NCERT Class 12 Maths

Solutions

Chapter - 5

Exercise 5.1

1. Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.

Sol. Given: $f(x) = 5x - 3$ ...(*i*)

Continuity at $x = 0$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (5x - 3) \quad \text{(By } (i))$$

Putting $x = 0$, $= 5(0) - 3 = 0 - 3 = -3$

Putting $x = 0$ in (*i*), $f(0) = 5(0) - 3 = -3$

$\therefore \lim_{x \to 0} f(x) = f(0) (= -3) \qquad \therefore f(x)$ is continuous at $x = 0$.

Continuity at $x = -3$

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} (5x - 3) \quad \text{(By } (i))$$

Putting $x = -3$, $= 5(-3) - 3 = -15 - 3 = -18$

Putting $x = -3$ in (*i*), $f(-3) = 5(-3) - 3 = -15 - 3 = -18$

$\therefore \lim_{x \to -3} f(x) = f(-3)(= -18)$

$\therefore f(x)$ is continuous at $x = -3$.

Continuity at $x = 5$

$$\lim_{x \to 5} f(x) = \lim_{x \to 5} (5x - 3) \quad \text{(By } (i))$$

Putting $x = 5$, $5(5) - 3 = 25 - 3 = 22$

Putting $x = 5$ in (*i*), $f(5) = 5(5) - 3 = 25 - 3 = 22$

$\therefore \lim_{x \to 5} (5x - 3) = f(5) (= 22) \qquad \therefore f(x)$ is continuous at $x = 5$.

2. Examine the continuity of the function

$f(x) = 2x^2 - 1$ at $x = 3$.

Sol. Given: $f(x) = 2x^2 - 1$ ...(*i*)

Continuity at $x = 3$

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} (2x^2 - 1) \quad \text{[By (i)]}$$

Putting $x = 3$, $= 2.3^2 - 1 = 2(9) - 1 = 18 - 1 = 17$

Putting $x = 3$ in (*i*), $f(3) = 2.3^2 - 1 = 18 - 1 = 17$

$\therefore \lim_{x \to 3} f(x) = f(3)$ (= 17) $\quad \therefore$ $f(x)$ is continuous at $x = 3$.

3. Examine the following functions for continuity:

(*a*) $f(x) = x - 5$ **(*b*) $f(x) = \dfrac{1}{x-5}$, $x \neq 5$**

(*c*) $f(x) = \dfrac{x^2 - 25}{x+5}$, $x \neq -5$ **(*d*) $f(x) = |x - 5|$.**

Sol. (*a*) **Given:** $f(x) = x - 5$ ...(*i*)

The domain of f is R

($\because$ $f(x)$ is real and finite for all $x \in$ R)

Let c be any real number (*i.e.*, $c \in$ domain of f).

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x - 5) \quad \text{[By (i)]}$$

Putting $x = c$, $= c - 5$

Putting $x = c$ in (*i*), $f(c) = c - 5$

$\therefore \lim_{x \to c} f(x) = f(c)$ $(= c - 5)$

$\therefore$ f is continuous at every point c in its domain (here R).

Hence f is continuous.

Or

Here $f(x) = x - 5$ is a polynomial function. We know that every polynomial function is continuous (see note below).

Hence $f(x)$ is continuous (in its domain R)

Very important Note. The following functions are continuous (for all x in their domain).

1. Constant function
2. Polynomial function.
3. Rational function $\dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomial functions of x and $g(x) \neq 0$.
4. Sine function ($\Rightarrow$ sin x).
5. cos x.
6. e^x.
7. e^{-x}.
8. log x ($x > 0$).
9. Modulus function.

(*b*) **Given:** $f(x) = \dfrac{1}{x-5}$, $x \neq 5$ ...(*i*)

Given: The domain f is R – ($x \neq 5$) *i.e.*, R – {5}

($\because$ For $x = 5$, $f(x) = \frac{1}{x-5} = \frac{1}{5-5} = \frac{1}{0} \to \infty$

$\therefore$ $5 \notin$ domain of f)

Let c be any real number such that $c \neq 5$

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{x-5} \qquad \text{[By } (i)]$$

Putting $x = c$, $= \frac{1}{c-5}$

Putting $x = c$ in (i), $f(c) = \frac{1}{c-5}$

$\therefore$ $\lim_{x \to c} f(x) = f(c) \left(= \frac{1}{c-5} \right)$

$\therefore$ $f(x)$ is continuous at every point c in the domain of f. Hence f is continuous.

Or

Here $f(x) = \frac{1}{x-5}$, $x \neq 5$ is a rational function $\left(= \frac{\text{Polynomial 1 of degree 0}}{\text{Polynomial } (x-5) \text{ of degree 1}} \right)$ and its denominator *i.e.,* $(x - 5) \neq 0$ ($\because$ $x \neq 5$). We know that every rational function is continuous (By Note below Solution of Q. No. 3(*a*)). Therefore f is continuous (in its domain R – {5}).

(*c*) $f(x) = \frac{x^2 - 25}{x+5}$, $x \neq -5$

Here $f(x) = \frac{x^2 - 25}{x+5}$, $x \neq -5$ is a rational function and denominator $x + 5 \neq 0$ ($\because$ $x \neq -5$).

(In fact $f(x) = \frac{x^2 - 25}{x+5}$, $(x \neq -5) = \frac{(x+5)(x-5)}{x+5}$ $= x - 5$, $(x \neq -5)$ is a polynomial function). We know that every rational function is continuous. Therefore f is continuous (in its domain R – {– 5}).

Or

Proceed as in Method I of Q. No. 3(b).

(*d*) **Given:** $f(x) = |x - 5|$

Domain of $f(x)$ is R ($\because$ $f(x)$ is real and finite for all real x in $(-\infty, \infty)$)

Here $f(x) = |x - 5|$ is a modulus function.

We know that every modulus function is continuous.

(By Note below Solution of Q. No. 3(*a*)). Therefore f is continuous in its domain R.

4. Prove that the function $f(x) = x^n$ is continuous at $x = n$ where n is a positive integer.

Sol. Given: $f(x) = x^n$ where n is a positive integer. ...(*i*)

Domain of $f(x)$ is R ($\because f(x)$ is real and finite for all real x)

Here $f(x) = x^n$, where n is a positive integer.

We know that every polynomial function of x is a continuous function. Therefore, f is continuous (in its whole domain R) and hence continuous at $x = n$ also.

Or

$$\lim_{x \to n} f(x) = \lim_{x \to n} x^n \qquad \text{[By (i)]}$$

Putting $x = n$, $= n^n$

Again putting $x = n$ in (*i*), $f(n) = n^n$

$\therefore \lim_{x \to n} f(x) = f(n)\ (= n^n)$ $\therefore$ $f(x)$ is continuous at $x = n$.

5. Is the function f defined by

$$f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$$

continuous at $x = 0$?, At $x = 1$?, At $x = 2$?

Sol. Given: $f(x) = \begin{cases} x, & \text{if } x \le 1 & ...(i) \\ 5, & \text{if } x > 1 & ...(ii) \end{cases}$

(Read Note **(on continuity)** before the solution of Q. No. 1 of this exercise)

Continuity at $x = 0$

Left Hand Limit $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x$ [By (*i*)]

($x \to 0^- \Rightarrow x <$ slightly less than $0 \Rightarrow x < 1$)

Putting $x = 0$, $= 0$

Right hand limit $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x$ [By (*i*)]

($x \to 0^+ \Rightarrow x$ is slightly greater than 0 say $x = 0.001 \Rightarrow x < 1$)

Putting $x = 0$, $\lim_{x \to 0^+} f(x) = 0$ $\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0$

$\therefore \lim_{x \to 0} f(x)$ exists and $= 0 = f(0)$

($\because$ Putting $x = 0$ in (*i*), $f(0) = 0$)

$\therefore$ $f(x)$ is continuous at $x = 0$.

Continuity at $x = 1$

Left Hand Limit $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x$ [By (*i*)]

Putting $x = 1$, $= 1$

Right Hand Limit $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 5$

Putting $x = 1$, $\lim_{x \to 1^+} f(x) = 5$

$\therefore \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$ $\therefore \lim_{x \to 1} f(x)$ does not exist.

$\therefore$ $f(x)$ is discontinuous at $x = 1$.

Continuity at $x = 2$

Left Hand Limit $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 5$ [By (ii)]

($x \to 2 -$ $\Rightarrow x$ is slightly $< 2 \Rightarrow x = 1.98$ (say) $\Rightarrow x > 1$)

Putting $x = 2$, $= 5$

Right Hand Limit $= \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 5$ [By (ii)]

($x \to 2 +$ $\Rightarrow$ x is slightly > 2 and hence $x > 1$ also)

Putting $x = 2$, $= 5$

$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ $(= 5)$

$\therefore \lim_{x \to 2} f(x)$ exists and $= 5 = f(2)$

(Putting $x = 2 > 1$ in (ii), $f(2) = 5$)

$\therefore$ $f(x)$ is continuous at $x = 2$

Answer. f is continuous at $x = 0$ and $x = 2$ but not continuous at $x = 1$.

Find all points of discontinuity of f, where f is defined by (Exercises 6 to 12)

6. $f(x) = \begin{cases} 2x + 3, & x \le 2 \\ 2x - 3, & x > 2 \end{cases}$.

Sol. Given: $f(x) = 2x + 3$, $x \le 2$ ...(i)

$= 2x - 3$ $x > 2$ ...(ii)

To find points of discontinuity of f (in its domain)

Here $f(x)$ is defined for $x \le 2$ *i.e.,* on $(-\infty, 2]$

and also for $x > 2$ *i.e.,* on $(2, \infty)$

$\therefore$ Domain of f is $(-\infty, 2] \cup (2, \infty) = (-\infty, \infty) = R$

By (i), for all $x < 2$ ($x = 2$ being partitioning point can't be mentioned here) $f(x) = 2x + 3$ is a polynomial and hence continuous.

By (ii), for all $x > 2$, $f(x) = 2x - 3$ is a polynomial and hence continuous. Therefore $f(x)$ is continuous on $R - \{2\}$.

Let us examine continuity of f at partitioning point $x = 2$

Left Hand Limit $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 3)$ [By (i)]

Putting $x = 2$, $= 2(2) + 3 = 4 + 3 = 7$

Right Hand Limit $= \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x - 3)$ [By (ii)]

Putting $x = 2$, $= 2(2) - 3 = 4 - 3 = 1$

$\therefore \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

$\therefore \lim_{x \to 2} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 2$ (only).

7. $f(x) = \begin{cases} |x| + 3, & \text{if} \quad x \le -3 \\ -2x, & \text{if} \quad -3 < x < 3 \\ 6x + 2, & \text{if} \quad x \ge 3 \end{cases}$.

Sol. Given: $f(x) = \begin{cases} |x| + 3, & \text{if} \quad x \le -3 & ...(i) \\ -2x, & \text{if} \quad -3 < x < 3 & ...(ii) \\ 6x + 2, & \text{if} \quad x \ge 3 & ...(iii) \end{cases}$

Here $f(x)$ is defined for $x \le -3$ *i.e.,* $(-\infty, -3]$ and also for $-3 < x < 3$ and also for $x \ge 3$ *i.e.,* on $[3, \infty)$.

$\therefore$ Domain of f is $(-\infty, -3] \cup (-3, 3) \cup [3, \infty) = (-\infty, \infty) = R$.

By (*i*), for all $x < -3$, $f(x) = |x| + 3 = -x + 3$

($\because$ $x < -3$ means x is negative and hence $|x| = -x$)

is a polynomial and hence continuous.

By (*ii*), for all x $(-3 < x < 3)$ $f(x) = -2x$ is a polynomial and hence continuous.

By (*iii*), for all $x > 3$, $f(x) = 6x + 2$ is a polynomial and hence continuous. Therefore, $f(x)$ is continuous on $R - \{-3, 3\}$.

From (*i*), (*ii*) and (*iii*) we can observe that $x = -3$ and $x = 3$ are partitioning points of the domain R.

Let us examine continuity of *f* at partitioning point $x = -3$

Left Hand Limit $= \lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (|x| + 3)$ [By (*i*)]

($\because$ $x \to -3^- \Rightarrow x < -3$)

$= \lim_{x \to -3^-} (-x + 3)$

($\because$ $x \to -3^- \Rightarrow x < -3$ means x is negative and hence $|x| = -x$)

Put $x = -3$, $= 3 + 3 = 6$

Right Hand Limit $= \lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (-2x)$ [By (*ii*)]

($\because$ $x \to -3^+ \Rightarrow x > -3$)

Putting $x = -3$, $= -2(-3) = 6$

$\therefore \lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} f(x)$ $(= 6)$

$\therefore \lim_{x \to -3} f(x)$ exists and $= 6$

Putting $x = -3$ in (*i*), $f(-3) = |-3| + 3 = 3 + 3 = 6$

$\therefore \lim_{x \to -3} f(x) = f(-3)$ $(= 6)$

$\therefore$ $f(x)$ is continuous at $x = -3$.

Now let us examine continuity of f at partitioning point $x = 3$

Left Hand Limit $= \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-2x)$ [By (*ii*)]

$(\because \; x \to 3^- \Rightarrow x < 3)$

Putting $x = 3$, $= -2(3) = -6$

Right Hand Limit $= \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x + 2)$ [By (*iii*)]

$(\because \; x \to 3^+ \Rightarrow x > 3)$

Putting $x = 3$, $= 6(3) + 2 = 18 + 2 = 20$

$\therefore \quad \lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

$\therefore \quad \lim_{x \to 3} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 3$ (only).

8. $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$.

Sol. Given: $f(x) = \dfrac{|x|}{x}$ if $x \neq 0$

[*i.e.*, $= \dfrac{x}{x} = 1$ if $x > 0$ $(\because$ For $x > 0$, $|x| = x)$

and $= -\dfrac{x}{x} = -1$ if $x < 0$ $(\because$ For $x < 0$, $|x| = -x)$

i.e., $f(x) = 1$ if $x > 0$ ...(*i*)

$= -1$ if $x < 0$ ...(*ii*)

$= 0$ if $x = 0$ ...(*iii*)

Clearly domain of $f(x)$ is R $(\because \; f(x)$ is defined for $x > 0$, for $x < 0$ and also for $x = 0)$

By (*i*), for all $x > 0$, $f(x) = 1$ is a constant function and hence continuous.

By (*ii*), for all $x < 0$, $f(x) = -1$ is a constant function and hence continuous.

Therefore $f(x)$ is continuous on R – {0}.

Let us examine continuity of f at the partitioning point $x = 0$

Left Hand Limit $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -1$ [By (*ii*)]

$(\because \; x \to 0^- \Rightarrow x < 0)$

Put $x = 0$, $= -1$

Right Hand Limit $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1$ [By (*i*)]

$(\because \; x \to 0^+ \Rightarrow x > 0)$

Put $x = 0$, $= 1$

$\therefore \quad \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

$\therefore \quad \lim_{x \to 0} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 0$ (only).

Note. It may be noted that the function given in Q. No. 8 is called a **signum function.**

9. $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$.

Sol. Given:

$$f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 = \dfrac{x}{-x} = -1 \text{ if } x < 0 \quad \ldots(i) \\ & (\because \text{ For } x < 0, |x| = -x) \\ -1 & \text{if } x \geq 0 \quad \ldots(ii) \end{cases}$$

Here $f(x)$ is defined for $x < 0$ *i.e.,* on $(-\infty, 0)$ and also for $x \geq 0$ *i.e.,* on $[0, \infty)$.

$\therefore$ Domain of f is $(-\infty, 0) \cup [0, \infty) = (-\infty, \infty) = \text{R}$.

From (i) and (ii), we find that

$f(x) = -1$ for all real x (< 0 as well as ≥ 0)

Here $f(x) = -1$ is a constant function.

We know that every constant function is continuous.

$\therefore$ f is continuous (for all real x in its domain R)

Hence no point of discontinuity.

10. $f(x) = \begin{cases} x + 1, & \text{if } x \geq 1 \\ x^2 + 1, & \text{if } x < 1 \end{cases}$.

Sol. Given: $\begin{cases} x + 1, & \text{if } x \geq 1 \quad \ldots(i) \\ x^2 + 1, & \text{if } x < 1 \quad \ldots(ii) \end{cases}$

Here $f(x)$ is defined for $x \geq 1$ *i.e.,* on $[1, \infty)$ and also for $x < 1$ *i.e.,* on $(-\infty, 1)$.

Domain of f is $(-\infty, 1) \cup [1, \infty) = (-\infty, \infty) = \text{R}$

By (i), for all $x > 1$, $f(x) = x + 1$ is a polynomial and hence continuous.

By (ii), for all $x < 1$, $f(x) = x^2 + 1$ is a polynomial and hence continuous. Therefore f is continuous on $\text{R} - \{1\}$.

Let us examine continuity of f at the partitioning point $x = 1$.

Left Hand Limit $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1)$ [By (ii)]

$(\because x \to 1^- \Rightarrow x < 1)$

Putting $x = 1$, $= 1^2 + 1 = 1 + 1 = 2$

Right Hand Limit = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1)$ [By (i)]

($\because$ $x \to 1^+ \Rightarrow x > 1$)

Putting $x = 1$, $= 1 + 1 = 2$

$\therefore$ $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$ $(= 2)$

$\therefore$ $\lim_{x \to 1} f(x)$ exists and $= 2$

Putting $x = 1$ in (i), $f(1) = 1 + 1 = 2$

$\therefore$ $\lim_{x \to 1} f(x) = f(1)$ $(= 2)$

$\therefore$ $f(x)$ is continuous at $x = 1$ also.

$\therefore$ f is be continuous on its whole domain (R here).

Hence no point of discontinuity.

11. $f(x) = \begin{cases} x^3 - 3, & \text{if } x \le 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$.

Sol. Given: $f(x) = \begin{cases} x^3 - 3, & \text{if } x \le 2 & ...(i) \\ x^2 + 1, & \text{if } x > 2 & ...(ii) \end{cases}$

Here $f(x)$ is defined for $x \le 2$ *i.e.*, on $(-\infty, 2]$ and also for $x > 2$ *i.e.*, on $(2, \infty)$.

$\therefore$ Domain of f is $(-\infty, 2] \cup (2, \infty) = (-\infty, \infty) = R$

By (i), for all $x < 2$, $f(x) = x^3 - 3$ is a polynomial and hence continuous.

By (ii), for all $x > 2$, $f(x) = x^2 + 1$ is a polynomial and hence continuous.

$\therefore$ f is continuous on R – {2}.

Let us examine continuity of f at the partitioning point $x = 2$.

Left Hand Limit = $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^3 - 3)$ [By (i)]

($\because$ $x \to 2^- \Rightarrow x < 2$)

Putting $x = 2$, $= 2^3 - 3 = 8 - 3 = 5$

Right Hand Limit = $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + 1)$ [By (ii)]

($\because$ $x \to 2^+ \Rightarrow x > 2$)

Putting $x = 2$, $= 2^2 + 1 = 4 + 1 = 5$

$\therefore$ $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ $(= 5)$

$\therefore$ $\lim_{x \to 2} f(x)$ exists and $= 5$

Putting $x = 2$ in (i), $f(2) = 2^3 - 3 = 8 - 3 = 5$

$\therefore$ $\lim_{x \to 2} f(x) = f(2)$ $(= 5)$

$\therefore$ $f(x)$ is continuous at $x = 2$ (also).

Hence no point of discontinuity.

12. $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$.

Sol. Given: $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 & ...(i) \\ x^2, & \text{if } x > 1 & ...(ii) \end{cases}$

Here $f(x)$ is defined for $x \le 1$ *i.e.,* on $(-\infty, 1]$ and also for $x > 1$ *i.e.,* on $(1, \infty)$.

$\therefore$ Domain of f is $(-\infty, 1] \cup (1, \infty) = (-\infty, \infty) = \mathbb{R}$

By (i), for all $x < 1$, $f(x) = x^{10} - 1$ is a polynomial and hence continuous.

By (ii), for all $x > 1$, $f(x) = x^2$ is a polynomial and hence continuous.

$\therefore$ $f(x)$ is continuous on $\mathbb{R} - \{1\}$.

Let us examine continuity of f at the partitioning point $x = 1$.

Left Hand Limit $= \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x^{10} - 1)$ [By (i)]

($\because$ $x \to 1^- \Rightarrow x < 1$)

Putting $x = 1$, $= (1)^{10} - 1 = 1 - 1 = 0$

Right Hand Limit $= \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} x^2$ [By (ii)]

Putting $x = 1$, $= 1^2 = 1$

$\therefore$ $\lim\limits_{x \to 1^-} f(x) \ne \lim\limits_{x \to 1^+} f(x)$

$\therefore$ $\lim\limits_{x \to 1} f(x)$ does not exist.

Hence the point of discontinuity is $x = 1$ (only).

13. Is the function defined by

$$f(x) = \begin{cases} x + 5 & \text{if } x \le 1 \\ x - 5 & \text{if } x > 1 \end{cases}$$

a continuous function?

Sol. Given: $f(x) = \begin{cases} x + 5, & \text{if } x \le 1 & ...(i) \\ x - 5, & \text{if } x > 1 & ...(ii) \end{cases}$

Here $f(x)$ is defined for $x \le 1$ *i.e.,* on $(-\infty, 1]$ and also for $x > 1$ *i.e.,* on $(1, \infty)$

$\therefore$ Domain of f is $(-\infty, 1] \cup (1, \infty] = (-\infty, \infty) = \mathbb{R}$.

By (i), for all $x < 1$, $f(x) = x + 5$ is a polynomial and hence continuous.

By (ii), for all $x > 1$, $f(x) = x - 5$ is a polynomial and hence continuous.

$\therefore$ f is continuous on R – {1}.

Let us examine continuity at the partitioning point $x = 1$.

Left Hand Limit $= \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x + 5)$ [By (*i*)]

Putting $x = 1$, $= 1 + 5 = 6$

Right Hand Limit $= \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x - 5)$ [By (*ii*)]

Putting $x = 1$, $= 1 - 5 = -4$

$\therefore \lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$

$\therefore \lim\limits_{x \to 1} f(x)$ does not exist.

Hence $f(x)$ is discontinuous at $x = 1$.

$\therefore$ $x = 1$ is the only point of discontinuity.

Discuss the continuity of the function, f, where f is defined by

14. $$f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}.$$

Sol. Given: $$f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 & \ldots(i) \\ 4, & \text{if } 1 < x < 3 & \ldots(ii) \\ 5, & \text{if } 3 \le x \le 10 & \ldots(iii) \end{cases}$$

From (*i*), (*ii*) and (*iii*), we can see that $f(x)$ is defined in [0, 1] $\cup$ (1, 3) $\cup$ [3, 10] *i.e.*, $f(x)$ is defined in [0, 10].

$\therefore$ Domain of $f(x)$ is [0, 10].

From (*i*), for $0 \le x < 1$, $f(x) = 3$ is a constant function and hence is continuous for $0 \le x < 1$.

From (*ii*), for $1 < x < 3$, $f(x) = 4$ is a constant function and hence is continuous for $1 < x < 3$.

From (*iii*), for $3 < x \le 10$, $f(x) = 5$ is a constant function and hence is continuous for $3 < x \le 10$.

Therefore, $f(x)$ is continuous in the domain [0, 10] – {1, 3}.

Let us examine continuity of f at the partitioning point $x = 1$.

Left Hand Limit $= \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} 3$ [By (*i*)]

($\because$ $x \to 1^- \Rightarrow x < 1$)

Putting $x = 1$; $= 3$

Right Hand Limit $= \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} 4$ [By (*ii*)]

($\because$ $x \to 1^+ \Rightarrow x > 1$)

Putting $x = 1$, $= 4$

$\therefore \lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$

$\therefore \quad \lim_{x \to 1} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 1$.

Let us examine continuity of f at the partitioning point $x = 3$.

Left Hand Limit $= \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 4$ [By (*ii*)]

($\because \quad x \to 3^- \Rightarrow x < 3$)

Putting $x = 3$, $= 4$

Right Hand Limit $= \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 5$ [By (*iii*)]

($\because \quad x \to 3^+ \Rightarrow x > 3$)

Putting $x = 3$; $= 5$

$\therefore \quad \lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

$\therefore \quad \lim_{x \to 3} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 3$ also.

$\therefore \quad x = 1$ and $x = 3$ are the two points of discontinuity of the function f in its domain [0, 10].

15. $$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}.$$

Sol. The domain of f is $\{x \in R : x < 0\} \cup \{x \in R : 0 \le x \le 1\} \cup \{x \in R : x > 1\} = R$

$x = 0$ and $x = 1$ are partitioning points for the domain of this function.

For all $x < 0$, $f(x) = 2x$ is a polynomial and hence continuous.

For $0 < x < 1$, $f(x) = 0$ is a constant function and hence continuous.

For all $x > 1$, $f(x) = 4x$ is a polynomial and hence continuous.

Let us discuss continuity at partitioning point $x = 0$.

At $x = 0$, $f(0) = 0$ [$\because f(x) = 0$ if $0 \le x \le 1$]

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 2x$ [$\because x \to 0^- \Rightarrow x < 0$ and $f(x) = 2x$ for $x < 0$]

$= 2 \times 0 = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 0$ [$\because x \to 0^+ \Rightarrow x > 0$ and $f(x) = 0$ if $0 \le x \le 1$]

$= 0$

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0$

Thus $\lim_{x \to 0} f(x) = 0 = f(0)$ and hence f is continuous at 0.

Let us discuss continuity at partitioning point $x = 1$.

At $x = 1$, $f(1) = 0$ [$\because f(x) = 0$ if $0 \le x \le 1$]

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 0 \quad [x \to 1- \Rightarrow x < 1 \text{ and } f(x) = 0 \text{ if } 0 \le x \le 1]$$

$$= 0$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4x \quad [x \to 1+ \Rightarrow x > 1 \text{ and } f(x) = 4x \text{ for } x > 1]$$

$$= 4 \times 1 = 4$$

The left and right hand limits of f at $x = 1$ do not coincide *i.e.,* are not equal.

$\therefore \lim_{x \to 1} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 1$.

Thus f is continuous at every point in the domain except $x = 1$. Hence, f is not a continuous function and $x = 1$ is the only point of discontinuity.

16. $f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 < x \le 1 \\ 2, & \text{if } x > 1 \end{cases}$.

Sol. Given: $f(x) = \begin{cases} -2, & \text{if } x \le -1 & ...(i) \\ 2x, & \text{if } -1 < x \le 1 & ...(ii) \\ 2, & \text{if } x > 1 & ...(iii) \end{cases}$

From (*i*), (*ii*) and (*iii*) we can see that $f(x)$ is defined for

$$\{x : x \le -1\} \cup \{x : -1 < x \le 1\} \cup \{x : x > 1\}$$

i.e., for $(-\infty, -1] \cup (-1, 1] \cup (1, \infty) = (-\infty, \infty) = R$

$\therefore$ Domain of $f(x)$ is R.

From (*i*), for $x < -1$, $f(x) = -2$ is a constant function and hence is continuous for $x < -1$.

From (*ii*), for $-1 < x < 1$, $f(x) = 2x$ is a polynomial function and hence is continuous for $-1 < x < 1$.

From (*iii*), for $x > 1$, $f(x) = 2$ is a constant function and hence is continuous for $x > 1$.

Therefore $f(x)$ is continuous in R $- \{-1, 1\}$.

Let us examine continuity of f at the partitioning point $x = -1$.

Left Hand Limit $= \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-2)$ [By (*i*)]

$(\because x \to -1^- \Rightarrow x < -1)$

Putting $x = -1$, $= -2$

Right Hand Limit $= \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} 2x$ (By (*ii*)]

$(\because x \to -1^+ \Rightarrow x > -1)$

Putting $x = -1$, $= 2(-1) = -2$

$\therefore \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$ $(= -2)$ $\therefore \lim_{x \to -1} f(x)$ exists and $= -2$.

Putting $x = -1$ in (*i*), $f(-1) = -2$

$\therefore \quad \lim_{x \to -1} f(x) = f(-1) \; (= -2) \quad \therefore \quad f(x)$ is continuous at $x = -1$.

Let us examine continuity of *f* at the partitioning point $x = 1$

Left Hand Limit $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x)$ [By (*ii*)]

$(\because \; x \to 1^- \;\Rightarrow\; x < 1)$

Putting $x = 1$, $= 2(1) = 2$

Right Hand Limit $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2$ [By (*iii*)]

$(\because \; x \to 1^+ \;\Rightarrow\; x > 1)$

Putting $x = 1$, $= 2$

$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \; (= 2) \quad \therefore \quad \lim_{x \to 1} f(x)$ exists and $= 2$.

Putting $x = 1$ in (*ii*), $f(1) = 2(1) = 2$

$\therefore \quad \lim_{x \to 1} f(x) = f(1) \; (= 2) \quad \therefore \quad f(x)$ is continuous at $x = 1$ also.

Therefore f is continuous for all x in its domain R.

17. Find the relationship between *a* and *b* so that the function *f* defined by

$$f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$$

is continuous at $x = 3$.

Sol. Given: $\quad f(x) = \begin{cases} ax + 1 & \text{if } x \le 3 \quad \text{...}(i) \\ bx + 3 & \text{if } x > 3 \quad \text{...}(ii) \end{cases}$

and $f(x)$ is continuous at $x = 3$.

Left Hand Limit $= \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax + 1)$ [By (*i*)]

$(x \to 3^- \;\Rightarrow\; x < 3)$

Putting $x = 3$, $= 3a + 1$ ...(*iii*)

Right Hand Limit $= \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (bx + 3)$ [By (*ii*)]

$(\because \; x \to 3^+ \;\Rightarrow\; x > 3)$

Putting $x = 3$, $= 3b + 3$ ...(*iv*)

Putting $x = 3$ in (*i*), $f(3) = 3a + 1$ ...(*v*)

Because $f(x)$ is continuous at $x = 3$ (given)

$\therefore \quad \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$

Putting values from (*iii*), (*iv*) and (*v*) we have

$3a + 1 = 3b + 3 \quad (= 3a + 1)$

$\therefore \quad 3a + 1 = 3b + 3 \quad$ [$\because$ First and third members are equal]

$\Rightarrow \quad 3a = 3b + 2$

Dividing by 3, $a = b + \dfrac{2}{3}$.

18. For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

continuous at $x = 0$? What about continuity at $x = 1$?

Sol. Given: $$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \quad ...(i) \\ 4x + 1, & \text{if } x > 0 \quad ...(ii) \end{cases}$$

Given: $f(x)$ is continuous at $x = 0$. To find λ.

Left Hand Limit $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \lambda(x^2 - 2x)$ [By (i)]

$(\because \ x \to 0^- \ \Rightarrow \ x < 0)$

Putting $x = 0$, $= \lambda(0 - 0) = 0$

Right Hand Limit $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (4x + 1)$ [By (ii)]

$(\because \ x \to 0^+ \ \Rightarrow \ x > 0)$

Putting $x = 0$, $= 4(0) + 1 = 1$

$\therefore \ \lim_{x \to 0^-} f(x) \ (= 0) \ne \lim_{x \to 0^+} f(x) \ (= 1)$

$\therefore \ \lim_{x \to 0} f(x)$ does not exist whatever λ may be

($\because$ Neither left limit nor right limit involves λ)

$\therefore$ For no value of λ, f is continuous at $x = 0$.

To examine continuity of f at $x = 1$

Left Hand Limit $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (4x + 1)$ [By (ii)]

$(x \to 1^- \ \Rightarrow \ x$ is slightly < 1 say $x = 0.99 > 0)$

Put $x = 1$, $= 4 + 1 = 5$

Right Hand Limit $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x + 1)$ [By (ii)]

$(x \to 1^+ \ \Rightarrow \ x$ is slightly > 1 say $x = 1.1 > 0)$

Put $x = 1$, $= 4 + 1 = 5$

$\therefore \ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \ (= 5)$

$\therefore \ \lim_{x \to 1} f(x)$ exists and $= 5$

Putting $x = 1$ in (ii) $(\because \ 1 > 0)$, $f(1) = 4 + 1 = 5)$

$\therefore \ \lim_{x \to 1} f(x) = f(1) \ (= 5)$

$\therefore \ f(x)$ is continuous at $x = 1$ (for all real values of λ).

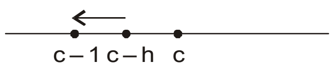
19. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

Sol. Given: $g(x) = x - [x]$

Let $x = c$ be any integer ($i.e.$, $c \in$ Z (= I))

Left Hand Limit $= \lim_{x \to c^-} g(x) = \lim_{x \to c^-} (x - [x])$

Put $x = c - h$, $h \to 0^+$



$= \lim_{h \to 0^+} (c - h - [c - h])$

$$= \lim_{h \to 0^+} (c - h - (c - 1))$$

[$\because$ If $c \in Z$ and $h \to 0^+$, then $[c - h] = c - 1$]

$$= \lim_{h \to 0^+} (c - h - c + 1) = \lim_{h \to 0^+} (1 - h)$$

Put $h = 0$, $= 1 - 0 = 1$

Right Hand Limit $= \lim_{x \to c^+} g(x) = \lim_{x \to c^+} (x - [x])$

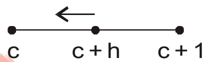
Put $x = c + h$, $h \to 0^+$

$$= \lim_{h \to 0^+} (c + h - [c + h]) \quad = \lim_{h \to 0^+} (c + h - c)$$

($\because$ If $c \in Z$ and $h \to 0^+$, then $[c + h] = c$)

$$= \lim_{h \to 0^+} h$$

Put $h = 0$; $= 0$



$\therefore \lim_{x \to c^-} g(x) \neq \lim_{x \to c^+} g(x)$

$\therefore \lim_{x \to c} g(x)$ does not exist and hence $g(x)$ is discontinuous at $x = c$ (any integer).

$\therefore$ $g(x) = x - [x]$ is discontinuous at all integral points.

Very Important Note. If two functions f and g are continuous in a common domain D,

then (*i*) $f + g$ (*ii*) $f - g$ (*iii*) fg are continuous in the same domain D.

(*iv*) $\dfrac{f}{g}$ is also continuous at all points of D except those where $g(x) = 0$.

20. Is the function $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

Sol. Given: $f(x) = x^2 - \sin x + 5 = (x^2 + 5) - \sin x$

$$= g(x) - h(x) \qquad ...(i)$$

where $g(x) = x^2 + 5$ and $h(x) = \sin x$

We know that $g(x) = x^2 + 5$ is a polynomial function and hence is continuous (for all real x)

Again $h(x) = \sin x$ being a sine function is continuous (for all real x)

$\therefore$ By (*i*) $f(x) = x^2 - \sin x + 5 = g(x) - h(x)$

being the difference of two continuous functions is also continuous for all real x (see Note above) and hence continuous at $x = \pi (\in R)$ also.

Or

Given: $f(x) = x^2 - \sin x + 5 \qquad ...(i)$

To examine continuity at $x = \pi$

$$\lim_{x \to \pi} f(x) = \lim_{x \to \pi} (x^2 - \sin x + 5) \qquad \text{[By } (i)]$$

Putting $x = \pi$, $= \pi^2 - \sin \pi + 5$

$= \pi^2 + 5$

[$\because$ $\sin \pi = \sin 180° = \sin (180° - 0°) = \sin 0° = 0$]

Again putting $x = \pi$ in (i), $f(\pi) = \pi^2 - \sin \pi + 5$

$= \pi^2 - 0 + 5 = \pi^2 + 5$

$\therefore \quad \lim_{x \to \pi} f(x) = f(\pi)$

$\therefore$ $f(x)$ is continuous at $x = \pi$.

21. Discuss the continuity of the following functions:

(*a*) $f(x) = \sin x + \cos x$ (*b*) $f(x) = \sin x - \cos x$

(*c*) $f(x) = \sin x \cdot \cos x$.

Sol. We know that $\sin x$ is a continuous function for all real x

Also we know that $\cos x$ is a continuous function for all real x (see solution of Q. No. 22(i) below)

$\therefore$ By Note at the end of solution of Q. No. 19,

(i) their sum function $f(x) = \sin x + \cos x$ is also continuous for all real x.

(ii) their difference function $f(x) = \sin x - \cos x$ is also continuous for all real x.

(iii) their product function $f(x) = \sin x \cdot \cos x$ is also continuous for all real x.

Note. To find $\lim_{x \to c} f(x)$, we can also start with putting $x = c + h$ where $h \to 0$ (and not only $h \to 0^+$)

$\therefore \quad \lim_{x \to c} f(x) = \lim_{h \to 0} f(c + h)$.

(Please note that this method of finding the limits makes us find both $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ simultaneously).

22. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

Sol. (i) Let $f(x)$ be the cosine function

i.e., $f(x) = \cos x$ ...(i)

Clearly, $f(x)$ is real and finite for all real values of x *i.e.,* $f(x)$ is defined for all real x. Therefore domain of $f(x)$ is R.

Let $x = c \in$ R.

$\lim_{x \to c} f(x) = \lim_{x \to c} \cos x$

Put $x = c + h$ where $h \to 0$

$= \lim_{h \to 0} \cos (c + h) = \lim_{h \to 0} (\cos c \cos h - \sin c \sin h)$

Putting $h = 0$, $= \cos c \cos 0 - \sin c \sin 0$

$= \cos c\,(1) - \sin c\,(0)$

$= \cos c$

$\therefore \quad \lim_{x \to c} f(x) = \cos c$

Putting $x = c$ in (*i*), $f(c) = \cos c$

$\therefore \quad \lim_{x \to c} f(x) = f(c) \ (= \cos c)$

$\therefore \quad f(x)$ is continuous at (every) $x = c \in \mathrm{R}$

$\therefore \quad f(x) = \cos x$ is continuous on R.

(*ii*) Let $f(x)$ be cosecant function

i.e., $f(x) = \operatorname{cosec} x = \dfrac{1}{\sin x}$

$f(x)$ is not finite *i.e.,* $\to \infty$

when $\sin x = 0$ *i.e.,* when $x = n\pi$, $n \in \mathrm{Z}$.

$\therefore$ Domain of $f(x) = \operatorname{cosec} x$ is $\mathrm{D} = \mathrm{R} - \{x = n\pi;\ n \in \mathrm{Z}\}$.

($\because$ $f(x)$ is real and finite $\forall\ x \in \mathrm{D}$).

Now $f(x) = \operatorname{cosec} x = \dfrac{1}{\sin x} = \dfrac{g(x)}{h(x)}$ ...(*i*)

Now $g(x) = 1$ being constant function is continuous on domain D and $h(x) = \sin x$ is non-zero and continuous on Domain D.

Therefore by (*i*), $f(x) = \operatorname{cosec} x \left(= \dfrac{1}{\sin x} = \dfrac{g(x)}{h(x)}\right)$ is continuous on domain $\mathrm{D} = \mathrm{R} - \{x = n\pi,\ n \in \mathrm{Z}\}$

(Also read **Note** at the end of solution of Q. No. 19).

(*iii*) Let $f(x)$ be the secant function

i.e., $f(x) = \sec x = \dfrac{1}{\cos x}$ $f(x)$ is not finite *i.e.,* $\to \infty$

When $\cos x = 0$ *i.e.,* when $x = (2n + 1)\dfrac{\pi}{2}$, $n \in \mathrm{Z}$.

$\therefore$ Domain of $f(x) = \sec x$ is

$$\mathrm{D} = \mathrm{R} - \{x = (2n + 1)\frac{\pi}{2};\ n \in \mathrm{Z}\}$$

Now $f(x) = \sec x = \dfrac{1}{\cos x} = \dfrac{g(x)}{h(x)}$ ...(*i*)

Now $g(x) = 1$ being constant function is continuous on domain D and $h(x) = \cos x$ is non-zero and continuous on domain D.

Therefore by (*i*), $f(x) = \sec x \left(= \dfrac{1}{\cos x} = \dfrac{g(x)}{h(x)}\right)$ is continuous on domain $\mathrm{D} = \mathrm{R} - \{x : x = (2n + 1)\dfrac{\pi}{2};\ n \in \mathrm{Z}\}$.

(*iv*) Let $f(x)$ be the cotangent function *i.e.,* $f(x) = \cot x = \dfrac{\cos x}{\sin x}$.

$f(x)$ is not finite *i.e.,* $\to \infty$

When $\sin x = 0$ *i.e.,* when $x = n\pi$, $n \in Z$.

$\therefore$ Domain of $f(x) = \cot x$ is

$$D = R - \{x = n\pi;\ n \in Z\}$$

Now $f(x) = \cot x = \dfrac{\cos x}{\sin x} = \dfrac{g(x)}{h(x)}$ ...(*i*)

Now $g(x) = \cos x$ being cosine function is continuous on D and is non-zero on D.

Therefore by (*i*), $f(x) = \cot x \left(= \dfrac{\cos x}{\sin x} = \dfrac{g(x)}{h(x)}\right)$ is continuous on domain $D = R - \{x : x = n\pi,\ n \in Z\}$.

23. Find all points of discontinuity of *f*, where

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0 \\ x + 1, & \text{if } x \geq 0 \end{cases}.$$

Sol. The domain of $f = \{x \in R : x < 0\} \cup \{x \in R : x \geq 0\} = R$

$x = 0$ is the partitioning point of the domain of the given function.

For all $x < 0$, $f(x) = \dfrac{\sin x}{x}$ (given)

Since $\sin x$ and x are continuous for $x < 0$ (in fact, they are continuous for all x) and $x \neq 0$

$\therefore f$ is continuous when $x < 0$

For all $x > 0$, $f(x) = x + 1$ is a polynomial and hence continuous.

$\therefore f$ is continuous when $x > 0$.

Let us discuss the continuity of $f(x)$ at the partitioning point $x = 0$.

At $x = 0$, $f(0) = 0 + 1 = 1$ $[\because f(x) = x + 1 \text{ for } x \geq 0]$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin x}{x}$$

$$\left[\because x \to 0^- \Rightarrow x < 0 \text{ and } f(x) = \frac{\sin x}{x} \text{ for } x < 0\right]$$

$$= 1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 1)$$

$$\left[\because x \to 0^+ \Rightarrow x > 0 \text{ and } f(x) = x + 1 \text{ for } x > 0\right]$$

$$= 0 + 1 = 1$$

Since $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = 1 \quad \therefore \lim\limits_{x \to 0} f(x) = 1$

Thus $\lim\limits_{x \to 0} f(x) = f(0)$ and hence f is continuous at $x = 1$.

Now f is continuous at every point in its domain and hence f is a continuous function.

24. Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a continuous function?

Sol. For all $x \neq 0$, $f(x) = x^2 \sin \dfrac{1}{x}$ being the product function of two continuous functions x^2 (polynomial function) and $\sin \dfrac{1}{x}$ (a sine function) is continuous for all real $x \neq 0$.

Now let us examine continuity at $x = 0$.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin \frac{1}{x}$$

Putting $x = 0$ $\quad = 0 \times$ A finite quantity between -1 and $1 = 0$

$$\left[\because \sin \frac{1}{x} (= \sin \theta) \text{ always lies between } -1 \text{ and } 1\right]$$

Also $\quad f(x) = 0$ at $x = 0$ *i.e.,* $f(0) = 0$

$\therefore \lim_{x \to 0} f(x) = f(0)$, therefore function f is continuous at $x = 0$ (also).

Hence $f(x)$ continuous on domain R of f.

25. Examine the continuity of f, where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}.$$

Sol. Given:

$$f(x) = \begin{cases} \sin x - \cos x & \text{if } x \neq 0 & \quad ...(i) \\ -1 & \text{if } x = 0 & \quad ...(ii) \end{cases}$$

From (i), $f(x)$ is defined for $x \neq 0$ and from (ii) $f(x)$ is defined for $x = 0$.

$\therefore$ Domain of $f(x)$ is $\{x : x \neq 0\} \cup \{0\} = \text{R}$.

From (i), for $x \neq 0$, $f(x) = \sin x - \cos x$ being the difference of two continuous functions $\sin x$ and $\cos x$ is continuous for all $x \neq 0$.

Hence $f(x)$ is continuous on $\text{R} - \{0\}$.

Now let us examine continuity at $x = 0$.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (\sin x - \cos x)$$

[By (i) as $x \to 0$ means $x \neq 0$]

Putting $x = 0$, $\quad = \sin 0 - \cos 0 = 0 - 1 = -1$

From (ii) $\quad f(x) = -1$ when $x = 0$

i.e., $\quad f(0) = -1$

$$\therefore \quad \lim_{x \to 0} f(x) = f(0) \; (= -1)$$

$\therefore \quad f(x)$ is continuous at $x = 0$ (also).

Hence $f(x)$ is continuous on domain R of f.

Find the values of k so that the function f is continuous at the indicated point in Exercises 26 to 29.

26. $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$ **at** $x = \dfrac{\pi}{2}$.

Sol. Left Hand Limit $= \lim\limits_{x \to \frac{\pi}{2}} f(x) = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{k \cos x}{\pi - 2x}$

Put $x = \dfrac{\pi}{2} - h$ where $h \to 0^+$

$$= \lim_{h \to 0^+} \frac{k \cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = \lim_{h \to 0^+} \frac{k \sin h}{\pi - \pi + 2h}$$

$$= \lim_{h \to 0^+} \frac{k \sin h}{2h} = \frac{k}{2} \times \lim_{h \to 0^+} \frac{\sin h}{h} = \frac{k}{2} \times 1 = \frac{k}{2} \qquad ...(i)$$

Right Hand Limit $= \lim\limits_{x \to \frac{\pi^+}{2}} f(x) = \lim\limits_{x \to \frac{\pi^+}{2}} \dfrac{k \cos x}{\pi - 2x}$

Put $x = \dfrac{\pi}{2} + h$ where $h \to 0^+$

$$= \lim_{h \to 0^+} \frac{k \cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = \lim_{h \to 0^+} \frac{-k \sin h}{\pi - \pi - 2h} = \lim_{h \to 0^+} \frac{-k \sin h}{-2h}$$

$$= \frac{k}{2} \times \lim_{h \to 0^+} \frac{\sin h}{h} = \frac{k}{2} \times 1 = \frac{k}{2} \qquad ...(ii)$$

Also $f\left(\dfrac{\pi}{2}\right) = 3 \quad ...(iii) \qquad \because \; f(x) = 3$ when $x = \dfrac{\pi}{2}$ (given)

Because $f(x)$ is continuous at $x = \dfrac{\pi}{2}$ (given)

$$\therefore \lim_{x \to \frac{\pi^-}{2}} f(x) = \lim_{x \to \frac{\pi^+}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

Putting values from (*i*), (*ii*), and (*iii*), $\dfrac{k}{2} = 3$ or $k = 6$.

27. $f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$ **at $x = 2$.**

Sol. Given: $f(x) = \begin{cases} kx^2, & \text{if } x \le 2 & \text{...}(i) \\ 3, & \text{if } x > 2 & \text{...}(ii) \end{cases}$

Given: $f(x)$ is continuous at $x = 2$.

Left Hand Limit $= \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} kx^2$ [By (*i*)]

($\because$ $x \to 2^- \Rightarrow x$ is < 2)

Put $x = 2$, $= k(2)^2 = 4k$

Right Hand Limit $= \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} 3$ [By (*ii*)]

($\because$ $x \to 2^+ \Rightarrow x > 2$)

Putting $x = 2$, $= 3$

Putting $x = 2$ in (*i*) $f(2) = k(2)^2 = 4k$.

Because $f(x)$ is continuous at $x = 2$ (given),

therefore $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = f(2)$

Putting values, $4k = 3 = 3 \Rightarrow k = \frac{3}{4}$.

28. $f(x) = \begin{cases} kx + 1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$ **at $x = \pi$.**

Sol. Given: $f(x) = \begin{cases} kx + 1, & \text{if } x \le \pi & \text{...}(i) \\ \cos x, & \text{if } x > \pi & \text{...}(ii) \end{cases}$

Given: $f(x)$ is continuous at $x = \pi$.

Left Hand Limit $= \lim\limits_{x \to \pi^-} f(x) = \lim\limits_{x \to \pi^-} (kx + 1)$ [By (*i*)]

($\because$ $x \to \pi^- \Rightarrow x < \pi$)

Putting $x = \pi$, $= k\pi + 1$

Right Hand Limit $= \lim\limits_{x \to \pi^+} f(x) = \lim\limits_{x \to \pi^+} \cos x$ [By (*ii*)]

($\because$ $x \to \pi^+ \Rightarrow x > \pi$)

Putting $x = \pi$, $= \cos \pi = \cos 180° = \cos (180° - 0)$

$= -\cos 0 = -1$

Putting $x = \pi$ in (*i*), $f(\pi) = k\pi + 1$

But $f(x)$ is continuous at $x = \pi$ (given), therefore

$$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi)$$

Putting values $k\pi + 1 = -1 = k\pi + 1$

$\Rightarrow$ $k\pi + 1 = -1$ [$\because$ First and third members are same]

$\Rightarrow$ $k\pi = -2 \Rightarrow k = -\frac{2}{\pi}$.

29. $f(x) = \begin{cases} kx + 1, & \text{if } x \le 5 \\ 3x - 5, & \text{if } x > 5 \end{cases}$ **at $x = 5$.**

Sol. Given: $f(x) = \begin{cases} kx + 1 & \text{if } x \le 5 & \text{...}(i) \\ 3x - 5 & \text{if } x > 5 & \text{...}(ii) \end{cases}$

Given: $f(x)$ is continuous at $x = 5$.

Left Hand Limit $= \lim\limits_{x \to 5^-} f(x) = \lim\limits_{x \to 5^-} (kx + 1)$ [By (*i*)]

Putting $x = 5$, $= k(5) + 1 = 5k + 1$

Right Hand Limit $= \lim\limits_{x \to 5^+} f(x) = \lim\limits_{x \to 5^+} (3x - 5)$ [By (*ii*)]

Putting $x = 5$, $= 3(5) - 5 \quad = 15 - 5 = 10$

Putting $x = 5$ in (*i*), $f(5) \quad = 5k + 1$

But $f(x)$ is continuous at $x \quad = 5$ (given)

$\therefore \quad \lim\limits_{x \to 5^-} f(x) = \lim\limits_{x \to 5^+} f(x) \quad = f(5)$

Putting values $5k + 1 = 10 \quad = 5k + 1$

$\Rightarrow \quad 5k + 1 = 10 \quad \Rightarrow \quad 5k = 9 \quad \Rightarrow \quad k = \dfrac{9}{5}.$

30. Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, & \text{if} \quad x \le 2 \\ ax + b, & \text{if} \quad 2 < x < 10 \\ 21, & \text{if} \quad x \ge 10 \end{cases}.$$

is a continuous function.

Sol. Given:

$$f(x) = \begin{cases} 5 & \text{if} \quad x \le 2 & ...(i) \\ ax + b & \text{if} \quad 2 < x < 10 & ...(ii) \\ 21 & \text{if} \quad x \ge 10 & ...(iii) \end{cases}$$

From (*i*), (*ii*) and (*iii*), $f(x)$ is defined for $\{x \le 2\} \cup \{2 < x < 10\} \cup \{x \ge 10\}$ *i.e.*, for $(-\infty, 2] \cup (2, 10) \cup [10, \infty)$ *i.e.*, for $(-\infty, \infty)$ *i.e.*, on R. $\therefore$ Domain of $f(x)$ is R.

Given: $f(x)$ is a continuous function (of course on its domain here R), therefore $f(x)$ is also continuous at partitioning points $x = 2$ and $x = 10$ of the domain.

Because $f(x)$ is continuous at partitioning point $x = 2$, therefore

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) \qquad ...(iv)$$

Now $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} 5$ [By (*i*)]

($\because \quad x \to 2^- \quad \Rightarrow \quad x < 2$)

Putting $x = 2$, $= 5$

Again $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (ax + b)$ [By (*ii*)]

($\because \quad x \to 2^+ \quad \Rightarrow \quad x > 2$)

Putting $x = 2$, $= 2a + b$

Putting $x = 2$ in (*i*), $f(2) = 5$.

Putting these values in eqn. (*iv*), we have

$$5 = 2a + b = 5 \quad \Rightarrow \quad 2a + b = 5 \qquad ...(v)$$

Again because $f(x)$ is continuous at partitioning point $x = 10$,

therefore $\lim_{x \to 10^-} f(x) = \lim_{x \to 10^+} f(x) = f(10)$ ...(*vi*)

Now $\lim_{x \to 10^-} f(x) = \lim_{x \to 10^-} (ax + b)$ [By (*ii*)]

$(x \to 10^- \Rightarrow x < 10)$

Putting $x = 10$, $= 10a + b$

Again $\lim_{x \to 10^+} f(x) = \lim_{x \to 10^+} 21$ [By (*iii*)]

$(\because x \to 10^+ \Rightarrow x > 10)$

Putting $x = 10$; $= 21$

Putting $x = 10$ in Eqn. (*iii*), $f(10) = 21$

Putting these values in eqn. (*vi*), we have

$10a + b = 21 = 21$

$\Rightarrow \quad 10a + b = 21$ ...(*vii*)

Let us solve eqns. (*v*) and (*vii*) for a and b.

Eqn. (*vii*) – eqn. (*v*) gives $8a = 16 \Rightarrow a = \frac{16}{8} = 2$

Putting $a = 2$ in (*v*), $4 + b = 5 \quad \therefore \quad b = 1$.

$\therefore \quad a = 2, b = 1$.

Very Important Result: Composite function of two continuous functions is continuous.

We know by definition that $(fog)x = f(g(x))$

and $(gof)x = g(f(x))$

31. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Sol. Given: $f(x) = \cos(x^2)$ ...(*i*)

$f(x)$ has a real and finite value for all $x \in$ R.

$\therefore$ Domain of $f(x)$ is R.

Let us take $g(x) = \cos x$ and $h(x) = x^2$.

Now $g(x) = \cos x$ is a cosine function and hence is continuous.

Again $h(x) = x^2$ is a polynomial function and hence is continuous.

$\therefore \quad (goh)x = g(h(x)) = g(x^2)$ $[\because h(x) = x^2]$

$= \cos(x^2)$ (Changing x to x^2 in $g(x) = \cos x$)

$= f(x)$ (By (*i*)) being the composite function of two continuous functions is continuous for all x in domain R.

Or

Take $g(x) = x^2$ and $h(x) = \cos x$.

Then $(hog)x = h(g(x)) = h(x^2)$

$= \cos(x^2) = f(x)$.

32. Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Sol. $f(x) = |\cos x|$ ...(*i*)

$f(x)$ has a real and finite value for all $x \in$ R.

$\therefore$ Domain of $f(x)$ is R.

Let us take $g(x) = \cos x$ and $h(x) = | x |$

We know that $g(x)$ and $h(x)$ being cosine function and modulus function are continuous for all real x.

Now $(goh)x = g(h(x)) = g(| x |) = \cos | x |$ being the composite function of two continuous functions is continuous (but $\neq f(x)$)

Again $(hog)x = h(g(x)) = h(\cos x)$

$= | \cos x | = f(x)$ [By (*i*)]

[Changing x to $\cos x$ in $h(x) = | x |$, we have $h(\cos x) = | \cos x |$]

Therefore $f(x) = | \cos x |$ $(= (hog)x)$ being the composite function of two continuous functions is continuous.

33. Examine that sin | *x* | is a continuous function.

Sol. Let $f(x) = \sin x$ and $g(x) = | x |$

We know that $\sin x$ and $| x |$ are continuous functions.

$\therefore$ f and g are continuous.

Now $(fog)(x) = f\{g(x)\} = \sin \{g(x)\} = \sin | x |$

We know that composite function of two continuous functions is continuous.

$\therefore$ fog is continuous. Hence, $\sin |x|$ is continuous.

34. Find all points of discontinuity of *f* defined by

$$f(x) = | x | - | x + 1 |.$$

Sol. Given: $f(x) = | x | - | x + 1 |$ ...(*i*)

This $f(x)$ is real and finite for every $x \in \mathbb{R}$.

$\therefore$ f is defined for all $x \in \mathbb{R}$ *i.e.,* domain of f is R.

Putting each expression within modulus equal to 0

i.e., $x = 0$ and $x + 1 = 0$ *i.e.,* $x = 0$ and $x = -1$.

Marking these values of x namely -1 and 0 (in proper ascending order) on the number line, domain R of f is divided into three sub-intervals $(-\infty, -1]$, $[-1, 0]$ and $[0, \infty)$.

On the sub-interval $(-\infty, -1]$ *i.e.,* for $x \leq -1$, (say for $x = -2$ etc.)

$x < 0$ and $(x + 1)$ is also < 0 and therefore

$| x | = -x$ and $| x + 1 | = -(x + 1)$

Hence (*i*) becomes $f(x) = | x | - | x + 1 |$

$= -x - (-(x + 1)) = -x + x + 1$

i.e., $f(x) = 1$ for $x \leq -1$ ...(*ii*)

On the sub-interval $[-1, 0]$ *i.e.,*for $-1 \leq x \leq 0$ $\left(\text{say for } x = \frac{-1}{2}\right)$

$x < 0$ and $(x + 1) > 0$ and therefore $| x | - x$ and $| x + 1 | = x + 1$.

Hence (*i*) becomes $f(x) = | x | - | x + 1 |$

$= -x - (x + 1) = -x - x - 1$

$$= -2x - 1$$

i.e., $f(x) = -2x - 1$ for $-1 \le x \le 0$ ...(*iii*)

On the sub-interval $[0, \infty)$ *i.e.,* for $x \ge 0$,

$x \ge 0$ and also $x + 1 > 0$ and therefore

$$|x| = x \text{ and } |x+1| = x+1$$

Hence (*i*) becomes $f(x) = |x| - |x+1| = x - (x+1)$

$$= x - x - 1 = -1$$

i.e., $f(x) = -1$ for $x \ge 0$ ...(*iv*)

From (*ii*), for $x < -1$, $f(x) = 1$ is a constant function and hence is continuous for $x < -1$.

From (*iii*), for $-1 < x < 0$, $f(x) = -2x - 1$ is a polynomial function and hence is continuous for $-1 < x < 0$.

From (*iv*), for $x > 0$, $f(x) = -1$ is a constant function and hence is continuous for $x > 0$.

$\therefore$ f is continuous in $R - \{-1, 0\}$.

Let us examine continuity of *f* at partitioning point *x* = – 1.

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} 1$$ [By (*ii*)]

($\because$ $x \to -1^- \Rightarrow x < -1$)

Putting $x = -1$, $= 1$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (-2x - 1)$$ (By (*iii*))

($\because$ $x \to -1^+ \Rightarrow x > -1$)

Putting $x = -1$, $= -2(-1) - 1 = 2 - 1 = 1$

$$\therefore \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) \; (= 1)$$

$\therefore$ $\lim_{x \to -1} f(x)$ exists and $= 1$.

Putting $x = -1$ in (*ii*) or (*iii*), $f(-1) = 1$

$$\therefore \lim_{x \to -1} f(x) = f(-1) \; (= 1)$$

$\therefore$ f is continuous at $x = -1$ also.

Let us examine continuity of *f* at partitioning point *x* = 0.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-2x - 1)$$ (By (*iii*))

($\because$ $x \to 0^- \Rightarrow x < 0$)

Putting $x = 0$, $= -2(0) - 1 = -1$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (-1)$$ [By (*iv*)]

($\ldots \to 0^+ \Rightarrow x > 0$)

Putting $x = 0$, $= -1$ $\therefore$ $\lim\limits_{x \to 0^-} f(x) = \underset{x \to 0^+}{Lt} f(x)$ $(= -1)$

$\therefore$ $\lim\limits_{x \to 0} f(x)$ exists and $= -1$

Putting $x = 0$ in (*iii*) or (*iv*), $f(0) = -1$

$$\therefore \quad \lim_{x \to 0} f(x) = f(0) \; (= -1)$$

$\therefore$ f is continuous at $x = 0$ also.

$\therefore$ f is continuous on the domain R.

$\therefore$ There is no point of discontinuity.

Second Solution

We know that every modulus function is continuous for all real x.
Therefore $|x|$ and $|x + 1|$ are continuous for all real x.
Also, we know that difference of two continuous functions is continuous.

$\therefore$ $f(x) = |x| - |x + 1|$ is also continuous for all real x.

$\therefore$ There is no point of discontinuity.