NCERT Class 12 Maths

Solutions

Chapter - 5

Continuity and Differentiability

Exercise 5.5

Note. Logarithmic Differentiation.
The process of differentiating a function after taking its logarithm is called **logarithmic differentiation.**
This process of differentiation is very useful in the following situations:
(i) The given function is of the form $(f(x))^{g(x)}$
(ii) The given function involves complicated (as per our thinking) products (or and) quotients (or and) powers.

Remark 1. $\log \dfrac{(a^m\, b^n\, c^p)}{d^q\, l^k}$

$$= m \log a + n \log b + p \log c - q \log d - k \log l$$

Remark 2. $\log (u + v) \neq \log u + \log v$

and $\log (u - v) \neq \log u - \log v$.

Differentiate the following functions given in Exercises 1 to 5 w.r.t. *x*.

1. cos *x* cos 2*x* cos 3*x*.

Sol. Let $y = \cos x \cos 2x \cos 3x$ ...(*i*)

Taking logs on both sides, we have (see Note, (*ii*) page 261)

$$\log y = \log (\cos x \cos 2x \cos 3x)$$
$$= \log \cos x + \log \cos 2x + \log \cos 3x$$

Differentiating both sides w.r.t. *x*, we have

$$\frac{d}{dx} \log y = \frac{d}{dx} \log \cos x + \frac{d}{dx} \log \cos 2x + \frac{d}{dx} \log \cos 3x$$

$$\therefore \quad \frac{1}{y} \frac{dy}{dx} = \frac{1}{\cos x} \frac{d}{dx} \cos x + \frac{1}{\cos 2x} \frac{d}{dx} \cos 2x$$
$$+ \frac{1}{\cos 3x} \frac{d}{dx} \cos 3x \left[\because \frac{d}{dx} \log f(x) = \frac{1}{f(x)} \frac{d}{dx} f(x)\right]$$
$$= \frac{1}{\cos x} (- \sin x) + \frac{1}{\cos 2x} (- \sin 2x) \frac{d}{dx} (2x)$$
$$+ \frac{1}{\cos 3x} (- \sin 3x) \frac{d}{dx} 3x$$
$$= - \tan x - (\tan 2x)\, 2 - \tan 3x\, (3)$$

$$\therefore \quad \frac{dy}{dx} = - y (\tan x + 2 \tan 2x + 3 \tan 3x).$$

Putting the value of *y* from (*i*),

$$\frac{dy}{dx} = - \cos x \cos 2x \cos 3x (\tan x + 2 \tan 2x + 3 \tan 3x).$$

2. $\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$.

Sol. Let $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} = \left[\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]^{1/2}$ ...(*i*)

Taking logs on both sides, we have

$$\log y = \frac{1}{2} [\log (x - 1) + \log (x - 2) - \log (x - 3)$$
$$- \log (x - 4) - \log (x - 5)] \text{ (By Remark I above)}$$

Differentiating both sides w.r.t. *x*, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1}\frac{d}{dx}(x-1) + \frac{1}{x-2}\frac{d}{dx}(x-2) - \frac{1}{x-3}\frac{d}{dx}(x-3) - \frac{1}{x-4}\frac{d}{dx}(x-4) - \frac{1}{x-5}\frac{d}{dx}(x-5)\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}y\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

Putting the value of y from (i),

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right].$$

3. $(\log x)^{\cos x}$**.**

Sol. Let $y = (\log x)^{\cos x}$ ...(i) [Form $(f(x))^{g(x)}$]

Taking logs on both sides of (i), we have (see Note (i) page 261)

$$\log y = \log (\log x)^{\cos x} = \cos x \log (\log x)$$

[$\because$ $\log m^n = n \log m$]

$$\therefore \quad \frac{d}{dx}\log y = \frac{d}{dx}[\cos x \,.\, \log(\log x)]$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \cos x \frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}\cos x$$

[By Product Rule]

$$= \cos x \frac{1}{\log x}\frac{d}{dx}\log x + \log(\log x)(-\sin x)$$

$$= \frac{\cos x}{\log x} \,.\, \frac{1}{x} - \sin x \log(\log x)$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{\cos x}{x \log x} - \sin x \log(\log x)\right].$$

Putting the value of y from (i),

$$\frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x \log x} - \sin x \log(\log x)\right].$$

Very Important Note.

To differentiate $y = (f(x))^{g(x)} \pm (l(x))^{m(x)}$

or $y = (f(x))^{g(x)} \pm h(x)$

or $y = (f(x))^{g(x)} \pm k$ where k is a constant;

Never start with taking logs of both sides, put one term $= u$ and the other $= v$

$$\therefore \quad y = u \pm v$$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

Now find $\frac{du}{dx}$ and $\frac{dv}{dx}$ by the methods already learnt.

4. $x^x - 2^{\sin x}$.

Sol. Let $\quad y = x^x - 2^{\sin x}$

Put $\quad u = x^x$ and $v = 2^{\sin x}$ (See Note)

$\therefore \quad y = u - v$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \qquad ...(i)$$

Now $\quad u = x^x$ $\qquad$ [Form $(f(x))^{g(x)}$]

$\therefore \quad \log u = \log x^x = x \log x$ $\qquad$ [$\because \quad \log m^n = n \log m$]

$$\therefore \quad \frac{d}{dx} \log u = \frac{d}{dx}(x \log x)$$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = x \frac{d}{dx} \log x + \log x \frac{d}{dx} x$$

$$= x \cdot \frac{1}{x} + \log x \cdot 1 = 1 + \log x$$

$$\therefore \quad \frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x) \qquad ...(ii)$$

Again $v = 2^{\sin x}$

$$\therefore \quad \frac{dv}{dx} = \frac{d}{dx} 2^{\sin x} = 2^{\sin x} \log 2 \frac{d}{dx} \sin x$$

$$\left[\because \frac{d}{dx} a^{f(x)} = a^{f(x)} \log a \frac{d}{dx} f(x)\right]$$

$$\Rightarrow \quad \frac{dv}{dx} = 2^{\sin x} (\log 2) \cos x = \cos x \cdot 2^{\sin x} \log 2 \qquad ...(iii)$$

Putting values from (*ii*) and (*iii*) in (*i*),

$$\frac{dy}{dx} = x^x(1 + \log x) - \cos x \cdot 2^{\sin x} \log 2.$$

5. $(x + 3)^2 (x + 4)^3 (x + 5)^4$.

Sol. Let $y = (x + 3)^2 (x + 4)^3 (x + 5)^4 \qquad ...(i)$

Taking logs on both sides of eqn. (*i*) (see Note (*ii*) page 261)

we have $\log y = 2 \log (x + 3) + 3 \log (x + 4)$

$+ 4 \log (x + 5)$ (By Remark I page 262)

$$\therefore \quad \frac{d}{dx} \log y = 2 \frac{d}{dx} \log (x + 3) + 3 \frac{d}{dx} \log (x + 4) + 4 \frac{d}{dx} \log (x + 5)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = 2 \cdot \frac{1}{x+3} \frac{d}{dx}(x + 3) + 3 \frac{1}{x+4} \frac{d}{dx}(x + 4) + 4 \cdot \frac{1}{x+5} \frac{d}{dx}(x + 5)$$

$$= \frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}$$

$$\therefore \quad \frac{dy}{dx} = y\left(\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right)$$

Putting the value of y from (i),

$$\frac{dy}{dx} = (x + 3)^2 (x + 4)^3 (x + 5)^4 \left(\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right).$$

Differentiate the following functions given in Exercises 6 to 11 w.r.t. *x*.

6. $\left(x + \frac{1}{x}\right)^x + x^{\left(1 + \frac{1}{x}\right)}$.

Sol. Let $y = \left(x + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Putting $\left(x + \frac{1}{x}\right)^x = u$ and $x^{\left(1+\frac{1}{x}\right)} = v$,

We have $y = u + v \quad \therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(i)

Now $u = \left(x + \frac{1}{x}\right)^x$

Taking logarithms, $\log u = \log \left(x + \frac{1}{x}\right)^x = x \log\left(x + \frac{1}{x}\right)$ [Form uv]

Differentiating w.r.t. x, we have

$$\frac{1}{u}\ \frac{du}{dx} = x \cdot \frac{1}{x + \frac{1}{x}}\ \frac{d}{dx}\left(x + \frac{1}{x}\right) + \log\left(x + \frac{1}{x}\right) . 1$$

$$\frac{1}{u}\ \frac{du}{dx} = x \cdot \frac{1}{x + \frac{1}{x}} \cdot \left(1 - \frac{1}{x^2}\right) + \log\left(x + \frac{1}{x}\right) . 1$$

$$\left[\because \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = (-1)\,x^{-2} = \frac{-1}{x^2}\right]$$

$$\Rightarrow \quad \frac{du}{dx} = u\left[\frac{x^2 - 1}{x^2 + 1} + \log\left(x + \frac{1}{x}\right)\right]$$

$$= \left(x + \frac{1}{x}\right)^x \left[\frac{x^2 - 1}{x^2 + 1} + \log\left(x + \frac{1}{x}\right)\right] \quad ...(ii)$$

Also $v = x^{\left(1+\frac{1}{x}\right)}$

Taking logarithms, $\log v = \log x^{\left(1+\frac{1}{x}\right)} = \left(1 + \frac{1}{x}\right) \log x$

Differentiating w.r.t. x, we have

$$\frac{1}{v} \cdot \frac{dv}{dx} = \left(1 + \frac{1}{x}\right) \cdot \frac{1}{x} + \log x \cdot \left(-\frac{1}{x^2}\right)$$

$$\left[\because \frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = (-1)\,x^{-2} = \frac{-1}{x^2}\right]$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left[\frac{1}{x}\left(1+\frac{1}{x}\right)-\frac{1}{x^2}\log x\right]$$

$$= x^{\left(1+\frac{1}{x}\right)}\left[\frac{1}{x}\left(1+\frac{1}{x}\right)-\frac{1}{x^2}\log x\right] \qquad ...(iii)$$

Putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*), we have

$$\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right] + x^{\left(1+\frac{1}{x}\right)}\left[\frac{1}{x}\left(1+\frac{1}{x}\right)-\frac{1}{x^2}\log x\right]$$

7. $(\log x)^x + x^{\log x}$.

Sol. Let $y = (\log x)^x + x^{\log x}$

$= u + v$ where $u = (\log x)^x$ and $v = x^{\log x}$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(i)$$

Now $u = (\log x)^x$ $\qquad [(f(x))^{g(x)}]$

$\therefore \log u = \log (\log x)^x = x \log (\log x)$ $[\because \log m^n = n \log m]$

$$\therefore \quad \frac{d}{dx}\log u = \frac{d}{dx}[x \log (\log x)]$$

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}x \text{ (By product rule)}$$

$$= x \cdot \frac{1}{\log x}\frac{d}{dx}\log x + \log(\log x) \cdot 1$$

$$= x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log(\log x)$$

$$\therefore \quad \frac{du}{dx} = u\left[\frac{1}{\log x} + \log(\log x)\right] = (\log x)^x\left(\frac{1}{\log x}+\log(\log x)\right)$$

$$= (\log x)^x \frac{(1+\log x \log(\log x))}{\log x}$$

$$= (\log x)^{x-1}(1 + \log x \log(\log x)) \qquad ...(ii)$$

Again $v = x^{\log x}$ $\qquad$ [Form $(f(x))^{g(x)}$]

$\therefore \log v = \log x^{\log x} = \log x \cdot \log x$ $\qquad [\because \log m^n = n \log m]$

$= (\log x)^2$

$$\therefore \quad \frac{d}{dx}\log v = \frac{d}{dx}(\log x)^2 \qquad \therefore \quad \frac{1}{v}\frac{dv}{dx} = 2(\log x)^1\frac{d}{dx}\log x$$

$$\left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)\right]$$

$$= 2\log x \cdot \frac{1}{x}$$

$\therefore \quad \frac{dv}{dx} = v\left(\frac{2}{x}\log x\right) = x^{\log x} \cdot \frac{2}{x} \log x$

$= 2x^{\log x - 1} \log x$ ...(iii)

Putting values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (ii) and (iii) in (i), we have

$\frac{dy}{dx} = (\log x)^{x-1} (1 + \log x \log (\log x)) + 2x^{\log x - 1} \log x.$

8. $\mathbf{(\sin x)^x + \sin^{-1} \sqrt{x}}$ **.**

Sol. Let $y = (\sin x)^x + \sin^{-1} \sqrt{x}$

$= u + v$ where $u = (\sin x)^x$ and $v = \sin^{-1} \sqrt{x}$

$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(i)

Now $u = (\sin x)^x$ [Form $(f(x))^{g(x)}$]

$\therefore \log u = \log (\sin x)^x = x \log \sin x$

$\therefore \quad \frac{d}{dx}(\log u) = \frac{d}{dx}(x \log \sin x)$

$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = x \frac{d}{dx} \log \sin x + \log \sin x \frac{d}{dx} x$

$= x \cdot \frac{1}{\sin x} \frac{d}{dx} \sin x + (\log \sin x) \cdot 1$

$= x \frac{1}{\sin x} \cos x + \log \sin x = x \cot x + \log \sin x$

$\therefore \quad \frac{du}{dx} = u\,(x \cot x + \log \sin x) = (\sin x)^x (x \cot x + \log \sin x)$...(ii)

Again $v = \sin^{-1} \sqrt{x}$

$\therefore \frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \frac{d}{dx}\sqrt{x} \quad \Big| \because \frac{d}{dx}\sin^{-1} f(x) = \frac{1}{\sqrt{1-(f(x))^2}} \frac{d}{dx} f(x)$

$= \frac{1}{\sqrt{1-x}} \frac{1}{2\sqrt{x}} \left[\because \frac{d}{dx}\sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}\right]$

or $\quad \frac{dv}{dx} = \frac{1}{2\sqrt{x}\,\sqrt{1-x}} = \frac{1}{2\sqrt{x(1-x)}} = \frac{1}{2\sqrt{x-x^2}}$ ...(iii)

Putting values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (ii) and (iii) in (i),

$\frac{dy}{dx} = (\sin x)^x (x \cot x + \log \sin x) + \frac{1}{2\sqrt{x-x^2}}$.

9. $\mathbf{x^{\sin x} + (\sin x)^{\cos x}}$**.**

Sol. Let $y = x^{\sin x} + (\sin x)^{\cos x}$

$= u + v$ where $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$

$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(*i*)

Now $u = x^{\sin x}$ [Form $(f(x))^{g(x)}$]

$\therefore \log u = \log x^{\sin x} = \sin x \log x$

$$\therefore \quad \frac{d}{dx} \log u = \frac{d}{dx} (\sin x \log x)$$

$$\Rightarrow \quad \frac{1}{u} \frac{du}{dx} = \sin x \frac{d}{dx} \log x + \log x \frac{d}{dx} \sin x$$

$$= \sin x \cdot \frac{1}{x} + (\log x) \cos x \quad = \frac{\sin x}{x} + \cos x \log x$$

$$\therefore \quad \frac{du}{dx} = u \left(\frac{\sin x}{x} + \cos x \log x \right)$$

$$= x^{\sin x} \left(\frac{\sin x}{x} + \cos x \log x \right) \quad \text{...}(ii)$$

Again $v = (\sin x)^{\cos x}$ [Form $f(x)^{g(x)}$]

$\therefore \quad \log v = \log (\sin x)^{\cos x} = \cos x \log \sin x$

$$\therefore \frac{d}{dx} (\log v) = \frac{d}{dx} [\cos x \log \sin x]$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = \cos x \frac{d}{dx} \log \sin x + \log \sin x \frac{d}{dx} \cos x$$

$$= \cos x \frac{1}{\sin x} \frac{d}{dx} (\sin x) + \log \sin x (-\sin x)$$

$$= \cot x \cdot \cos x - \sin x \log \sin x$$

$$\therefore \quad \frac{dv}{dx} = v (\cos x \cot x - \sin x \log \sin x)$$

$$= (\sin x)^{\cos x} (\cos x \cot x - \sin x \log \sin x) \quad \text{...}(iii)$$

Putting values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*),

we have $\frac{dy}{dx} = x^{\sin x} \left(\frac{\sin x}{x} + \cos x \log x \right)$

$$+ (\sin x)^{\cos x} (\cos x \cot x - \sin x \log \sin x)$$

10. $\boldsymbol{x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}}$.

Sol. Let $y = x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$

Putting $x^{x \cos x} = u$ and $\frac{x^2 + 1}{x^2 - 1} = v$

We have $y = u + v \quad \therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(*i*)

Now $\quad u = x^{x \cos x}$

Taking logarithms, $\log u = \log x^{x \cos x} = x \cos x \log x$

Differentiating w.r.t. x, we have

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x \cos x \log x)$$

$$= \frac{d}{dx}(x) \cdot \cos x \log x + x \frac{d}{dx}(\cos x) \cdot \log x$$

$$+ x \cos x \frac{d}{dx}(\log x)$$

$$\left[\because \frac{d}{dx}(uvw) = \frac{du}{dx} vw + u \frac{dv}{dx} \cdot w + uv \frac{dw}{dx}\right]$$

$$= 1 \cos x \log x + x(-\sin x) \log x + x \cos x \cdot \frac{1}{x}$$

$$\Rightarrow \quad \frac{du}{dx} = u [\cos x \log x - x \sin x \log x + \cos x]$$

$$= x^{x \cos x} [\cos x \log x - x \sin x \log x + \cos x] \qquad ...(ii)$$

Also $\quad v = \dfrac{x^2 + 1}{x^2 - 1}$. Using quotient rule, we have

$$\frac{dv}{dx} = \frac{(x^2 - 1)\frac{d}{dx}(x^2 + 1) - (x^2 + 1) \cdot \frac{d}{dx}(x^2 - 1)}{(x^2 - 1)^2}$$

$$= \frac{(x^2 - 1) \cdot 2x - (x^2 + 1) \cdot 2x}{(x^2 - 1)^2} = \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2 - 1)^2}$$

$$= -\frac{4x}{(x^2 - 1)^2} \qquad ...(iii)$$

Putting the values of $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*), we have

$$\frac{dy}{dx} = x^{x \cos x} [\cos x \log x - x \sin x \log x + \cos x] - \frac{4x}{(x^2 - 1)^2}.$$

11. $(x \cos x)^x + (x \sin x)^{1/x}$.

Sol. Let $y = (x \cos x)^x + (x \sin x)^{1/x}$

Putting $(x \cos x)^x = u$ and $(x \sin x)^{1/x} = v$,

We have $y = u + v \qquad \therefore \quad \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx} \qquad ...(i)$

Now $\quad u = (x \cos x)^x$

Taking logarithms, $\log u = \log (x \cos x)^x = x \log (x \cos x)$

$= x (\log x + \log \cos x)$

Differentiating w.r.t. x, we have

$$\frac{1}{u} \cdot \frac{du}{dx} = x \left[\frac{1}{x} + \frac{1}{\cos x} \cdot (-\sin x)\right] + (\log x + \log \cos x) \cdot 1$$

$$\Rightarrow \quad \frac{du}{dx} = u\,[1 - x \tan x + \log (x \cos x)]$$

$[\because \ \log x + \log \cos x = \log (x \cos x)]$

$$= (x \cos x)^x\,[1 - x \tan x + \log (x \cos x)] \qquad ...(ii)$$

Also $\quad v = (x \sin x)^{1/x}$

Taking logarithms, $\log v = \log (x \sin x)^{1/x} = \dfrac{1}{x} \log (x \sin x)$

$$= \frac{1}{x}\,(\log x + \log \sin x)$$

Differentiating w.r.t. x, we have

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{1}{x}\left[\frac{1}{x} + \frac{1}{\sin x} \cdot \cos x\right] + (\log x + \log \sin x)\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log (x \sin x)}{x^2}\right]$$

$$= (x \sin x)^{1/x} \cdot \left[\frac{1 + x \cot x - \log (x \sin x)}{x^2}\right] \qquad ...(iii)$$

Putting the values of $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$ from (ii) and (iii) in (i), we have

$$\frac{dy}{dx} = (x \cos x)^x\,[1 - x \tan x + \log (x \cos x)] + (x \sin x)^{1/x}\left[\frac{1 + x \cot x - \log (x \sin x)}{x^2}\right].$$

Find $\dfrac{dy}{dx}$ of the functions given in Exercises 12 to 15:

12. $x^y + y^x = 1$.

Sol. Given : $x^y + y^x = 1$

$\Rightarrow \quad u + v = 1 \quad$ where $\quad u = x^y$ and $\quad v = y^x$

$$\therefore \quad \frac{d}{dx}(u) + \frac{d}{dx}(v) = \frac{d}{dx}(1)$$

$$i.e., \quad \frac{du}{dx} + \frac{dv}{dx} = 0 \qquad ...(i)$$

Now $\quad u = x^y \qquad [(\text{Variable})^{\text{variable}} = (f(x))^{g(x)}]$

$\therefore \quad \log u = \log x^y = y \log x$

$$\therefore \quad \frac{d}{dx} \log u = \frac{d}{dx}\,(y \log x)$$

$$\Rightarrow \quad \frac{1}{u}\,\frac{du}{dx} = y \frac{d}{dx} \log x + \log x \,\frac{dy}{dx} \qquad = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

$$\therefore \quad \frac{du}{dx} = u\left(\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right)$$

or $\quad \dfrac{du}{dx} = x^y\left(\dfrac{y}{x} + \log x \dfrac{dy}{dx}\right) = x^y \dfrac{y}{x} + x^y \log x \dfrac{dy}{dx}$

or $\quad \dfrac{du}{dx} = x^{y-1}y + x^y \log x \dfrac{dy}{dx}$ ...(*ii*) $\quad \left[\because \dfrac{x^y}{x} = \dfrac{x^y}{x^1} = x^{y-1}\right]$

Again $\quad v = y^x$

$\therefore \quad \log v = \log y^x = x \log y \qquad \therefore \dfrac{d}{dx} \log v = \dfrac{d}{dx}(x \log y)$

$\Rightarrow \quad \dfrac{1}{v}\dfrac{dv}{dx} = x\dfrac{d}{dx}(\log y) + \log y \dfrac{d}{dx}x = x \cdot \dfrac{1}{y}\dfrac{dy}{dx} + \log y \cdot 1$

$\Rightarrow \quad \dfrac{dv}{dx} = v\left(\dfrac{x}{y}\dfrac{dy}{dx} + \log y\right)$

$= y^x\left(\dfrac{x}{y}\dfrac{dy}{dx} + \log y\right) = y^x \dfrac{x}{y}\dfrac{dy}{dx} + y^x \log y$

$\Rightarrow \quad \dfrac{dv}{dx} = y^{x-1}x\dfrac{dy}{dx} + y^x \log y$ ...(*iii*)

Putting values of $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*), we have

$x^{y-1}y + x^y \log x \dfrac{dy}{dx} + y^{x-1}x\dfrac{dy}{dx} + y^x \log y = 0$

or $\quad \dfrac{dy}{dx}(x^y \log x + y^{x-1}x) = -x^{y-1}y - y^x \log y$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{(x^{y-1}\, y + y^x \log y)}{x^y \log x + y^{x-1}\, x}$.

13. $\boldsymbol{y^x = x^y}$**.**

Sol. Given: $y^x = x^y \Rightarrow x^y = y^x$.

| Form on both sides is $(f(x))^{g(x)}$

Taking logarithms, $\log x^y = \log y^x \Rightarrow y \log x = x \log y$

Differentiating w.r.t. x, we have

$y \cdot \dfrac{1}{x} + \log x \cdot \dfrac{dy}{dx} = x \cdot \dfrac{1}{y} \cdot \dfrac{dy}{dx} + \log y \cdot 1$

$\Rightarrow \left(\log x - \dfrac{x}{y}\right)\dfrac{dy}{dx} = \log y - \dfrac{y}{x}$

$\Rightarrow \dfrac{y \log x - x}{y} \cdot \dfrac{dy}{dx} = \dfrac{x \log y - y}{x} \quad \therefore \dfrac{dy}{dx} = \dfrac{y(x \log y - y)}{x(y \log x - x)}$.

14. $\boldsymbol{(\cos x)^y = (\cos y)^x}$**.**

Sol. Given: $(\cos x)^y = (\cos y)^x$ [Form on both sides is $(f(x))^{g(x)}$]

$\therefore$ Taking logs on both sides, we have

$\log (\cos x)^y = \log (\cos y)^x$

$\Rightarrow \quad y \log \cos x = x \log \cos y \quad [\because \log m^n = n \log m]$

Differentiating both sides w.r.t. x, we have

$$\frac{d}{dx}(y \log \cos x) = \frac{d}{dx}(x \log \cos y)$$

Applying Product Rule on both sides,

$$\Rightarrow \quad y\frac{d}{dx}\log \cos x + \log \cos x \frac{dy}{dx}$$
$$= x \frac{d}{dx}\log \cos y + \log \cos y \frac{d}{dx}x$$

$$\Rightarrow \quad y \cdot \frac{1}{\cos x}\frac{d}{dx}\cos x + \log \cos x \frac{dy}{dx}$$
$$= x \cdot \frac{1}{\cos y}\frac{d}{dx}\cos y + \log \cos y$$

$$\Rightarrow \quad y\frac{1}{\cos x}(-\sin x) + \log \cos x \frac{dy}{dx}$$
$$= x\frac{1}{\cos y}\left(-\sin y \frac{dy}{dx}\right) + \log \cos y$$

$$\Rightarrow \quad -y \tan x + \log \cos x \cdot \frac{dy}{dx} = -x \tan y \frac{dy}{dx} + \log \cos y$$

$$\Rightarrow \quad x \tan y \frac{dy}{dx} + \log \cos x \cdot \frac{dy}{dx} = y \tan x + \log \cos y$$

$$\Rightarrow \quad \frac{dy}{dx}(x \tan y + \log \cos x) = y \tan x + \log \cos y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}.$$

15. $\boldsymbol{xy = e^{x-y}}$**.**

Sol. Given: $xy = e^{x-y}$

Taking logs on both sides, we have

$$\log (xy) = \log e^{x-y}$$

$$\Rightarrow \quad \log x + \log y = (x - y) \log e$$

$$\Rightarrow \quad \log x + \log y = x - y \qquad (\because \log e = 1)$$

Differentiating both sides w.r.t. x, we have

$$\frac{d}{dx}\log x + \frac{d}{dx}\log y = \frac{d}{dx}x - \frac{d}{dx}y$$

$$\Rightarrow \quad \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} + \frac{dy}{dx} = 1 - \frac{1}{x} \quad \Rightarrow \frac{dy}{dx}\left(\frac{1}{y}+1\right) = \frac{x-1}{x}$$

$$\Rightarrow \quad \left(\frac{1+y}{y}\right)\frac{dy}{dx} = \frac{x-1}{x}$$

Cross-multiplying $x(1 + y)\frac{dy}{dx} = y(x - 1)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y(x-1)}{x(1+y)}.$$

16. Find the derivative of the function given by $f(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$ and hence find $f'(1)$.

Sol. Given: $f(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$ ...(*i*)

Taking logs on both sides, we have

$$\log f(x) = \log (1 + x) + \log (1 + x^2) + \log (1 + x^4) + \log (1 + x^8)$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{f(x)} \frac{d}{dx} f(x) = \frac{1}{1+x} \frac{d}{dx}(1 + x) + \frac{1}{1+x^2} \frac{d}{dx} (1 + x^2) + \frac{1}{1+x^4} \frac{d}{dx} (1 + x^4) + \frac{1}{1+x^8} \frac{d}{dx} (1 + x^8)$$

$$\Rightarrow \quad \frac{1}{f(x)} f'(x) = \frac{1}{1+x} . 1 + \frac{1}{1+x^2} . 2x + \frac{1}{1+x^4} . 4x^3 + \frac{1}{1+x^8}\ 8x^7$$

$$\therefore \quad f'(x) = f(x) \left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

Putting the value of $f(x)$ from (*i*),

$$f'(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8) \left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

Putting $x = 1$,

$$f'(1) = (1 + 1)(1 + 1)(1 + 1)(1 + 1) \left[\frac{1}{1+1} + \frac{2}{1+1} + \frac{4}{1+1} + \frac{8}{1+1}\right]$$

$$= 2.2.2.2 \left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right] = 16\left[\frac{15}{2}\right] = 8 \times 15 = 120.$$

17. Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below:

(*i*) by using product rule.

(*ii*) by expanding the product to obtain a single polynomial.

(*iii*) by logarithmic differentiation.

Do they all give the same answer?

Sol. Given: Let $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$ ...(1)

(*i*) **To find $\frac{dy}{dx}$ by using Product Rule**

$$\frac{dy}{dx} = (x^2 - 5x + 8) \frac{d}{dx}(x^3 + 7x + 9)$$

$$+ (x^3 + 7x + 9)\frac{d}{dx}(x^2 - 5x + 8)$$

$$= (x^2 - 5x + 8)(3x^2 + 7) + (x^3 + 7x + 9)(2x - 5)$$

$$= 3x^4 + 7x^2 - 15x^3 - 35x + 24x^2 + 56$$

$$+ 2x^4 - 5x^3 + 14x^2 - 35x + 18x - 45$$

$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11 \qquad ...(2)$$

(*ii*) **To find $\frac{dy}{dx}$ by expanding the product to obtain a single polynomial.**

From (*i*), $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

$$= x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$$

or $\quad y = x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$

$$\therefore \quad \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11 \qquad ...(3)$$

(*iii*) **To find $\frac{dy}{dx}$ by logarithmic differentiation**

Taking logs on both sides of (*i*), we have

$$\log y = \log (x^2 - 5x + 8) + \log (x^3 + 7x + 9)$$

$$\therefore \frac{d}{dx}\log y = \frac{d}{dx}\log (x^2 - 5x + 8) + \frac{d}{dx}\log (x^3 + 7x + 9)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 - 5x + 8}\frac{d}{dx}(x^2 - 5x + 8) + \frac{1}{x^3 + 7x + 9}\cdot\frac{d}{dx}(x^3 + 7x + 9)$$

$$= \frac{1}{x^2 - 5x + 8}(2x - 5) + \frac{1}{x^3 + 7x + 9}(3x^2 + 7)$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{(2x - 5)}{x^2 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9}\right]$$

$$= y\left[\frac{(2x - 5)(x^3 + 7x + 9) + (3x^2 + 7)(x^2 - 5x + 8)}{(x^2 - 5x + 8)(x^3 + 7x + 9)}\right]$$

$$= y\,\frac{[2x^4 + 14x^2 + 18x - 5x^3 - 35x - 45 + 3x^4 - 15x^3 + 24x^2 + 7x^2 - 35x + 56]}{(x^2 - 5x + 8)(x^3 + 7x + 9)}$$

$$\text{or} \quad \frac{dy}{dx} = y\,\frac{(5x^4 - 20x^3 + 45x^2 - 52x + 11)}{(x^2 - 5x + 8)(x^3 + 7x + 9)}$$

Putting the value of y from (*i*),

$$\frac{dy}{dx} = (x^2 - 5x + 8)(x^3 + 7x + 9)\,\frac{(5x^4 - 20x^3 + 45x^2 - 52x + 11)}{(x^2 - 5x + 8)(x^3 + 7x + 9)}$$

$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11 \qquad ...(4)$$

From (2), (3) and (4), we can say that value of $\frac{dy}{dx}$ is same obtained by three different methods used in (*i*), (*ii*) and (*iii*).

18. If *u, v* and *w* are functions of *x*, then show that

$$\frac{d}{dx}(u \, . \, v \, . \, w) = \frac{du}{dx} \, v \, . \, w + u \, . \, \frac{dv}{dx} \, . \, w + u \, . \, v \, \frac{dw}{dx}$$

in two ways-first by repeated application of product rule, second by logarithmic differentiation.

Sol. Given: *u*, *v* and *w* are functions of *x*.

To prove: $\frac{d}{dx}(u \, . \, v \, . \, w) = \frac{du}{dx} \, . \, v \, . \, w + u \, . \, \frac{dv}{dx} \, . \, w + u \, . \, v \, . \, \frac{dw}{dx}$ ...(*i*)

(*i*) **To prove eqn. (*i*): By repeated application of product rule**

L.H.S. $= \frac{d}{dx}(u \, . \, v \, . \, w)$

Let us treat the product *uv* as a single function

$$= \frac{d}{dx}[(uv)w] \qquad = uv\frac{d}{dx}(w) + w\frac{d}{dx}(uv)$$

Again Applying Product Rule on $\frac{d}{dx}(uv)$

$$\text{L.H.S.} = \frac{d}{dx}(uvw) = uv\frac{dw}{dx} + w\left[u\frac{d}{dx}v + v\frac{d}{dx}u\right]$$

$$= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

Rearranging terms

or $\frac{d}{dx}(uvw) = \frac{du}{dx} \, . \, v \, . \, w + u \, . \, \frac{dv}{dx} \, . \, w + u \, . \, v \, . \, \frac{dw}{dx}$

which proves eqn. (*i*)

(*ii*) **To prove eqn. (*i*): By Logarithmic differentiation**

Let $y = uvw$

Taking logs on both sides

$$\log y = \log (u \, . \, v \, . \, w) = \log u + \log v + \log w$$

$$\therefore \frac{d}{dx}\log y = \frac{d}{dx}\log u + \frac{d}{dx}\log v + \frac{d}{dx}\log w$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left[\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right]$$

Putting $y = uvw$, $\frac{d}{dx}(uvw) = uvw\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$

$= \dfrac{du}{dx} \,.\, v \,.\, w + u \,.\, \dfrac{dv}{dx} \,.\, w + u \,.v. \dfrac{dw}{dx}$ which proves eqn. (*i*).

Remark. The result of eqn. (*i*) can be used as a formula for derivative of product of three functions.

It can be used as a formula for doing Q. No. 1 and Q. No. 5 of this Exercise 5.5.