Exercise 6.3

1. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.

Sol. Given: Equation of the curve is $y = 3x^4 - 4x$ ...(*i*)

$\therefore$ Slope of the tangent to the curve $y = f(x)$ at the point (x, y)

$$= \text{Value of } \frac{dy}{dx} \text{ at the point } (x, y)$$

$$= 3(4x^3) - 4 = 12x^3 - 4$$

$\therefore$ Slope of the tangent at (point) $x = 4$ to curve (*i*) is

$$12(4)^3 - 4 = 12 \times 64 - 4 = 768 - 4 = 764.$$

2. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.

Sol. Given: Equation of the curve is $y = \frac{x-1}{x-2}$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = \frac{(x-2)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x-2)}{(x-2)^2}$$

or $$\frac{dy}{dx} = \frac{(x-2)-(x-1)}{(x-2)^2} = \frac{x-2-x+1}{(x-2)^2} = \frac{-1}{(x-2)^2} \quad ...(ii)$$

Putting $x = 10$ (given) in (*ii*), slope of the tangent to the given curve (*i*), at $x = 10$ (= value of $\frac{dy}{dx}$ at $x = 10$)

$$= \frac{-1}{(10-2)^2} = \frac{-1}{(8)^2} = \frac{-1}{64}.$$

3. Find the slope of the tangent to the curve $y = x^3 - x + 1$ at the point whose x-coordinate is 2.

Sol. Given: Equation of the curve is $y = x^3 - x + 1$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 1$$

Slope of the tangent to curve (*i*) at $x = 2$ (given)

$$= \text{Value of } \frac{dy}{dx} \text{ (at } x = 2) = 3.2^2 - 1 = 3(4) - 1$$

$$= 12 - 1 = 11.$$

4. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.

Sol. Given: Equation of the curve is $y = x^3 - 3x + 2$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 3$$

Slope of the tangent of curve (*i*) at $x = 3$ (given)

$$= \text{Value of } \frac{dy}{dx} \text{ (at } x = 3) = 3 \cdot 3^2 - 3 = 3 \cdot 9 - 3$$

$$= 27 - 3 = 24.$$

5. Find the slope of the normal to the curve

$$x = a\cos^3\theta,\ y = a\sin^3\theta \quad \text{at} \quad \theta = \frac{\pi}{4}.$$

Sol. Given: Equations of the curve are

$$x = a\cos^3\theta,\ y = a\sin^3\theta$$

$$\therefore \quad \frac{dx}{d\theta} = a\frac{d}{d\theta}(\cos\theta)^3 \quad \text{and} \quad \frac{dy}{d\theta} = a\frac{d}{d\theta}(\sin\theta)^3$$

$$= a.3(\cos\theta)^2\frac{d}{d\theta}(\cos\theta) \quad \text{and}$$

$$= a.3(\sin\theta)^2\frac{d}{d\theta}\sin\theta$$

$$\text{or} \quad \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

Slope of the tangent at $\theta = \frac{\pi}{4}$ (given) = value of $\frac{dy}{dx}$ at $\left(\theta = \frac{\pi}{4}\right)$

$$= -\tan\frac{\pi}{4} = -1.$$

$\therefore$ Slope of the normal $\left(\text{at } \theta = \frac{\pi}{4}\right)$ = negative reciprocal of slope of tangent = 1. $\left(\because \frac{-1}{m} = \frac{-1}{-1} = 1\right)$.

6. Find the slope of the normal to the curve

$$x = 1 - a\sin\theta,\ y = b\cos^2\theta \text{ at } \theta = \frac{\pi}{2}.$$

Sol. Given: Equations of the curve are

$$x = 1 - a\sin\theta,\ y = b\cos^2\theta$$

$$\therefore \quad \frac{dx}{d\theta} = 0 - a\cos\theta \quad \text{and} \quad \frac{dy}{d\theta} = b\frac{d}{d\theta}(\cos\theta)^2$$

$$\Rightarrow \quad \frac{dx}{d\theta} = -a\cos\theta \quad \text{and} \quad \frac{dy}{d\theta} = b.2(\cos\theta)\frac{d}{d\theta}\cos\theta$$

$$= -2b\cos\theta\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2b\cos\theta\sin\theta}{-a\cos\theta} = \frac{2b}{a}\sin\theta$$

Slope of the tangent (at $\theta = \frac{\pi}{2}$ (given))

$$= \text{value of } \frac{dy}{dx}\left(\text{at } \theta = \frac{\pi}{2}\right)$$

$$= \frac{2b}{a}\sin\frac{\pi}{2} = \frac{2b}{a}(1)$$

$$= \frac{2b}{a} \ (= m \text{ say})$$

$\therefore$ Slope of the normal $\left(\text{at } \theta = \frac{\pi}{2}\right) = \frac{-1}{m} = -\frac{a}{2b}$.

7. Find the points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis.

Sol. Equation of curve is $y = x^3 - 3x^2 - 9x + 7$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = 3x^2 - 6x - 9$ = Slope of tangent at (x, y)

Since the tangent is parallel to the x-axis, $\frac{dy}{dx} = 0$

$\Rightarrow \quad 3x^2 - 6x - 9 = 0$ or $x^2 - 2x - 3 = 0$

or $\quad (x - 3)(x + 1) = 0 \qquad \therefore \quad x = 3, -1$

When $x = 3$, from (*i*), $\quad y = 27 - 27 - 27 + 7 = -20$

When $x = -1$, from (*i*), $\quad y = -1 - 3 + 9 + 7 = 12$.

$\therefore$ The required points are $(3, -20)$ and $(-1, 12)$.

8. Find a point on the curve $y = (x - 2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

Sol. Let A(2, 0) and B(4, 4) be the given points.

Slope of chord AB $= \frac{4-0}{4-2} = 2 \qquad \left[\because \ m = \frac{y_2 - y_1}{x_2 - x_1}\right]$

Equation of curve is $y = (x - 2)^2$

$\therefore$ Slope of tangent at $(x, y) = \frac{dy}{dx} = 2(x - 2)$.

If the tangent is parallel to the chord AB, then

slope of tangent = slope of chord

$\Rightarrow \quad 2(x - 2) = 2 \ \Rightarrow 2x - 4 = 2 \ \Rightarrow 2x = 6 \ \Rightarrow x = 3$

$\therefore \qquad y = (3 - 2)^2 = 1$

Hence, the required point is (3, 1).

9. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

Sol. Equation of curve is $\quad y = x^3 - 11x + 5$ ...(*i*)

Equation of tangent is $\quad y = x - 11$

or $\quad x - y - 11 = 0$ ...(*ii*)

From (*i*), $\quad \frac{dy}{dx} = 3x^2 - 11$

$\Rightarrow$ Slope of tangent at (x, y) is $3x^2 - 11$.

But slope of tangent from (*ii*) is $\frac{-a}{b} = \frac{-1}{-1} = 1$.

$\therefore \quad 3x^2 - 11 = 1$ or $3x^2 = 12$ or $x^2 = 4 \quad \therefore \quad x = \pm 2$

From (*i*), when $x = 2, \quad y = 8 - 22 + 5 = -9$

when $\quad x = -2 \quad y = -8 + 22 + 5 = 19$

$\therefore$ We get two points $(2, -9)$ and $(-2, 19)$. Of these, $(-2, 19)$

does not satisfy eqn. (*ii*) while (2, – 9) does. Hence, the required point is (2, – 9).

10. Find the equation of all lines having slope – 1 that are tangents to the curve $y = \dfrac{1}{x-1}$, $x \neq 1$.

Sol. Given: Equation of the curve is $y = \dfrac{1}{x-1} = (x-1)^{-1}$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = (-1)(x-1)^{-2} \frac{d}{dx}(x-1) = \frac{-1}{(x-1)^2}$$

= Slope of the tangent to the given curve at any point (*x*, *y*).

But the slope is given to be – 1

$$\therefore \quad \frac{-1}{(x-1)^2} = -1 \quad \Rightarrow \quad -(x-1)^2 = -1$$

$$\Rightarrow \quad (x-1)^2 = 1 \quad \Rightarrow \quad x - 1 = \pm 1 \quad \Rightarrow \quad x = 1 \pm 1$$

$$\Rightarrow \quad x = 1 + 1 = 2 \quad \text{or} \quad x = 1 - 1 = 0$$

Putting $x = 2$ in (*i*), $y = \dfrac{1}{2-1} = \dfrac{1}{1} = 1$

∴ One point of contact is (2, 1).

∴ Equation of one required tangent is $y - 1 = -1(x-2)$

$[\because y - y_1 = m(x - x_1)]$

i.e., $y - 1 = -x + 2$ or $x + y - 3 = 0$

Putting $x = 0$ in (*i*), $y = \dfrac{1}{0-1} = \dfrac{1}{-1} = -1$

∴ The other point of contact is (0, – 1).

∴ Equation of the other tangent is

$$y - (-1) = -1(x - 0) \quad \text{or} \quad y + 1 = -x$$

or $$x + y + 1 = 0$$

∴ Equations of required tangents are

$$x + y - 3 = 0 \quad \text{and} \quad x + y + 1 = 0.$$

11. Find the equations of all lines having slope 2 which are tangents to the curve $y = \dfrac{1}{x-3}$, $x \neq 3$.

Sol. Equation of curve is $y = \dfrac{1}{x-3} = (x-3)^{-1}$

Differentiating w.r.t. *x*, we get

$$\frac{dy}{dx} = (-1)(x-3)^{-2} = \frac{-1}{(x-3)^2}$$

= Slope of tangent to the given curve at any point (*x*, *y*)

But the slope is given to be 2.

$$\therefore \quad \frac{-1}{(x-3)^2} = 2 \quad \text{or} \quad 2(x-3)^2 = -1 \quad \text{or} \quad (x-3)^2 = -\frac{1}{2} < 0$$

which is not possible since $(x-3)^2 > 0$.

Hence, there is no tangent to the given curve having slope 2.

12. Find the equations of all lines having slope 0 which are tangents to the curve

$$\boldsymbol{y = \frac{1}{x^2 - 2x + 3}}.$$

Sol. Equation of curve is $y = \dfrac{1}{x^2 - 2x + 3}$ ...(*i*)

Differentiating w.r.t. x, we have

$= \dfrac{dy}{dx} = \dfrac{d}{dx}[(x^2 - 2x + 3)^{-1}] = -(x^2 - 2x + 3)^{-2} \cdot (2x - 2)$

$= \dfrac{-2(x-1)}{(x^2 - 2x + 3)^2}$

= Slope of tangent to the given curve at any point (x, y)

But the slope (of tangent) is given to be 0

$\therefore \quad \dfrac{-2(x-1)}{(x^2 - 2x + 3)^2} = 0 \quad \Rightarrow -2(x - 1) = 0$

$\Rightarrow \quad x - 1 = 0 \quad \Rightarrow \quad x = 1$

Putting $x = 1$ in (*i*), we have $y = \dfrac{1}{1 - 2 + 3} = \dfrac{1}{2}$

Thus the point on the curve at which tangent has slope 0 is $\left(1, \dfrac{1}{2}\right)$.

$\therefore$ Equation of tangent is $y - \dfrac{1}{2} = 0\,(x - 1)$

or $y - \dfrac{1}{2} = 0$ or $y = \dfrac{1}{2}$.

13. Find the points on the curvem $\boldsymbol{\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1}$ at which the tangents are

(*i*) parallel to *x*-axis **(*ii*) parallel to *y*-axis.**

Sol. **Given:** Equation of the curve is $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ ...(*i*)

Differentiating both sides of eqn. (*i*) w.r.t. x, we have

$\dfrac{2x}{9} + \dfrac{2y}{16}\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad \dfrac{2y}{16}\dfrac{dy}{dx} = -\dfrac{2x}{9}$

$\Rightarrow \quad 18y\dfrac{dy}{dx} = -32x \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{-32x}{18y} = \dfrac{-16x}{9y}$ ...(*ii*)

(*i*) If tangent is **parallel to *x*-axis,** $\Rightarrow$ Slope of tangent = 0

$\Rightarrow \quad \boldsymbol{\dfrac{dy}{dx} = 0} \quad \Rightarrow$ From (*ii*), $\dfrac{-16x}{9y} = 0 \Rightarrow -16x = 0$

$\Rightarrow \quad x = \dfrac{0}{-16} = 0$

Putting $x = 0$ in (i), $\frac{y^2}{16} = 1$ or $y^2 = 16$. Therefore, $y = \pm 4$.

$\therefore$ The points on curve (i) where tangents are parallel to x-axis are $(0, \pm 4)$.

(ii) If the tangent is **parallel to y-axis**

$$\Rightarrow \text{ Slope of the tangent} = \pm \infty \quad \Rightarrow \quad \frac{dy}{dx} = \pm \infty$$

$$\Rightarrow \quad \frac{dx}{dy} = 0$$

$$\therefore \text{ From } (ii), \quad \frac{9y}{-16x} = 0 \quad \Rightarrow \quad 9y = 0 \quad \Rightarrow \quad y = \frac{0}{9} = 0$$

Putting $y = 0$ in (i), $\frac{x^2}{9} = 1$ or $x^2 = 9$

$\therefore$ $x = \pm 3$.

Hence the points on the curve at which the tangent are parallel to y-axis are $(\pm 3, 0)$.

14. Find the equations of the tangent and normal to the given curves at the indicated points:

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5).

(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3).

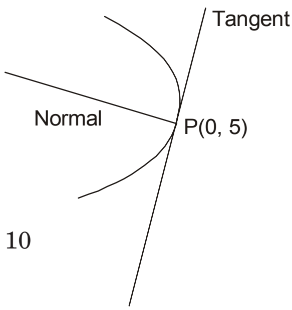
(iii) $y = x^3$ at (1, 1).

(iv) $y = x^2$ at (0, 0).

(v) $x = \cos t$, $y = \sin t$ at $t = \frac{\pi}{4}$.

Sol. (i) **Given:** Equation of the curve is

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5 \quad ...(i)$$



$\therefore$ $\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$ = Slope of the tangent at point (x, y)

$\therefore$ Slope of tangent at (0, 5)

$= \text{Value of } \frac{dy}{dx}$ at (0, 5)

(Putting $x = 0$)

$= 4(0)^3 - 18(0)^2 + 26(0) - 10$

$= -10$ ($= m$ say)

$\therefore$ Slope of the normal at (0, 5)

$$= \frac{-1}{m} = \frac{-1}{-10} = \frac{1}{10}$$

$\therefore$ Equation of the tangent at (0, 5) is

$y - 5 = -10(x - 0)$ | $y - y_1 = m(x - x_1)$

i.e., $y - 5 = -10x$ or $10x + y = 5$

and equation of the normal at (0, 5) is

$$y - 5 = \frac{1}{10}(x - 0)$$

$\Rightarrow \quad 10y - 50 = x \quad$ *i.e.,* $\quad - x + 10y - 50 = 0$

or $\quad x - 10y + 50 = 0.$

(*ii*) **Given:** Equation of the curve is

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5 \qquad ...(i)$$

$$\therefore \qquad \frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

$$= \text{Slope of the tangent at the point } (x, y)$$

$\therefore$ Slope of the tangent at (1, 3) = Value of $\frac{dy}{dx}$ at (1, 3).

(Putting $x = 1$) $= 4(1)^3 - 18(1)^2 + 26(1) - 10$

$= 4 - 18 + 26 - 10 = 30 - 28 = 2$ (= m say)

$\therefore$ Slope of the normal at (1, 3) $= \frac{-1}{m} = \frac{-1}{2}$

$\therefore$ Equation of the tangent at (1, 3) is $y - 3 = 2(x - 1)$

$\Rightarrow \quad y - 3 = 2x - 2 \quad \Rightarrow \quad y = 2x + 1$

and equation of the normal at (1, 3) is $y - 3 = \frac{-1}{2}(x - 1)$

$\Rightarrow \quad 2(y - 3) = -(x - 1) \quad \Rightarrow \quad 2y - 6 = -x + 1$

$\Rightarrow \quad x + 2y - 7 = 0.$

(*iii*) **Given:** Equation of the curve is

$$y = x^3 \qquad ...(i)$$

$\therefore \quad \frac{dy}{dx} = 3x^2$ = Slope of the tangent at the point (x, y).

$\therefore$ Slope of the tangent at (1, 1) = Value of $\frac{dy}{dx}$ at (1, 1).

(Putting $x = 1$) $= 3.1^2 = 3 = m$(say)

$\therefore$ Slope of the normal at (1, 1) $= \frac{-1}{m} = \frac{-1}{3}$

$\therefore$ Equation of the tangent at (1, 1) is $y - 1 = 3(x - 1)$

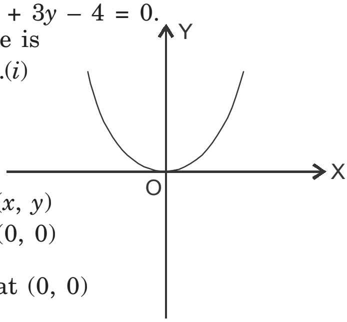
$\Rightarrow \quad y - 1 = 3x - 3 \quad \Rightarrow \quad y = 3x - 2$

and equation of the normal at (1, 1) is $y - 1 = \frac{-1}{3}(x - 1)$

$\Rightarrow \quad 3y - 3 = -x + 1 \quad \Rightarrow \quad x + 3y - 4 = 0.$

(*iv*) **Given:** Equation of the curve is

$$y = x^2 \qquad ...(i)$$



$$\therefore \qquad \frac{dy}{dx} = 2x$$

= Slope of the tangent at (x, y)

$\therefore$ Slope of the tangent at (0, 0)

$= \text{Value of } \frac{dy}{dx} \text{ at } (0, 0)$

(Putting $x = 0$) $= 2 \times 0 = 0$ (= m say)

$\therefore$ Tangent at (0, 0) to curve (*i*) is $(y - 0) = 0\ (x - 0)$ or $y = 0$ *i.e.* x-axis and hence normal at (0, 0) to curve (*i*) is y-axis.

(*v*) **Given:** Equations of the curve are

$$x = \cos t, \quad y = \sin t$$

$$\therefore \quad \frac{dx}{dt} = -\sin t \text{ and } \frac{dy}{dt} = \cos t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t$$

$$= \text{Slope of the tangent at } (x, y)$$

$$\therefore \quad \text{Slope of the tangent at } t = \frac{\pi}{4} \text{ is value of } \frac{dy}{dx} \text{ at } t = \frac{\pi}{4}$$

$$= -\cot\frac{\pi}{4} = -1 \ (= m \text{ say})$$

$$\therefore \quad \text{Slope of the normal at } t = \frac{\pi}{4} \text{ is } \frac{-1}{m} = \frac{-1}{-1} = 1$$

$$\text{Point } t = \frac{\pi}{4} \quad \Rightarrow \quad \text{Point } (x, y) = (\cos t, \sin t)$$

$$= \left(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\therefore \quad \text{Equation of the tangent is } y - \frac{1}{\sqrt{2}} = -1\left(x - \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad y - \frac{1}{\sqrt{2}} = -x + \frac{1}{\sqrt{2}} \quad \Rightarrow \quad x + y = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

$$\text{or} \quad x + y = \sqrt{2} \qquad \left[\because \frac{2}{\sqrt{2}} = \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}} = \sqrt{2}\right]$$

and equation of the normal at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is

$$y - \frac{1}{\sqrt{2}} = 1\left(x - \frac{1}{\sqrt{2}}\right) \quad \text{or} \quad y - \frac{1}{\sqrt{2}} = x - \frac{1}{\sqrt{2}}$$

$$\text{or} \quad y = x.$$

15. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is

(*a*) parallel to the line $2x - y + 9 = 0$

(*b*) perpendicular to the line $5y - 15x = 13$.

Sol. Given: Equation of the curve is $y = x^2 - 2x + 7$ ...(*i*)

$\therefore$ Slope of the tangent $= \frac{dy}{dx} = 2x - 2$ ...(*ii*)

(*a*) Slope of the given line $2x - y + 9 = 0$ is

$$\frac{-\text{coeff. of } x}{\text{coeff. of } y}\left(\frac{-a}{b}\right) = \frac{-2}{-1} = 2$$

$\therefore$ Slope of tangent parallel to this line is also = 2

($\because$ Parallel lines have same slope)

$\Rightarrow$ (By (*ii*)), $2x - 2 = 2 \Rightarrow 2x = 2 + 2 = 4$

$$\Rightarrow \quad x = \frac{4}{2} = 2$$

Putting $x = 2$ in (*i*), $y = 4 - 4 + 7 = 7$

$\therefore$ Point of contact is (2, 7)

$\therefore$ Equation of the tangent at (2, 7) is

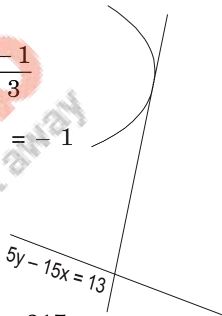
$y - 7 = 2(x - 2)$ or $y - 7 = 2x - 4$

or $\quad y - 2x - 3 = 0.$

(*b*) Slope of the given line

$5y - 15x = 13$ *i.e.*, $-15x + 5y = 13$

is $\frac{-a}{b} = -\left(\frac{-15}{5}\right) = 3 = (m \text{ say})$



$\therefore$ Slope of the required tangent

perpendicular to this line $= \frac{-1}{m} = \frac{-1}{3}$

$\Rightarrow$ (By (*ii*)) $2x - 2 = \frac{-1}{3} \Rightarrow 6x - 6 = -1$

$\Rightarrow \quad 6x = 6 - 1 = 5 \Rightarrow x = \frac{5}{6}$

Putting $x = \frac{5}{6}$ in (*i*), $y = \frac{25}{36} - \frac{5}{3} + 7$

$$= \frac{25 - 60 + 252}{36} = \frac{277 - 60}{36} = \frac{217}{36}$$

$\therefore$ Point of contact is $\left(\frac{5}{6}, \frac{217}{36}\right)$

$\therefore$ Equation of the required tangent at $\left(\frac{5}{6}, \frac{217}{36}\right)$ is

$$y - \frac{217}{36} = \frac{-1}{3}\left(x - \frac{5}{6}\right)$$

$\Rightarrow 3y - \frac{217}{12} = -x + \frac{5}{6} \quad \Rightarrow x + 3y = \frac{217}{12} + \frac{5}{6}$

$\Rightarrow \quad x + 3y = \frac{217 + 10}{12} = \frac{227}{12}$

Cross-multiplying, $12x + 36y = 227$.

16. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel.

Sol. Given: Equation of the given curve is $y = 7x^3 + 11$

$\therefore \quad \frac{dy}{dx} = 21x^2$ = Slope of the tangent to the curve at (x, y)

Putting $x = 2$, slope of the tangent $= 21(2)^2 = 21 \times 4 = 84$

Putting $x = -2$, slope of the tangent $= 21(-2)^2 = 21 \times 4 = 84$
Since the slopes of the two tangents are equal (each = 84), therefore, tangents at $x = 2$ and $x = -2$ are parallel.

17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

Sol. **Given:** Equation of the curve is $y = x^3$ ...(*i*)

$\therefore \dfrac{dy}{dx} = 3x^2$ = Slope of the tangent at the point (x, y) ...(*ii*)

Given: Slope of the tangent = y-coordinate of the point.
Putting values from (*ii*) and (*i*),

$$3x^2 = x^3 \Rightarrow 3x^2 - x^3 = 0 \Rightarrow x^2(3 - x) = 0$$

$\therefore$ Either $x^2 = 0$ *i.e.,* $x = 0$ or $3 - x = 0$ *i.e.,* $x = 3$
Putting $x = 0$ in (*i*), $y = 0$ $\therefore$ Point is (0, 0)
Putting $x = 3$ in (*i*), $y = 3^3 = 27$ $\therefore$ Point is (3, 27)
$\therefore$ The required points are (0, 0) and (3, 27).

18. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.

Sol. Equation of curve is

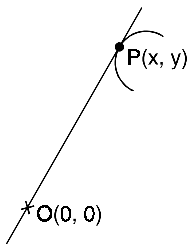
$$y = 4x^3 - 2x^5 \quad ...(i)$$

Let the required point be P(x, y), the tangent at which passes through the origin O(0, 0).

Differentiating both sides of eqn. (*i*) w.r.t. x, $\dfrac{dy}{dx} = 12x^2 - 10x^4$

$\therefore$ Slope of the tangent OP at P(x, y) = $\dfrac{dy}{dx} = 12x^2 - 10x^4 = \dfrac{y - 0}{x - 0}$

or $\dfrac{y}{x} = 12x^2 - 10x^4$ or $y = 12x^3 - 10x^5$



Putting this value of y in eqn. (*i*), we have

$$12x^3 - 10x^5 = 4x^3 - 2x^5 \text{ or } 8x^3 - 8x^5 = 0$$

or $8x^3(1 - x^2) = 0$
$\therefore$ Either $x = 0$ or $1 - x^2 = 0$
i.e., $x^2 = 1$ $\therefore$ $x = \pm 1$
Putting $x = 0$ in (*i*), $y = 0$
Putting $x = 1$ in (*i*), $y = 4 - 2 = 2$
Putting $x = -1$ in (*i*), $y = -4 + 2 = -2$
Hence, the required points are (0, 0), (1, 2) and (– 1, – 2).

19. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.

Sol. Equation of curve is $x^2 + y^2 - 2x - 3 = 0$ ...(*i*)
Differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - 2 = 0 \quad \text{or} \quad 2y\frac{dy}{dx} = 2 - 2x$$

Dividing by 2, $y\dfrac{dy}{dx} = 1 - x$

$\therefore \quad \frac{dy}{dx} = \frac{1-x}{y}$

Now the tangent is parallel to the x-axis if the slope of tangent is zero

i.e., $\frac{dy}{dx} = 0$ or $\frac{1-x}{y} = 0$ or $x = 1$

Putting $x = 1$ in (i), we get $1 + y^2 - 2 - 3 = 0$

or $y^2 = 4 \therefore y = \pm 2$

Hence, the required points are (1, 2) and (1, – 2).

20. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

Sol. Given: Equation of the curve is $ay^2 = x^3$ ...(*i*)

Differentiating both sides of (*i*) w.r.t. x,

$$a\frac{d}{dx}y^2 = \frac{d}{dx}x^3 \Rightarrow a.2y\frac{dy}{dx} = 3x^2$$

$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2ay}$ = Slope of the tangent at the point (x, y)

$\therefore$ Slope of the tangent at the point (am^2, am^3)

(Putting $x = am^2$, $y = am^3$) $= \frac{3(am^2)^2}{2a.am^3} = \frac{3a^2m^4}{2a^2m^3} = \frac{3m}{2}$

$\therefore$ Slope of the normal at the point $(am^2, am^3) = -\frac{2}{3m}$

(Negative reciprocal)

$\therefore$ Equation of the normal at (am^2, am^3) is

$$y - am^3 = -\frac{2}{3m}(x - am^2)$$

$\Rightarrow \quad 3m(y - am^3) = -2(x - am^2)$

$\Rightarrow \quad 3my - 3am^4 = -2x + 2am^2$

or $\quad 2x + 3my - 2am^2 - 3am^4 = 0$

or $\quad 2x + 3my - am^2(2 + 3m^2) = 0.$

21. Find the equations of the normal to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

Sol. Equation of curve is $y = x^3 + 2x + 6$ ...(*i*)

Differentiating w.r.t. x, we get

Slope of tangent to the curve at $(x, y) = \frac{dy}{dx} = 3x^2 + 2$

$\Rightarrow$ Slope of normal to the curve at (x, y)

$$= \frac{-1}{3x^2+2} \quad ...(ii)$$

Now the slope of given line $x + 14y + 4 = 0$ is $-\frac{1}{14}$. Since the normal is parallel to this line, the slope of normal is also $-\frac{1}{14}$ as parallel lines have equal slopes.

$\therefore$ By (*ii*), we have $\dfrac{-1}{3x^2+2} = -\dfrac{1}{14}$

or $3x^2 + 2 = 14$ or $3x^2 = 12$ or $x^2 = 4$ $\therefore$ $x = \pm 2$

Putting $x = 2$ in (*i*), $y = 8 + 4 + 6 = 18$

Putting $x = -2$ in (*i*), $y = -8 - 4 + 6 = -6$

$\therefore$ The coordinates of the feet of normals (*i.e.,* points of contact) are (2, 18) and (– 2, – 6).

$\therefore$ Equation of normal at (2, 18) is

$$y - 18 = -\frac{1}{14}(x - 2)$$

or $14y - 252 = -x + 2$ or $x + 14y - 254 = 0$

and equation of normal at (– 2, – 6) is

$$y + 6 = -\frac{1}{14}(x + 2)$$

or $14y + 84 = -x - 2$ or $x + 14y + 86 = 0.$

22. Find the equation of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Sol. Given: Equation of the parabola is $y^2 = 4ax$ ...(*i*)

Differentiating both sides of (*i*) w.r.t. x, we have

$$\frac{d}{dx}y^2 = 4a\frac{d}{dx}(x) \quad \Rightarrow \quad 2y\frac{dy}{dx} = 4a$$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{4a}{2y} = \dfrac{2a}{y}$ = Slope of the tangent at the point (x, y)

$\therefore$ Slope of the tangent at the point $(at^2, 2at)$ is

(Putting $x = at^2$, $y = 2at$) $= \dfrac{2a}{2at} = \dfrac{1}{t}$

$\therefore$ Slope of the normal $= -t$ (Negative reciprocal)

$\therefore$ Equation of the tangent at the point $(at^2, 2at)$ is

$$y - 2at = \frac{1}{t}(x - at^2) \quad \text{or} \quad ty - 2at^2 = x - at^2$$

$\Rightarrow$ $ty = x + at^2$

Again equation of the normal at the point $(at^2, 2at)$ is

$$y - 2at = -t(x - at^2) \quad \text{or} \quad y - 2at = -tx + at^3$$

or $tx + y = 2at + at^3$.

23. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

Sol. Equations of curves are $x = y^2$ ...(*i*) and $xy = k$ ...(*ii*)

To find the point(s) of intersection, we solve them simultaneously for x and y.

Putting $x = y^2$ from eqn. (*i*) in eqn. (*ii*),

we have $y^2 \cdot y = k$ or $y^3 = k$ $\therefore$ $y = k^{1/3}$

Putting this value of y in (*i*), $x = (k^{1/3})^2 = k^{2/3}$

$\therefore$ The point of intersection is $(k^{2/3}, k^{1/3}) = (x, y)$ (say) ...(*iii*)

Differentiating (*i*), w.r.t. x, $1 = 2y\frac{dy}{dx}$

or $\frac{dy}{dx} = \frac{1}{2y} = m_1$ ...(*iv*)

Differentiating (*ii*) w.r.t. x, $x\frac{dy}{dx} + y = 0$

or $\frac{dy}{dx} = -\frac{y}{x} = m_2$ ...(*v*)

Because the curves (*i*) and (*ii*) cut at right angles at their point of intersection (x, y), therefore $m_1m_2 = -1$.

Putting values of m_1 and m_2 from (*iv*) and (*v*), we have

$$\frac{1}{2y}\left(-\frac{y}{x}\right) = -1 \quad \text{or} \quad \frac{1}{2x} = 1$$

or $2x = 1$. But from (*iii*), $x = k^{2/3}$ $\therefore$ $2.k^{2/3} = 1$

Cubing both sides, $8k^2 = 1$.

24. Find the equation of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

Sol. Equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ...(*i*)

Differentiating w.r.t. x, we have $\frac{2x}{a^2} - \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$

or $\frac{-2y}{b^2}\frac{dy}{dx} = \frac{-2x}{a^2}$ or $\frac{dy}{dx} = \frac{b^2x}{a^2y}$ ...(*ii*)

Putting $x = x_0$ and $y = y_0$ in (*ii*), slope of tangent at (x_0, y_0) is $\frac{b^2x_0}{a^2y_0}$

$\therefore$ Equation of tangent at (x_0, y_0) is $y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$

or $yy_0 - y_0^2 = \frac{b^2}{a^2}(xx_0 - x_0^2)$ or $\frac{yy_0}{b^2} - \frac{y_0^2}{b^2} = \frac{xx_0}{a^2} - \frac{x_0^2}{a^2}$

or $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}$ ...(*iii*)

Since (x_0, y_0) lies on the hyperbola (*i*), $\therefore$ $\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$

Putting this value in R.H.S. of equation (*iii*), equation of tangent at (x_0, y_0) becomes $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$.

Now, slope of tangent at (x_0, y_0) is $\dfrac{b^2x_0}{a^2y_0}$

$\Rightarrow$ Slope of normal at (x_0, y_0) is $-\dfrac{a^2y_0}{b^2x_0}$. (Negative reciprocal)

$\therefore$ Equation of normal at (x_0, y_0) is

$$y - y_0 = -\frac{a^2y_0}{b^2x_0}(x - x_0)$$

or $$b^2x_0 (y - y_0) = -a^2y_0 (x - x_0)$$

Dividing every term by $a^2b^2x_0y_0$,

$$\frac{y - y_0}{a^2y_0} = -\frac{(x - x_0)}{b^2x_0} \quad \text{or} \quad \frac{(x - x_0)}{b^2x_0} + \frac{(y - y_0)}{a^2y_0} = 0$$

25. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$.

Sol. Given: Equation of the curve is $y = \sqrt{3x - 2}$ ...(*i*)

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(3x - 2)^{1/2} = \frac{1}{2}(3x - 2)^{-1/2}\frac{d}{dx}(3x - 2) = \frac{1}{2\sqrt{3x - 2}}\cdot 3$$

= Slope of the tangent at point (x, y) of curve (*i*) ...(*ii*)

Again slope of the given line $4x - 2y + 5 = 0$ is $\dfrac{-a}{b} = \dfrac{-4}{-2} = 2$ ...(*iii*)

Since required tangent is parallel to the given line, therefore

$$\frac{3}{2\sqrt{3x - 2}} = 2$$ [Parallel lines have same slope]

Cross-multiplying, $4\sqrt{3x - 2} = 3$

Squaring both sides, $16(3x - 2) = 9 \Rightarrow 48x - 32 = 9$

$$\Rightarrow 48x = 32 + 9 = 41 \Rightarrow x = \frac{41}{48}$$

Putting $x = \dfrac{41}{48}$ in (*i*), $y = \sqrt{3\left(\dfrac{41}{48}\right) - 2}$

$$= \sqrt{\frac{41}{16} - 2} = \sqrt{\frac{41 - 32}{16}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

$\therefore$ Point of contact is $(x, y) = \left(\dfrac{41}{48}, \dfrac{3}{4}\right)$

$\therefore$ Equation of the required tangent is $y - \dfrac{3}{4} = 2\left(x - \dfrac{41}{48}\right)$

$$\Rightarrow y - \frac{3}{4} = 2x - \frac{41}{24} \Rightarrow y = 2x + \frac{3}{4} - \frac{41}{24} \Rightarrow y = 2x + \frac{18 - 41}{24}$$

$\Rightarrow \quad 24y = 48x - 23 \quad$ or $\quad -48x + 24y = -23$

Dividing by -1, $48x - 24y = 23$.

Choose the correct answer in Exercise 26 and 27.

26. The slope of the normal to the curve $y = 2x^2 + 3 \sin x$ at $x = 0$ is

(A) 3 (B) $\frac{1}{3}$ (C) – 3 (D) $\frac{-1}{3}$.

Sol. Given: Equation of the curve is $y = 2x^2 + 3 \sin x$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = 4x + 3 \cos x$ = Slope of the tangent at the point (x, y)

Putting $x = 0$ (given), slope of the tangent (at $x = 0$)

$= 4(0) + 3 \cos 0 = 3 = m$ (say)

$\therefore$ Slope of the normal at $x = 0$ is $\frac{-1}{m} = \frac{-1}{3}$

$\therefore$ Option (D) is the correct answer.

27. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point

(A) (1, 2) (B) (2, 1) (C) (1, – 2) (D) (– 1, 2).

Sol. Given: Equation of the curve is $y^2 = 4x$ ...(*i*)

Differentiating both sides of (*i*) w.r.t. x,

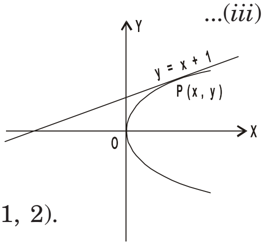
$$2y \frac{dy}{dx} = 4 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

= Slope of tangent to curve (*i*) at point(x, y) ...(*ii*)

Again slope of the given (tangent) line $y = x + 1$

i.e., $-x + y - 1 = 0$ *i.e.*, $x - y + 1 = 0$ is

$$-\frac{a}{b} = \frac{-1}{-1} = 1 \quad ...(iii)$$



From (*ii*) and (*iii*), $\frac{2}{y} = 1$

| $\because$ Both are slopes of the same line

$\therefore \quad y = 2$

Putting $y = 2$ in (*i*), $4 = 4x$ or $x = 1$

$\therefore$ Required point of contact is $P(x, y) = (1, 2)$.

$\therefore$ Option (A) is the correct answer.