

$\Rightarrow \quad x^2 - 9x - 3x + 27 = 0 \quad \Rightarrow \quad x(x - 9) - 3(x - 9) = 0$

$\Rightarrow \quad (x - 9)(x - 3) = 0$

$\therefore$ Either $\quad x - 9 = 0$ or $x - 3 = 0$

i.e., $\quad x = 9$ or $x = 3$

But $x = 9$ is rejected because for $x = 9$,

length of box $= 18 - 2x = 18 - 18 = 0$ which is clearly impossible.

$\therefore$ $x = 3$ is the only turning point.

At $x = 3$, $\frac{d^2z}{dx^2} = 24x - 144 = 72 - 144 = -72$ (Negative)

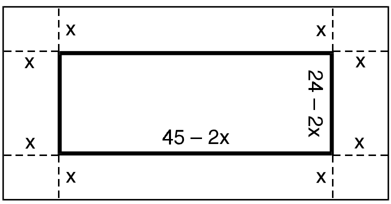
$\therefore$ z is maximum at $x = 3$.

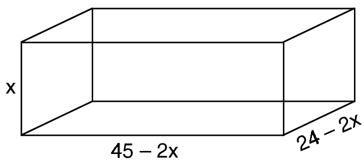
i.e., side of (each) square to be cut off from each corner for maximum volume is 3 cm.

Remark. The reader is suggested to take a paper sheet in square shape and cut off four equal squares from four corners and fold the flaps to form a box for himself or herself.

18. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off a square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum?

Sol. Dimensions of the rectangular sheet of tin are 45 cm and 24 cm. Let the side of the square cut off from each corner be x cm. Therefore, dimensions of the box are $45 - 2x$, $24 - 2x$ and x cm.





The volume V of the box in cubic cm is given by

$$V = (45 - 2x)(24 - 2x)(x) \qquad \text{[product of three dimensions]}$$

$$= x(1080 - 138x + 4x^2) = 1080x - 138x^2 + 4x^3$$

$\therefore \quad \dfrac{dV}{dx} = 1080 - 276x + 12x^2$ and $\dfrac{d^2V}{dx^2} = -276 + 24x$

For max. or min. put $\dfrac{dV}{dx} = 0$

$\Rightarrow \quad 1080 - 276x + 12x^2 = 0,$

Dividing by 12, $x^2 - 23x + 90 = 0$

$\Rightarrow \quad (x - 5)(x - 18) = 0 \quad \therefore x = 5, 18$

But $x = 18$ is impossible because otherwise the dimension

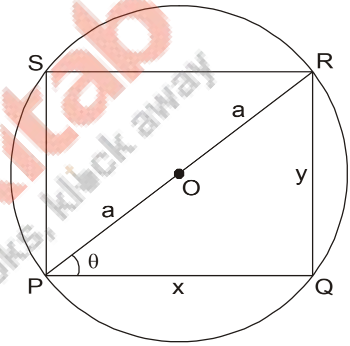
$24 - 2x = 24 - 36 = -12$ is negative. $\therefore x = 5$

$$\left[\frac{d^2V}{dx^2}\right]_{x=5} = -276 + 120 = -156 < 0$$

$\Rightarrow$ V is maximum when $x = 5$

Hence, the box with maximum volume is obtained by cutting off equal squares of side 5 cm.

19. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.



Sol. Let PQRS be the rectangle inscribed in a given circle with centre O and radius a.

Let x and y be the length and breadth of the rectangle $(\therefore x > 0$ and $y > 0)$

In right angled triangle PQR, By Pythagoras Theorem,

$$PQ^2 + QR^2 = PR^2$$

or $\quad x^2 + y^2 = (2a)^2 \quad$ (given condition)

or $\quad y^2 = 4a^2 - x^2 \quad \therefore y = \sqrt{4a^2 - x^2} \quad ...(i)$

Let A denote the area of the rectangle.

$\therefore \quad A = xy \quad ...(ii)$ (A is to be maximised)

To express A in terms of one independent variable, putting the value of y from (i) in (ii), we have

$$A = x\sqrt{4a^2 - x^2}$$

Let $\quad z = A^2 = x^2(4a^2 - x^2) = 4a^2x^2 - x^4 \quad ...(iii)$

Let us maximise $\quad z \; (= A^2)$

From (iii), $\quad \dfrac{dz}{dx} = 8a^2x - 4x^3$

and $\quad \dfrac{d^2z}{dx^2} = 8a^2 - 12x^2$

For max. or min.; put $\dfrac{dz}{dx} = 0$

$\therefore\ 8a^2x - 4x^3 = 0 \quad$ or $\quad 4x(2a^2 - x^2) = 0$

But x being side of rectangle cannot be zero.

$\therefore\ 2a^2 - x^2 = 0 \quad$ or $\quad x^2 = 2a^2$

$\therefore \qquad x = \sqrt{2}\ .\ a \qquad\qquad (\because x > 0)$

At $\qquad x = \sqrt{2}\ a, \quad \dfrac{d^2z}{dx^2} = 8a^2 - 12(2a^2) = 8a^2 - 24a^2$

$= -\ 16a^2$ is negative.

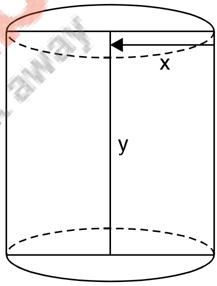
$\therefore \qquad z\ (= A^2)$ is maximum when $x = \sqrt{2}\,a$.

Putting $x = \sqrt{2}\,a$ in (i), $y = \sqrt{4a^2 - 2a^2} = \sqrt{2a^2} = \sqrt{2}\ .\ a$

$\therefore\ x = y = \sqrt{2}\,a\ \therefore$ A is maximum when $x = y = \sqrt{2}\,a$.

Hence, the area of the inscribed rectangle is maximum when it is a square.

20. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.



Sol. Let x be the radius of the circular base and y be the height of closed right circular cylinder.

Total surface area of cylinder is given $(x > 0, y > 0)$

$\Rightarrow$ It is constant = S (say)

$\therefore$ Curved surface area + area of two ends = S

$\Rightarrow 2\pi xy + 2\pi x^2 = S$ (Given condition)

Dividing every term by 2π

[To get a simpler relation in independent variables x and y]

$$xy + x^2 = \frac{S}{2\pi} = k \text{ (say)}$$

$$\therefore \qquad xy = k - x^2 \text{ or } y = \frac{k - x^2}{x} \qquad ...(i)$$

Let z denote the volume of cylinder

$$\therefore \qquad z = \pi x^2 y \qquad ...(ii)$$

[Here z is to be maximised]. Putting the value of y from (i) in (ii) [to express z in terms of one independent variable x]

$$z = \pi x^2 \left(\frac{k - x^2}{x}\right) \quad \text{or} \quad z = \pi x\,(k - x^2) = \pi\,(kx - x^3)$$

$$\therefore\ \frac{dz}{dx} = \pi\,(k - 3x^2) \text{ and } \frac{d^2z}{dx^2} = \pi\,(-\,6x) = -\,6\pi x$$

For max. or min. put $\dfrac{dz}{dx} = 0\ \therefore\ \ \pi\,(k - 3x^2) = 0$

But $\pi \neq 0 \;\therefore\; k - 3x^2 = 0$ or $3x^2 = k$ or $x^2 = \frac{k}{3} \;\therefore\; x = \sqrt{\frac{k}{3}}$

At $\quad x = \sqrt{\frac{k}{3}}, \quad \frac{d^2z}{dx^2} = -6\pi x = -6\pi\sqrt{\frac{k}{3}}$ is negative

$\therefore \quad z$ is max. at $\quad x = \sqrt{\frac{k}{3}}$ ...(*iii*)

Putting $\quad x = \sqrt{\frac{k}{3}}$ in (*i*), $y = \dfrac{k - \frac{k}{3}}{\sqrt{\frac{k}{3}}} = \dfrac{2\frac{k}{3}}{\sqrt{\frac{k}{3}}}$

$$= 2\sqrt{\frac{k}{3}} \qquad \left[\because \frac{t}{\sqrt{t}} = \sqrt{t}\right]$$

or $\quad y = 2\sqrt{\frac{k}{3}} = 2x$ [By (*iii*)]

i.e., Height = Diameter

Hence, the volume of cylinder is maximum when its height is equal to the diameter of its base.

Remark 1. Right circular cylinder $\Rightarrow$ Closed right circular cylinder.

Remark 2. Total surface area of open cylinder = $2\pi xy + \pi x^2$.

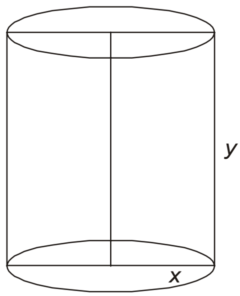
21. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?

Sol. Let x cm be the radius and y cm be the height of closed cylinder.

Given: Volume of closed cylinder = 100 cu cm

$\Rightarrow \quad \pi x^2 y = 100$

$\Rightarrow \quad y = \dfrac{100}{\pi x^2}$ ...(*i*)



Let z denote the surface area of cylinder.

$$\therefore \quad z = \underbrace{2\pi xy}_{\text{(Curved surface area)}} + \underbrace{2\pi x^2}_{\text{(Area of two ends)}}$$

or $\quad z = 2\pi(xy + x^2)$

Putting $y = \dfrac{100}{\pi x^2}$ from (*i*),

$$z = 2\pi\left(x \cdot \frac{100}{\pi x^2} + x^2\right) = 2\pi\left(\frac{100}{\pi x} + x^2\right) = 2\pi\left(\frac{100}{\pi}x^{-1} + x^2\right)$$

Now z is a function of x alone.

$$\therefore \quad \frac{dz}{dx} = 2\pi\left(-\frac{100}{\pi}x^{-2} + 2x\right)$$

and $\dfrac{d^2z}{dx^2} = 2\pi\left(\dfrac{200}{\pi}x^{-3} + 2\right)$

Putting $\dfrac{dz}{dx} = 0$ to find turning points, we have

$$2\pi\left(\frac{-100}{\pi x^2} + 2x\right) = 0. \text{ But } 2\pi \neq 0$$

$\therefore \quad \dfrac{-100}{\pi x^2} + 2x = 0 \Rightarrow \dfrac{-100}{\pi x^2} = -2x$

Cross-multiplying, $2\pi x^3 = 100$

$\Rightarrow \quad x^3 = \dfrac{100}{2\pi} = \dfrac{50}{\pi}$

$\therefore \quad x = \left(\dfrac{50}{\pi}\right)^{1/3}$

At $\quad x = \left(\dfrac{50}{\pi}\right)^{1/3}, \dfrac{d^2z}{dx^2} = 2\pi\left(\dfrac{200}{\pi x^3} + 2\right)$

$$= 2\pi\left(\frac{200}{\pi\left(\frac{50}{\pi}\right)} + 2\right) = 2\pi(4 + 2) = 12\pi \text{ (positive)}$$

$\therefore$ z is minimum (local) when radius $x = \left(\dfrac{50}{\pi}\right)^{1/3}$ cm ...(*ii*)

(See Note at the end of solution of Q. No. 5)

Putting $x = \left(\dfrac{50}{\pi}\right)^{1/3}$ in (*i*), $y = \dfrac{100}{\pi\left(\frac{50}{\pi}\right)^{2/3}}$

$\Rightarrow$ Height $y = 2 \cdot \dfrac{\frac{50}{\pi}}{\left(\frac{50}{\pi}\right)^{2/3}} = 2\left(\dfrac{50}{\pi}\right)^{1-2/3} = 2\left(\dfrac{50}{\pi}\right)^{1/3}$ cm.

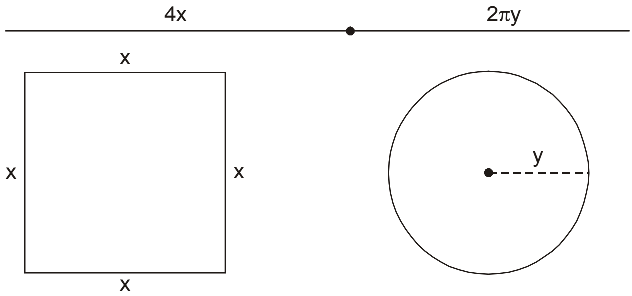
Remark. $y = 2x$ (By (*ii*))

22. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

Sol. Let x metres be the side of the square and y metres, the radius of the circle.

Length of wire = Perimeter of square + Circumference of circle

$= 4x + 2\pi y$



According to the question, $4x + 2\pi y = 28$ (Given condition)

Dividing by 2, $\quad 2x + \pi y = 14 \quad \therefore\ y = \dfrac{14 - 2x}{\pi}$ ...(*i*)

Area of square $= x^2$ sq. m. Area of circle $= \pi y^2$ sq. m

Let A denote their combined area, then

$$A = x^2 + \pi y^2$$ [Here A is to be minimised]

Putting the value of y from eqn. (*i*),

[To express A in terms of one independent variable x]

$$A = x^2 + \pi\left(\frac{14-2x}{\pi}\right)^2 = x^2 + \pi\left(\frac{2(7-x)}{\pi}\right)^2 = x^2 + \pi\,.\,\frac{4}{\pi^2}(7-x)^2$$

or $\quad A = x^2 + \dfrac{4}{\pi}(7-x)^2$

$\therefore \quad \dfrac{dA}{dx} = 2x - \dfrac{8}{\pi}(7-x)$ and $\dfrac{d^2A}{dx^2} = 2 + \dfrac{8}{\pi}$

For max. or min., $\dfrac{dA}{dx} = 0 \quad \therefore\ 2x - \dfrac{8}{\pi}(7-x) = 0$

$\therefore \quad 2x = \dfrac{8}{\pi}(7-x)$

or $\quad 2\pi x = 56 - 8x$ or $(2\pi + 8)x = 56$

$\therefore \quad x = \dfrac{56}{2\pi + 8} = \dfrac{56}{2(\pi + 4)} = \dfrac{28}{\pi + 4}$.

Also $\dfrac{d^2A}{dx^2} = 2 + \dfrac{8}{\pi}$ is +ve.

$\therefore$ A is minimum when $\quad x = \dfrac{28}{\pi + 4}$. ...(*ii*)

Hence, the wire should be cut at a distance $4x = \dfrac{112}{\pi + 4}$ m from one end.

Note 1. Length of circle $= 2\pi y$.

Putting the value of y from (*i*),

$$= 28 - 4x$$ ...(*iii*)

Putting the value of x from (*ii*),

Length of circle $= 28 - \dfrac{112}{\pi + 4} = \dfrac{28\pi}{\pi + 4}$

$\therefore$ The length of two parts (square and circle) are respectively $\dfrac{112}{\pi + 4}$ m and $\dfrac{28\pi}{\pi + 4}$ m.

Note 2. Side of square $= x = \dfrac{28}{\pi + 4}$

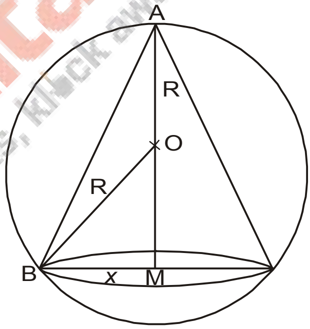
From (i), Radius of circle $= y = \dfrac{14 - 2x}{\pi} = \dfrac{14 - 2\left(\dfrac{28}{\pi + 4}\right)}{\pi}$

$$= \frac{14\pi + 56 - 56}{\pi(\pi + 4)} = \frac{14\pi}{\pi(\pi + 4)} = \frac{14}{\pi + 4},$$

Therefore, diameter of circle $= \dfrac{28}{\pi + 4}$

$\therefore$ Side of square = Diameter of circle.

23. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\dfrac{8}{27}$ of the volume of the sphere.



Sol. Let O be the centre and R be the radius of the given sphere.

Let BM $= x$ and AM $= y$ be the radius and height of **any** cone inscribed in the given sphere.

In right angled triangle OMB,

By Pythagoras Theorem,

$$\text{OM}^2 + \text{BM}^2 = \text{OB}^2$$

$\Rightarrow \quad (y - \text{R})^2 + x^2 = \text{R}^2 \quad [\because \ \text{OM} = \text{AM} - \text{OA} = y - \text{R}]$

$\Rightarrow \quad y^2 + \text{R}^2 - 2\text{R}y + x^2 = \text{R}^2 \quad \Rightarrow \quad x^2 + y^2 - 2\text{R}y = 0$

$\Rightarrow \quad x^2 = 2\text{R}y - y^2 \qquad ...(i)$

Let z denote the volume of any cone inscribed in the given sphere.

$\therefore \quad z = \dfrac{1}{3}\pi x^2 y$

Putting the value of x^2 from (i),

$$z = \frac{\pi}{3}(2\text{R}y - y^2)y = \frac{\pi}{3}(2\text{R}y^2 - y^3) \qquad ...(ii)$$

Now z is a function of y alone.

$\therefore \quad \dfrac{dz}{dy} = \dfrac{\pi}{3}(4\text{R}y - 3y^2)$ and $\dfrac{d^2z}{dy^2} = \dfrac{\pi}{3}(4\text{R} - 6y)$

Putting $\dfrac{dz}{dy} = 0$ to find turning points, we have $\dfrac{\pi}{3}(4\text{R}y - 3y^2) = 0$

But $\frac{\pi}{3} \neq 0$. Therefore $4Ry - 3y^2 = 0 \Rightarrow -3y^2 = -4Ry$

Dividing both sides by $-y \neq 0$,

$$3y = 4R \Rightarrow y = \frac{4R}{3}$$

At $\quad y = \frac{4R}{3}, \frac{d^2z}{dy^2} = \frac{\pi}{3}(4R - 6y) = \frac{\pi}{3}(4R - 8R)$

$$= \frac{\pi}{3}(-4R) = -\frac{4R}{3} \quad \text{(Negative)}$$

$\therefore$ z is maximum at $y = \frac{4R}{3}$ ...(*iii*)

Putting $y = \frac{4R}{3}$ from (*iii*) in (*i*), we have

$$x^2 = 2R \cdot \frac{4R}{3} - \left(\frac{4R}{3}\right)^2 = \frac{8R^2}{3} - \frac{16R^2}{9}$$

$$= 8R^2\left(\frac{1}{3} - \frac{2}{9}\right) = 8R^2\left(\frac{3-2}{9}\right) \Rightarrow x^2 = \frac{8R^2}{9}$$

$\therefore$ Maximum volume z of the cone

$$= \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi \cdot \frac{8R^2}{9} \cdot \frac{4R}{3} = \frac{8}{27} \cdot \frac{4\pi}{3} R^3$$

$$= \frac{8}{27}\text{(Volume of the sphere).}$$

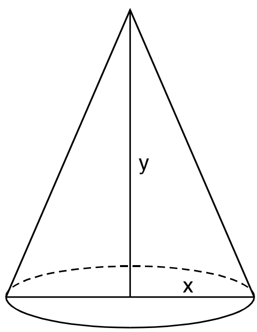
24. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

Sol. Let x be the base radius and y be the height of cone.

Given Volume $\Rightarrow$ Volume of the cone is constant and = V (say)

$\therefore \quad \frac{1}{3}\pi x^2 y = V$ (Given condition)

$\therefore \quad x^2 y = \frac{3V}{\pi} = k$ (say) ...(*i*)



Let S denote the curved surface of the cone

$\therefore$ $S = \pi x \sqrt{x^2 + y^2}$ **(formula S = πrl)**

(S is to be minimised here)

Let $\quad z = S^2 = \pi^2 x^2 (x^2 + y^2)$ ...(*ii*)

Putting $x^2 = \frac{k}{y}$ from (*i*) in (*ii*) to get z as a function of single independent variable y.

[Here, we are getting z as a simpler function of y as compared to z as a function of x]

$\therefore \quad z = \pi^2 \dfrac{k}{y}\left(\dfrac{k}{y} + y^2\right) = \pi^2 k\left(\dfrac{k}{y^2} + y\right)$

or $\quad z = \pi^2 k\ (ky^{-2} + y)$

$\therefore \quad \dfrac{dz}{dy} = \pi^2 k\ [-2ky^{-3} + 1]$ and $\dfrac{d^2z}{dy^2} = \pi^2 k\ [6k\ y^{-4}] = \dfrac{6\pi^2 k^2}{y^4}$

For max. or min., put $\dfrac{dz}{dy} = 0$

$\therefore \quad \pi^2 k\left(-\dfrac{2k}{y^3} + 1\right) = 0 \qquad$ But $\quad \pi^2 k \neq 0$

$\therefore \quad -\dfrac{2k}{y^3} + 1 = 0 \qquad$ or $\quad \dfrac{2k}{y^3} = 1$

$\therefore \quad y^3 = 2k \qquad \therefore \quad y = (2k)^{1/3}$

At $\quad y = (2k)^{1/3}, \dfrac{d^2z}{dy^2} = \dfrac{6\pi^2k^2}{(2k)^{4/3}}$ which is positive.

$\therefore \quad z$ is least when $y = (2k)^{1/3}$ ...(*iii*)

$\therefore \quad$ From (*i*), $\quad x^2 = \dfrac{k}{y} = \dfrac{k}{(2k)^{1/3}}$ [Using (*iii*)]

or $\quad x^2 = \dfrac{2k}{2(2k)^{1/3}} = \dfrac{(2k)^{2/3}}{2} = \dfrac{y^2}{2}$ [By (*iii*)]

or $\quad y^2 = 2x^2 \qquad \therefore \quad y = \sqrt{2}\,x$

$\therefore \quad z$ or S is least when height $= \sqrt{2}$ (radius of base).

25. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

Sol. Let x be the base radius, y the height, l the given slant height and θ, the semi-vertical angle of cone. ($x > 0, y > 0$)

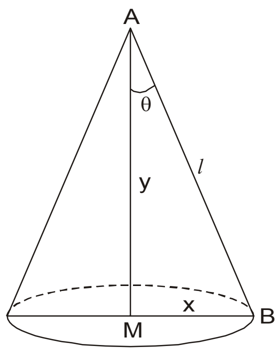
In ΔAMB,

By Pythagoras Theorem, $\quad x^2 + y^2 = l^2$

$\therefore \quad x^2 = l^2 - y^2$ ...(*i*)

Let V denote the volume of cone, then

$$V = \frac{1}{3}\pi x^2 y$$ ...(*ii*) [V is to be maximised here]



Putting $x^2 = l^2 - y^2$ from (*i*) in (*ii*), to express V as a function of single independent variable y.

$$V = \frac{1}{3}\pi\ (l^2 - y^2)\ y$$

or $\quad V = \dfrac{\pi}{3}\ (l^2 y - y^3)$

$\therefore \quad \dfrac{dV}{dy} = \dfrac{\pi}{3}\ (l^2 - 3y^2)$

and $\dfrac{d^2V}{dy^2} = \dfrac{\pi}{3}(-6y) = -2\pi y$

For max. or min. put $\dfrac{dV}{dy} = 0$

$\therefore \dfrac{\pi}{3}(l^2 - 3y^2) = 0$ But $\dfrac{\pi}{3} \neq 0$

$\therefore \quad l^2 - 3y^2 = 0$ or $3y^2 = l^2$ or $y^2 = \dfrac{l^2}{3}$

$\therefore \quad y = \dfrac{l}{\sqrt{3}}$ $\quad (\because y > 0)$

At $y = \dfrac{l}{\sqrt{3}}$, $\dfrac{d^2V}{dy^2} = -2\pi y = -\dfrac{2\pi l}{\sqrt{3}}$ which is negative.

$\therefore$ V is maximum at $y = \dfrac{l}{\sqrt{3}}$.

Putting $y = \dfrac{l}{\sqrt{3}}$ in eqn. (i), $x^2 = l^2 - \dfrac{l^2}{3} = \dfrac{2l^2}{3}$

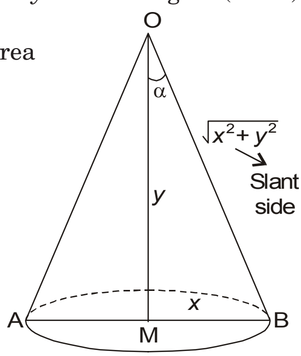
$\therefore \quad x = \sqrt{2}\dfrac{l}{\sqrt{3}}$

In right angled ΔAMB, $\tan\theta = \dfrac{MB}{AM} = \dfrac{x}{y} = \dfrac{\sqrt{2}\dfrac{l}{\sqrt{3}}}{\dfrac{l}{\sqrt{3}}} = \sqrt{2}$

$\therefore$ Semi-vertical angle $\theta = \tan^{-1}\sqrt{2}$.

26. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\dfrac{1}{3}\right)$.

Sol. Let x be the radius of base of cone and y be its height. (Total) surface area of cone is given.



$\therefore \underset{\text{(Curved Surface area)}}{\underset{\downarrow}{\pi rl}} + \underset{\text{(Area of base)}}{\underset{\downarrow}{\pi r^2}} = \text{Given Surface area}$

$\Rightarrow \quad \pi x\sqrt{x^2+y^2} + \pi x^2 = S$ (say)

Dividing both sides by π,

$x\sqrt{x^2+y^2} + x^2 = \dfrac{S}{\pi} = k$ (say)

$\Rightarrow \quad x\sqrt{x^2+y^2} = k - x^2$

Squaring both sides, we have $x^2(x^2+y^2) = (k-x^2)^2$

or $\quad x^4 + x^2y^2 = k^2 + x^4 - 2kx^2$

$\Rightarrow \quad x^2y^2 + 2kx^2 = k^2 \qquad\qquad \Rightarrow \quad x^2(y^2 + 2k) = k^2$

$$\Rightarrow \quad x^2 = \frac{k^2}{y^2 + 2k} \qquad \text{...}(i)$$

Let z denote the volume of the cone.

$$\therefore \quad z = \frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{\pi x^2 y}$$

Putting the value of x^2 from (i),

$$z = \frac{1}{3}\pi \; \frac{k^2}{y^2 + 2k} y = \frac{1}{3}\pi k^2 \; \frac{y}{y^2 + 2k}$$

$$\therefore \quad \frac{dz}{dy} = \frac{1}{3}\pi k^2 \; \frac{d}{dy} \; \frac{y}{y^2 + 2k}$$

$$\text{or} \quad \frac{dz}{dy} = \frac{1}{3}\pi k^2 \left[\frac{(y^2 + 2k) \,.\, 1 - y \,.\, 2y}{(y^2 + 2k)^2}\right] \text{(By quotient rule)}$$

$$\therefore \quad \frac{dz}{dy} = \frac{1}{3}\pi k^2 \; \frac{(2k - y^2)}{(y^2 + 2k)^2} \qquad \text{...}(ii)$$

Putting $\frac{dz}{dy} = 0$ to find turning points, we have

$$\frac{\pi k^2 \,(2k - y^2)}{3\,(y^2 + 2k)^2} = 0 \;\Rightarrow\; \pi k^2 \,(2k - y^2) = 0$$

But $\pi k^2 \neq 0 \quad \therefore \quad 2k - y^2 = 0 \;\Rightarrow\; y^2 = 2k$

$\therefore \quad y = \pm \sqrt{2k}$

Rejecting negative sign because height y of cone can't be negative.

$\therefore \quad y = \sqrt{2k}$ is the only turning point.

Let us apply first derivative test.

(because finding $\frac{d^2z}{dx^2}$ looks to be tedious)

Now in R.H.S. of (ii), $\frac{\pi k^2}{3(y^2 + 2k)^2} > 0$ clearly.

When y is slightly $< \sqrt{2k}$; then $y^2 < 2k$

$\Rightarrow \quad 0 < 2k - y^2 \;\Rightarrow\; 2k - y^2 > 0,$

therefore from (ii), $\frac{dz}{dy} > 0$ *i.e.,* (positive)

When y is slightly $> \sqrt{2k}$, then $y^2 > 2k \;\Rightarrow\; 0 > 2k - y^2$

$\Rightarrow \quad 2k - y^2 < 0$; therefore from (ii) $\frac{dz}{dy} < 0$ *i.e.,* (negative)

$\therefore \quad \frac{dz}{dy}$ changes sign from (+) to (–) as y increases through $\sqrt{2k}$

$\therefore \quad$ Volume z is maximum at $y = \sqrt{2k}$

Putting $y = \sqrt{2k}$ in (i), $x^2 = \dfrac{k^2}{2k + 2k} = \dfrac{k^2}{4k} = \dfrac{k}{4}$

$\therefore \qquad x = \sqrt{\dfrac{k}{4}} = \dfrac{\sqrt{k}}{2}$

Let α be the semi-vertical angle of the cone.
In right angled ΔOMB,

$$\sin \alpha = \frac{\text{MB}}{\text{OB}} = \frac{x}{\sqrt{x^2 + y^2}}$$

Putting values of x and y, $\sin \alpha = \dfrac{\frac{\sqrt{k}}{2}}{\sqrt{\frac{k}{4} + 2k}} = \dfrac{\frac{\sqrt{k}}{2}}{\sqrt{\frac{9k}{4}}} = \dfrac{\frac{\sqrt{k}}{2}}{3\frac{\sqrt{k}}{2}} = \dfrac{1}{3}$

$\therefore \qquad \alpha = \sin^{-1} \dfrac{1}{3}$.

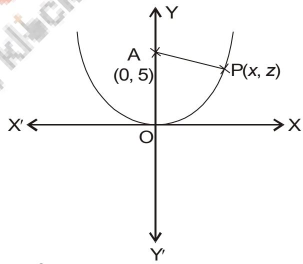
Choose the correct answer in the Exercises 27 to 29.

27. The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is

(A) $(2\sqrt{2}, 4)$ (B) $(2\sqrt{2}, 0)$ (C) (0, 0) (D) (2, 2).

Sol. Equation of the curve (upward parabola here) is

$$x^2 = 2y \qquad ...(i)$$



The given point is A(0, 5).
Let P(x, y) be any point on curve (i).
$\therefore$ Distance z = AP

$$= \sqrt{(x - 0)^2 + (y - 5)^2}$$

| Distance formula

Let $\qquad Z = z^2 = x^2 + (y - 5)^2$

Putting $\qquad x^2 = 2y$ from (i),

$\qquad Z = 2y + (y - 5)^2 = 2y + y^2 + 25 - 10y$

or $\qquad Z = y^2 - 8y + 25$

$\therefore \qquad \dfrac{dZ}{dy} = 2y - 8$ and $\dfrac{d^2Z}{dy^2} = 2$

Putting $\qquad \dfrac{dZ}{dy} = 0$ to get turning point(s), we have

$\qquad \dfrac{dZ}{dy} = 0$ *i.e.,* $2y - 8 = 0 \Rightarrow 2y = 8 \Rightarrow y = 4$

At $\qquad y = 4$, $\dfrac{d^2Z}{dy^2} = 2$ is (+ ve)

$\therefore$ Z$(= z^2)$ is minimum and hence z is minimum at $y = 4$.

Putting $y = 4$ in (i), $x^2 = 8 \quad \therefore \quad x = \pm \sqrt{8} = \pm 2\sqrt{2}$.

$\therefore \quad (2\sqrt{2}, 4)$ and $(-2\sqrt{2}, 4)$ are two points on curve (i) which are nearest to the given point (0, 5).

$\therefore$ Option (A) is correct answer.

28. For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is

(A) 0 **(B) 1** **(C) 3** **(D) $\frac{1}{3}$.**

Sol. Given: Let $f(x) = \frac{1-x+x^2}{1+x+x^2}$ ...(*i*)

$$\therefore \quad f'(x) = \frac{(1+x+x^2)\frac{d}{dx}(1-x+x^2)-(1-x+x^2)\frac{d}{dx}(1+x+x^2)}{(1+x+x^2)^2}$$

$$= \frac{(1+x+x^2)(-1+2x)-(1-x+x^2)(1+2x)}{(1+x+x^2)^2}$$

or $$f'(x) = \frac{-1+2x-x+2x^2-x^2+2x^3-1-2x+x+2x^2-x^2-2x^3}{(1+x+x^2)^2}$$

or $$f'(x) = \frac{-2+2x^2}{(1+x+x^2)^2} = \frac{-2(1-x^2)}{(1+x+x^2)^2}$$

Let us put $f'(x) = 0$ to get turning points.

Therefore $\frac{-2(1-x^2)}{(1+x+x^2)^2} = 0 \Rightarrow -2(1-x^2) = 0$

But $-2 \neq 0$. Therefore, $1 - x^2 = 0$ or $-x^2 = -1$

$\therefore \quad x^2 = 1 \Rightarrow x = \pm 1$

$\therefore \quad x = -1$ and $x = 1$ are two turning points.

Let us find values of $f(x)$ at these two turning points only because no closed interval is given to be domain of $f(x)$.

Putting $x = -1$ in (*i*), $f(-1) = \frac{1+1+1}{1-1+1} = 3$

Putting $x = 1$ in (*i*), $f(1) = \frac{1-1+1}{1+1+1} = \frac{1}{3}$

Therefore, minimum value of $f(x)$ is $\frac{1}{3}$.

$\therefore$ Option (D) is the correct answer.

Note. Maximum value of $f(x)$ for the above question is 3.

29. The maximum value of $[x(x-1)+1]^{1/3}$, $0 \le x \le 1$ is

(A) $\left(\frac{1}{3}\right)^{1/3}$ **(B) $\frac{1}{2}$** **(C) 1** **(D) $\frac{1}{3}$.**

Sol. Let $f(x) = (x(x-1)+1)^{1/3} = (x^2 - x + 1)^{1/3}$, $0 \le x \le 1$ ...(*i*)

$$\therefore \quad f'(x) = \frac{1}{3}(x^2 - x + 1)^{-2/3} \frac{d}{dx}(x^2 - x + 1)$$

or $$f'(x) = \frac{(2x-1)}{3(x^2-x+1)^{2/3}}$$

Putting $f'(x) = 0$ to get turning points, we have

$$\frac{2x - 1}{3(x^2 - x + 1)^{2/3}} = 0$$

Cross-multiplying $2x - 1 = 0 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$

This turning point $x = \frac{1}{2}$ belongs to the given closed interval $0 \le x \le 1$ *i.e.,* $[0, 1]$.

Now let us find values of $f(x)$ at the turning point $x = \frac{1}{2}$ and end points $x = 0$ and $x = 1$ of given closed interval $[0, 1]$.

Putting $x = \frac{1}{2}$ in (i),

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{4} - \frac{1}{2} + 1\right)^{1/3} = \left(\frac{1 - 2 + 4}{4}\right)^{1/3} = \left(\frac{3}{4}\right)^{1/3} < 1.$$

Putting $x = 0$ in (i), $f(0) = (1)^{1/3} = 1$

Putting $x = 1$ in (i), $f(1) = (1 - 1 + 1)^{1/3} = (1)^{1/3} = 1$

$\therefore$ Maximum value of $f(x)$ is 1.

$\therefore$ Option (C) is the correct answer.

Note. Minimum value of $f(x)$ for the above question is $\left(\frac{3}{4}\right)^{1/3}$.