Exercise 5.7

Find the second order derivatives of the functions given in Exercises 1 to 5.

1. $x^2 + 3x + 2$.

Sol. Let $y = x^2 + 3x + 2$

$$\therefore \quad \frac{dy}{dx} = 2x + 3.1 + 0 = 2x + 3$$

Again differentiating w.r.t. x, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2(1) + 0 = 2.$

2. x^{20}.

Sol. Let $y = x^{20}$

$$\therefore \quad \frac{dy}{dx} = 20x^{19}$$

Again differentiating w.r.t. x, $\frac{d^2y}{dx^2} = 20.19x^{18} = 380x^{18}.$

3. $x \cos x$.

Sol. Let $y = x \cos x$

$$\therefore \quad \frac{dy}{dx} = x\frac{d}{dx}\cos x + \cos x \frac{d}{dx} x \qquad \text{[By Product Rule]}$$

$$= -x \sin x + \cos x$$

Again differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}(x \sin x) + \frac{d}{dx}\cos x$$

$$= -\left[x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}(x)\right] - \sin x$$

$$= -(x \cos x + \sin x) - \sin x = -x \cos x - \sin x - \sin x$$

$$= -x \cos x - 2 \sin x = -(x \cos x + 2 \sin x).$$

4. $\log x$.

Sol. Let $y = \log x$ $\quad \therefore \quad \frac{dy}{dx} = \frac{1}{x}$

Again differentiating w.r.t. x, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1}$

$$= (-1)\, x^{-2} = \frac{-1}{x^2}.$$

5. $x^3 \log x$.

Sol. Let $y = x^3 \log x$

$$\therefore \quad \frac{dy}{dx} = x^3 \frac{d}{dx}\log x + \log x \frac{d}{dx} x^3 \qquad \text{[By Product Rule]}$$

$$= x^3 . \frac{1}{x} + (\log x)\, 3x^2$$

$$= x^2 + 3x^2 \log x$$

Again differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}x^2 + 3\frac{d}{dx}(x^2 \log x)$$

$$= 2x + 3\left[x^2 \frac{d}{dx}\log x + \log x \frac{d}{dx}x^2\right]$$

$$= 2x + 3\left(x^2 \cdot \frac{1}{x} + (\log x)\, 2x\right)$$

$$= 2x + 3(x + 2x \log x)$$

$$= 2x + 3x + 6x \log x = 5x + 6x \log x$$

$$= x(5 + 6 \log x).$$

Find the second order derivatives of the functions given in exercises 6 to 10.

6. $\boldsymbol{e^x \sin 5x}$**.**

Sol. Let $y = e^x \sin 5x$

$$\therefore \quad \frac{dy}{dx} = e^x \frac{d}{dx}\sin 5x + \sin 5x \frac{d}{dx}e^x \quad \text{[By Product Rule]}$$

$$= e^x \cos 5x \frac{d}{dx}5x + \sin 5x \,.\, e^x = e^x \cos 5x \,.\, 5 + e^x \sin 5x$$

or $$\frac{dy}{dx} = e^x(5 \cos 5x + \sin 5x)$$

Again applying Product Rule of derivatives

$$\frac{d^2y}{dx^2} = e^x \frac{d}{dx}(5 \cos 5x + \sin 5x) + (5 \cos 5x + \sin 5x)\frac{d}{dx}e^x$$

$$= e^x(5(-\sin 5x) \,.\, 5 + (\cos 5x) \,.\, 5) + (5 \cos 5x + \sin 5x)\, e^x$$

$$= e^x(-25 \sin 5x + 5 \cos 5x + 5 \cos 5x + \sin 5x)$$

$$= e^x(10 \cos 5x - 24 \sin 5x)$$

$$= 2e^x(5 \cos 5x - 12 \sin 5x).$$

7. $\boldsymbol{e^{6x} \cos 3x}$**.**

Sol. Let $y = e^{6x} \cos 3x$

$$\therefore \quad \frac{dy}{dx} = e^{6x}\frac{d}{dx}\cos 3x + \cos 3x \frac{d}{dx}e^{6x}$$

$$= e^{6x}(-\sin 3x)\frac{d}{dx}(3x) + \cos 3x \,.\, e^{6x}\frac{d}{dx}6x$$

$$= -e^{6x}\sin 3x \,.\, 3 + \cos 3x \,.\, e^{6x} \,.\, 6$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{6x}(-3 \sin 3x + 6 \cos 3x)$$

Again applying Product Rule of derivatives,

$$\frac{d^2y}{dx^2} = e^{6x}\frac{d}{dx}(-3 \sin 3x + 6 \cos 3x)$$

$$+ (-3 \sin 3x + 6 \cos 3x)\frac{d}{dx}e^{6x}$$

$= e^{6x} [- 3 . \cos 3x . 3 - 6 \sin 3x . 3]$
$+ (- 3 \sin 3x + 6 \cos 3x) e^{6x} . 6$
$= e^{6x} (- 9 \cos 3x - 18 \sin 3x - 18 \sin 3x + 36 \cos 3x)$
$= e^{6x} (27 \cos 3x - 36 \sin 3x)$
$= 9e^{6x} (3 \cos 3x - 4 \sin 3x).$

8. $\tan^{-1} x$.

Sol. Let $y = \tan^{-1} x$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+x^2}$$

Again differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{(1+x^2)\frac{d}{dx}(1) - 1\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)0-(2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}.$$

9. log (log x).

Sol. Let $y = \log (\log x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\log x}\frac{d}{dx}\log x \qquad \left[\because \frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)\right]$$

$$= \frac{1}{\log x}\frac{1}{x} = \frac{1}{x \log x}$$

Again differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{(x\log x)\frac{d}{dx}(1) - 1\frac{d}{dx}(x\log x)}{(x\log x)^2}$$

$$= \frac{(x\log x)\,0 - \left[x\frac{d}{dx}\log x + \log x\frac{d}{dx}(x)\right]}{(x\log x)^2}$$

$$= -\frac{\left[x.\frac{1}{x} + \log x . 1\right]}{(x\log x)^2} = -\frac{(1+\log x)}{(x\log x)^2}.$$

10. sin (log x).

Sol. Let $y = \sin (\log x)$

$$\therefore \quad \frac{dy}{dx} = \cos(\log x)\frac{d}{dx}(\log x) = \cos(\log x) . \frac{1}{x}$$

$$= \frac{\cos(\log x)}{x}$$

Again differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{x\frac{d}{dx}\cos(\log x) - \cos(\log x)\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x[-\sin(\log x)]\frac{d}{dx}\log x - \cos(\log x)}{x^2}$$

$$= \frac{-x\sin(\log x)\cdot\frac{1}{x} - \cos(\log x)}{x^2} = \frac{-[\sin(\log x) + \cos(\log x)]}{x^2}.$$

11. If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$.

Sol. Given: $y = 5\cos x - 3\sin x$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = -5\sin x - 3\cos x$

Again differentiating w.r.t. x, $\frac{d^2y}{dx^2} = -5\cos x + 3\sin x$

$= -(5\cos x - 3\sin x) - y$ (By (*i*))

or $\quad \frac{d^2y}{dx^2} = -y \quad \therefore \quad \frac{d^2y}{dx^2} + y = 0.$

12. If $y = \cos^{-1} x$. Find $\frac{d^2y}{dx^2}$ in terms of y alone.

Sol. Given: $y = \cos^{-1} x \quad \Rightarrow \quad x = \cos y$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} = \frac{-1}{\sqrt{1-\cos^2 y}}$ (By (*i*))

$= \frac{-1}{\sqrt{\sin^2 y}} = \frac{-1}{\sin y} = -\operatorname{cosec} y$

or $\quad \frac{dy}{dx} = -\operatorname{cosec} y$ ...(*ii*)

Again differentiating both sides w.r.t. x,

$\frac{d^2y}{dx^2} = -\frac{d}{dx}(\operatorname{cosec} y) = -\left[-\operatorname{cosec} y \cot y \frac{dy}{dx}\right]$

$= \operatorname{cosec} y \cot y\,(-\operatorname{cosec} y)$ (By (*ii*))

$= -\operatorname{cosec}^2 y \cot y.$

13. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$.

Sol. Given: $y = 3\cos(\log x) + 4\sin(\log x)$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = (y_1) = -3\sin(\log x)\frac{d}{dx}\log x + 4\cos(\log x)\frac{d}{dx}\log x$

or $\quad y_1 = -3\sin(\log x)\cdot\frac{1}{x} + 4\cos(\log x)\cdot\frac{1}{x}$

Multiplying both sides by L.C.M. $= x$,

$xy_1 = -3\sin(\log x) + 4\cos(\log x)$

Again differentiating both sides w.r.t. x,

$$\frac{d}{dx}(xy_1) = -3\cos(\log x)\frac{d}{dx}\log x - 4\sin(\log x)\frac{d}{dx}\log x$$

$$\Rightarrow \quad x\frac{d}{dx}y_1 + y_1\frac{d}{dx}x = -3\cos(\log x)\cdot\frac{1}{x} - 4\sin(\log x)\cdot\frac{1}{x}$$

(By Product Rule)

$$\Rightarrow \quad xy_2 + y_1 = -\frac{[3\cos(\log x) + 4\sin(\log x)]}{x}$$

Cross-multiplying

$$x(xy_2 + y_1) = -[3\cos(\log x) + 4\sin(\log x)]$$

$$\Rightarrow \quad x^2y_2 + xy_1 = -y \quad \text{(By (i))}$$

$$\Rightarrow \quad x^2y_2 + xy_1 + y = 0.$$

14. If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m + n)\frac{dy}{dx} + mny = 0.$

Sol. Given: $y = Ae^{mx} + Be^{nx}$ ...(i)

$$\therefore \quad \frac{dy}{dx} = Ae^{mx}\frac{d}{dx}(mx) + Be^{nx}\frac{d}{dx}(nx)\left[\because \frac{d}{dx}e^{f(x)} = e^{f(x)}\frac{d}{dx}f(x)\right]$$

or $\frac{dy}{dx} = Am\,e^{mx} + Bn\,e^{nx}$ ...(ii)

$$\therefore \quad \frac{d^2y}{dx^2} = Am.e^{mx}.m + Bne^{nx}.n$$

$$= Am^2e^{mx} + Bn^2e^{nx} \quad \text{...(iii)}$$

Putting values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from (i), (ii) and (iii) in

$$\text{L.H.S.} = \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$= Am^2e^{mx} + Bn^2e^{nx} - (m + n)(Am\,e^{mx} + Bn\,e^{nx}) + mn(Ae^{mx} + Be^{nx})$

$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmn\,e^{nx} - Amn\,e^{mx} - Bn^2e^{nx} + Amn\,e^{mx} + Bmn\,e^{nx} = 0 = \text{R.H.S.}$

15. If $y = 500\,e^{7x} + 600\,e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$.

Sol. Given: $y = 500\,e^{7x} + 600\,e^{-7x}$ ...(i)

$$\therefore \quad \frac{dy}{dx} = 500\,e^{7x}(7) + 600\,e^{-7x}(-7) = 500(7)\,e^{7x} - 600(7)\,e^{-7x}$$

$$\therefore \quad \frac{d^2y}{dx^2} = 500(7)\,e^{7x}(7) - 600(7)e^{-7x}(-7)$$

$$= 500(49)\,e^{7x} + 600(49)\,e^{-7x}$$

or $\frac{d^2y}{dx^2} = 49[500\,e^{7x} + 600\,e^{-7x}] = 49y$ (By (i))

or $\frac{d^2y}{dx^2} = 49y.$

16. If e^y $(x + 1) = 1$, show that $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$.

Sol. Given: $e^y (x + 1) = 1$

$$\Rightarrow \qquad e^y = \frac{1}{x+1}$$

Taking logs of both sides, $\log e^y = \log \dfrac{1}{x+1}$

or $\quad y \log e = \log 1 - \log (x + 1)$

or $\qquad y = -\log (x + 1) \quad [\because \log e = 1 \text{ and } \log 1 = 0]$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{x+1}\frac{d}{dx}(x + 1) = \frac{-1}{x+1} = -(x + 1)^{-1}$$

$$\therefore \quad \frac{d^2y}{dx^2} = -(-1)(x + 1)^{-2}\frac{d}{dx}(x + 1)$$

$$\left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)\right]$$

$$= \frac{1}{(x+1)^2} \qquad \left[\because \frac{d}{dx}(x+1) = 1 + 0 = 1\right]$$

$$\text{L.H.S.} = \frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$$

$$\text{R.H.S.} = \left(\frac{dy}{dx}\right)^2 = \left(\frac{-1}{x+1}\right)^2 = \frac{1}{(x+1)^2}$$

$\therefore$ L.H.S. = R.H.S. *i.e.*, $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$.

17. If $y = (\tan^{-1} x)^2$, show that $(x^2 + 1)^2 y_2 + 2x(x^2 + 1)y_1 = 2$.

Sol. Given: $y = (\tan^{-1} x)^2$ ...(*i*)

$$\therefore \quad y_1 = 2(\tan^{-1} x)\frac{d}{dx}\tan^{-1}x \quad \left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)\right]$$

$$\Rightarrow \quad y_1 = 2(\tan^{-1} x)\frac{1}{1+x^2} \qquad \Rightarrow \quad y_1 = \frac{2\tan^{-1}x}{1+x^2}$$

Cross-multiplying, $(1 + x^2) y_1 = 2 \tan^{-1} x$

Again differentiating both sides w.r.t. x,

$$(1 + x^2)\frac{d}{dx}y_1 + y_1\frac{d}{dx}(1 + x^2) = 2 . \frac{1}{1+x^2}$$

$$\Rightarrow \qquad (1 + x^2) y_2 + y_1 . 2x = \frac{2}{1+x^2}$$

Multiplying both sides by $(1 + x^2)$,

$$(x^2 + 1)^2 y_2 + 2xy_1 (1 + x^2) = 2.$$