NCERT Class 12 Maths Solutions

Chapter - 5

Exercise 5.6

If x and y are connected parametrically by the equations given in Exercises 1 to 5, without eliminating the parameter, find $\frac{dy}{dx}$.

1. $x = 2at^2$, $y = at^4$.

Sol. Given: $x = 2at^2$ and $y = at^4$

Differentiating both eqns. w.r.t. t, we have

$$\frac{dx}{dt} = \frac{d}{dt}(2at^2) = 2a\frac{d}{dt}t^2 = 2a.2t = 4at$$

and

$$\frac{dy}{dt} = \frac{d}{dt}(at^4) = a\frac{d}{dt}t^4 = a.4t^3 = 4at^3$$

We know that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4at^3}{4at} = t^2$.

2. $x = a \cos \theta$, $y = b \cos \theta$.

Sol. Given: $x = a \cos \theta$ and $y = b \cos \theta$

Differentiating both eqns. w.r.t. θ, we have

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a \cos \theta) = a\frac{d}{d\theta} \cos \theta = -a \sin \theta$$

and

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b \cos \theta) = b\frac{d}{d\theta} \cos \theta = -b \sin \theta$$

We know that $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$.

3. $x = \sin t$, $y = \cos 2t$.

Sol. Given: $x = \sin t$ and $y = \cos 2t$

Differentiating both eqns. w.r.t. t, we have

$$\frac{dx}{dt} = \cos t$$

and

$$\frac{dy}{dt} = -\sin 2t\frac{d}{dt}(2t) = -(\sin 2t)\,2 = -2\sin 2t$$

We know that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{2\sin 2t}{\cos t}$

$$= -2 \cdot \frac{2\sin t\cos t}{\cos t} = -4\sin t.$$

4. $\boldsymbol{x = 4t,\ y = \dfrac{4}{t}}$**.**

Sol. Given: $x = 4t$ and $y = \dfrac{4}{t}$

$$\therefore \quad \frac{dx}{dt} = \frac{d}{dt}(4t) \quad \text{and} \quad \frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right)$$

$$= 4\frac{d}{dt}t \qquad\qquad = \frac{t\dfrac{d}{dt}(4) - 4\dfrac{d}{dt}t}{t^2}$$

$$= 4(1) = 4 \qquad\qquad = \frac{t(0) - 4(1)}{t^2} = -\frac{4}{t^2}$$

We know that $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\left(-\dfrac{4}{t^2}\right)}{4} = \dfrac{-1}{(t^2)}$.

5. $\boldsymbol{x = \cos\theta - \cos 2\theta,\ y = \sin\theta - \sin 2\theta}$**.**

Sol. Given: $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$

$$\therefore \quad \frac{dx}{d\theta} = \frac{d}{d\theta}(\cos\theta) - \frac{d}{d\theta}\cos 2\theta \quad \text{and} \quad \frac{dy}{d\theta} = \cos\theta - \frac{d}{d\theta}\sin 2\theta$$

$$= -\sin\theta - (-\sin 2\theta)\frac{d}{d\theta}2\theta \qquad = \cos\theta - \cos 2\theta\frac{d}{d\theta}2\theta$$

$$= -\sin\theta + (\sin 2\theta)\,2 \qquad = \cos\theta - \cos 2\theta(2)$$

$$= 2\sin 2\theta - \sin\theta \qquad = \cos\theta - 2\cos 2\theta.$$

We know that $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$.

If x and y are connected parametrically by the equations given in Exercises 6 to 10, without eliminating the parameter, find $\dfrac{dy}{dx}$.

6. $\boldsymbol{x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)}$**.**

Sol. $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

Differentiating both eqns. w.r.t. θ, we have

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\theta - \sin\theta) \quad \text{and} \quad \frac{dy}{d\theta} = a\frac{d}{d\theta}(1 + \cos\theta)$$

$$= a\left[\frac{d}{d\theta}\theta - \frac{d}{d\theta}\sin\theta\right] \quad \text{and} \quad \frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}\cos\theta\right]$$

$$\Rightarrow \frac{dx}{d\theta} = a(1 - \cos\theta) \quad \text{and} \quad \frac{dy}{d\theta} = a(0 - \sin\theta) = -a\sin\theta$$

We know that $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{-a\sin\theta}{a(1 - \cos\theta)}$

$$= -\frac{\sin\theta}{1 - \cos\theta} = -\frac{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}{2\sin^2\dfrac{\theta}{2}} = -\frac{\cos\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}} = -\cot\frac{\theta}{2}.$$

7. $x = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}, \; y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}.$

Sol. We have $x = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}$

$$\therefore \quad \frac{dx}{dt} = \frac{\sqrt{\cos 2t} \cdot \frac{d}{dt}(\sin^3 t) - \sin^3 t \cdot \frac{d}{dt}(\sqrt{\cos 2t})}{(\sqrt{\cos 2t})^2}$$

[By Quotient Rule]

$$= \frac{\sqrt{\cos 2t} \cdot 3\sin^2 t \frac{d}{dt} \cdot (\sin t) - \sin^3 t \cdot \frac{1}{2}(\cos 2t)^{-1/2} \cdot \frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t} \cdot 3\sin^2 t \cos t - \frac{\sin^3 t}{2\sqrt{\cos 2t}} \cdot (-2\sin 2t)}{\cos 2t}$$

$$= \frac{3\sin^2 t \cos t \cos 2t + \sin^3 t \sin 2t}{(\cos 2t)^{3/2}}$$

$$= \frac{3\sin^2 t \cos t \cos 2t + \sin^3 t \cdot 2\sin t \cos t}{(\cos 2t)^{3/2}}$$

$$= \frac{\sin^2 t \cos t \,(3\cos 2t + 2\sin^2 t)}{(\cos 2t)^{3/2}}$$

and $$\frac{dy}{dt} = \frac{\sqrt{\cos 2t} \cdot \frac{d}{dt}(\cos^3 t) - \cos^3 t \cdot \frac{d}{dt}(\sqrt{\cos 2t})}{(\sqrt{\cos 2t})^2}$$

[By Quotient Rule]

$$= \frac{\sqrt{\cos 2t} \cdot 3\cos^2 t \cdot \frac{d}{dt}(\cos t) - \cos^3 t \cdot \frac{1}{2}(\cos 2t)^{-1/2} \cdot \frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t} \cdot 3\cos^2 t\,(-\sin t) - \frac{\cos^3 t}{2\sqrt{\cos 2t}}(-2\sin 2t)}{\cos 2t}$$

$$= \frac{-3\cos^2 t \sin t \cos 2t + \cos^3 t \cdot \sin 2t}{(\cos 2t)^{3/2}}$$

$$= \frac{-3\cos^2 t \sin t \cos 2t + \cos^3 t \cdot 2\sin t \cos t}{(\cos 2t)^{3/2}}$$

$$= \frac{\sin t \cos^2 t \,(2\cos^2 t - 3\cos 2t)}{(\cos 2t)^{3/2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{\sin t \cos^2 t\,(2\cos^2 t - 3\cos 2t)}{(\cos 2t)^{3/2}} \cdot \frac{(\cos 2t)^{3/2}}{\sin^2 t \cos t\,(3\cos 2t + 2\sin^2 t)}$$

$$= \frac{\cos t\,[2\cos^2 t - 3(2\cos^2 t - 1)]}{\sin t\,[3(1 - 2\sin^2 t) + 2\sin^2 t]} = \frac{\cos t\,(3 - 4\cos^2 t)}{\sin t\,(3 - 4\sin^2 t)}$$

$$= \frac{-(4\cos^3 t - 3\cos t)}{3\sin t - 4\sin^3 t} = \frac{-\cos 3t}{\sin 3t} = -\cot 3t$$

Hence $\frac{dy}{dx} = -\cot 3t.$

8. $\boldsymbol{x = a\left(\cos t + \log \tan \frac{t}{2}\right), y = a \sin t.}$

Sol. $x = a\left[\cos t + \log\left(\tan \frac{t}{2}\right)\right]$

$$\Rightarrow \frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \frac{d}{dt}\left(\tan \frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} \cdot \frac{1}{\cos^2 \frac{t}{2}} \cdot \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin \frac{t}{2} \cos \frac{t}{2}}\right]$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = a\left[\frac{1}{\sin t} - \sin t\right]$$

$$= a\left(\frac{1 - \sin^2 t}{\sin t}\right) = \frac{a\cos^2 t}{\sin t}$$

$$y = a \sin t \Rightarrow \frac{dy}{dt} = a \cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{\left(\frac{a\cos^2 t}{\sin t}\right)} = \frac{\sin t}{\cos t} = \tan t.$$

9. $\boldsymbol{x = a \sec \theta, y = b \tan \theta.}$

Sol. $x = a \sec \theta$ and $y = b \tan \theta$

Differentiating both eqns. w.r.t. θ, we have

$$\frac{dx}{d\theta} = a \sec \theta \tan \theta \quad \text{and} \quad \frac{dy}{d\theta} = b \sec^2 \theta$$

We know that $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{b\sec^2\theta}{a\sec\theta\tan\theta} = \dfrac{b\sec\theta}{a\tan\theta}$

$$= \frac{b\,.\,\frac{1}{\cos\theta}}{a\,.\,\frac{\sin\theta}{\cos\theta}} = \frac{b}{\cos\theta}\,.\,\frac{\cos\theta}{a\sin\theta} = \frac{b}{a\sin\theta} = \frac{b}{a}\ \text{cosec}\ \theta.$$

10. $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$.

Sol. We have $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

$$\therefore \quad \frac{dx}{d\theta} = a(-\sin\theta + \theta\cos\theta + \sin\theta\,.\,1) = a\theta\cos\theta$$

and $$\frac{dy}{d\theta} = a[\cos\theta - (\theta(-\sin\theta) + \cos\theta\,.\,1)]$$

$$= a\,[\cos\theta + \theta\sin\theta - \cos\theta] = a\theta\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta.$$

11. If $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$, show that $\dfrac{dy}{dx} = -\dfrac{y}{x}$.

Sol. Given: $x = \sqrt{a^{\sin^{-1}t}} = (a^{\sin^{-1}t})^{1/2} = a^{1/2\sin^{-1}t}$ ...(*i*)

$$\therefore \quad \frac{dx}{dt} = a^{1/2\sin^{-1}t}\ \log a\ \frac{d}{dt}\left(\frac{1}{2}\sin^{-1}t\right)$$

$$\left[\because \frac{d}{dx}a^x = a^x\log a \text{ and } \frac{d}{dx}a^{f(x)} = a^{f(x)}\log a\,\frac{d}{dx}f(x)\right]$$

$$\Rightarrow \quad \frac{dx}{dt} = a^{1/2\sin^{-1}t}\ \log a\,.\,\frac{1}{2}\ \frac{1}{\sqrt{1-t^2}} \quad \text{...(ii)}$$

Again **given:** $y = \sqrt{a^{\cos^{-1}t}} = (a^{\cos^{-1}t})^{1/2} = a^{1/2\cos^{-1}t}$ ...(*iii*)

$$\therefore \quad \frac{dy}{dt} = a^{1/2\cos^{-1}t}\ \log a\ \frac{d}{dt}\left(\frac{1}{2}\cos^{-1}t\right)$$

$$= a^{1/2\cos^{-1}t}\ \log a\,.\,\frac{1}{2}\left(\frac{-1}{\sqrt{1-t^2}}\right) \quad \text{...(iv)}$$

We know that $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

Putting values from (*iv*) and (*ii*),

$$\frac{dy}{dx} = \frac{a^{1/2\cos^{-1}t}\ \log a\,\frac{1}{2}\left(\frac{-1}{\sqrt{1-t^2}}\right)}{a^{1/2\sin^{-1}t}\ \log a\,.\,\frac{1}{2}\ \frac{1}{\sqrt{1-t^2}}} = \frac{-a^{1/2\cos^{-1}t}}{a^{1/2\sin^{-1}t}} = -\frac{y}{x}$$

(By (*iii*) and (*i*))