NCERT Class 12 Maths Solutions

Chapter - 4

Determinants

Exercise 4.4

Note. Minor (M_{ij}) and Cofactor (A_{ij}) of an element a_{ij} of a determinant Δ are defined **not for the value** of the element but for **(*i*, *j*)th position** of the element.

Def. 1. Minor M_{ij} of an element a_{ij} of a determinant Δ is the determinant obtained by omitting its ith row and jth column in which element a_{ij} lies.

Def. 2. Cofactor A_{ij} of an element a_{ij} of Δ is defined as $A_{ij} = (-1)^{i+j} M_{ij}$ where M_{ij} is the minor of a_{ij}.

1. Write minors and cofactors of the elements of the following determinants:

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ **(ii)** $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Sol. (i) Let $\Delta = \begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

M_{11} = Minor of a_{11} = | 3 | = 3;

$A_{11} = (-1)^{1+1} M_{11} = (-1)^{1+1} (3) = (-1)^2\, 3 = 3$

(Omit first row and first column of Δ)

M_{12} = Minor of a_{12} = | 0 | = 0

$A_{12} = (-1)^{1+2} M_{12} = (-1)^{1+2} (0) = (-1)^3 . 0 = 0$

M_{21} = Minor of a_{21} = | – 4 | = – 4,

$A_{21} = (-1)^{2+1} M_{21} = (-1)^{2+1}(-4) = (-1)^3(-4) = 4$

M_{22} = Minor of a_{22} = | 2 | = 2,

$A_{22} = (-1)^{2+2} M_{22} = (-1)^{2+2}\, 2 = (-1)^4\, 2 = 2$

(ii) Let $\Delta = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

M_{11} = Minor of a_{11} = | d | = d,

$A_{11} = (-1)^{1+1} d = (-1)^2 d = d$

M_{12} = Minor of a_{12} = | b | = b,

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 b = -b$

M_{21} = Minor of a_{21} = | c | = c,

$A_{21} = (-1)^{2+1} c = (-1)^3 c = -c$

M_{22} = Minor of a_{22} = | a | = a,

$A_{22} = (-1)^{2+2} a = (-1)^4 a = a$.

2. Write Minors and Cofactors of the elements of the following determinants:

(i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ **(ii)** $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Sol. (i) Let $\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$\therefore$ M_{11} = Minor of a_{11} = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ = 1 – 0 = 1

$A_{11} = (-1)^{1+1} M_{11} = (-1)^2\, 1 = 1$

M_{12} = Minor of a_{12} = $\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$ = 0 – 0 = 0

(Omitting first row and second column of Δ)

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3\ 0 = 0$

$M_{13} = \text{Minor of } a_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0,$

$A_{13} = (-1)^{1+3} M_{13} = (-1)^4\ 0 = 0$

$M_{21} = \text{Minor of } a_{21} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0,$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3\ 0 = 0$

$M_{22} = \text{Minor of } a_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1,$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4\ 1 = 1$

$M_{23} = \text{Minor of } a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0,$

$A_{23} = (-1)^{2+3} M_{23} = (-1)^5\ 0 = 0$

$M_{31} = \text{Minor of } a_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0,$

$A_{31} = (-1)^{3+1} M_{31} = (-1)^4\ 0 = 0$

$M_{32} = \text{Minor of } a_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0,$

$A_{32} = (-1)^{3+2} M_{32} = (-1)^5\ 0 = 0$

$M_{33} = \text{Minor of } a_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1,$

$A_{33} = (-1)^{3+3} M_{33} = (-1)^6\ 1 = 1.$

(*ii*) Let $\Delta = \begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

$M_{11} = \text{Minor of } a_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 - (-1) = 10 + 1 = 11,$

$A_{11} = (-1)^{1+1} M_{11} = (-1)^2\ 11 = 11$

$M_{12} = \text{Minor of } a_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6,$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3\ 6 = -6$

$M_{13} = \text{Minor of } a_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3,$

$A_{13} = (-1)^{1+3} M_{13} = (-1)^4\ 3 = 3$

M_{21} = Minor of $a_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4,$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$

M_{22} = Minor of $a_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2,$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4\, 2 = 2$

M_{23} = Minor of $a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1,$

$A_{23} = (-1)^{2+3} M_{23} = (-1)^5\, 1 = -1$

M_{31} = Minor of $a_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20,$

$A_{31} = (-1)^{3+1} M_{31} = (-1)^4 (-20) = -20$

M_{32} = Minor of $a_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13,$

$A_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-13) = 13$

M_{33} = Minor of $a_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5,$

$A_{33} = (-1)^{3+3} M_{33} = (-1)^6\, 5 = 5.$

Note. Two Most Important Results

1. Sum of the products of the elements of any row or column of a determinant Δ with their corresponding factors is = Δ.
   i.e., $\boldsymbol{\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}}$ **etc.**
2. Sum of the products of the elements of any row or column of a determinant Δ with the cofactors of any other row or column of Δ is zero.
   For example, $\boldsymbol{a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0}$.

3. Using Cofactors of elements of second row, evaluate

$$\boldsymbol{\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}.}$$

Sol. $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Elements of second row of Δ are $a_{21} = 2,\ a_{22} = 0,\ a_{23} = 1$

A_{21} = Cofactor of $a_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix}$ $[\because\ A_{ij} = (-1)^{i+j} M_{ij}]$

↓ ↓

(determinant obtained by omitting second row and first column of Δ)

$= (-1)^3 (9 - 16) = -(-7) = 7$

$$A_{22} = \text{Cofactor of } a_{22} = (-1)^{2+2}\begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = (-1)^4(15-8) = 7$$

$$A_{23} = \text{Cofactor } a_{23} = (-1)^{2+3}\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = (-1)^5(10-3) = -7$$

Now by Result I of Note after the solution of Q. No. 2,

$$\boldsymbol{\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}}$$
$$= 2(7) + 0(7) + 1(-7) = 14 - 7 = 7.$$

Remark. The above method of finding the value of Δ is equivalent to expanding Δ along second row.

4. Using Cofactors of elements of third column, evaluate

$$\boldsymbol{\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}.}$$

Sol. $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

Here elements of third column of Δ are

$a_{13} = yz$, $a_{23} = zx$, $a_{33} = xy$

$$A_{13} = \text{Cofactor of } a_{13} = (-1)^{1+3}\begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix}$$
$$= (-1)^4(z-y) = z - y$$

↓

(determinant obtained by omitting first row and third column of Δ)

$$A_{23} = \text{Cofactor of } a_{23} = (-1)^{2+3}\begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = (-1)^5(z-x) = -(z-x)$$

$$A_{33} = \text{Cofactor of } a_{33} = (-1)^{3+3}\begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = (-1)^6(y-x) = y - x$$

Now by Result I of Note after the solution of Q. NO. 2,

$$\boldsymbol{\Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}}$$
$$= yz(z-y) + zx[-(z-x)] + xy(y-x)$$
$$= yz^2 - y^2z - z^2x + zx^2 + xy^2 - x^2y$$
$$= (yz^2 - y^2z) + (xy^2 - xz^2) + (zx^2 - x^2y)$$
$$= yz(z-y) + x(y^2 - z^2) - x^2(y-z)$$
$$= -yz(y-z) + x(y+z)(y-z) - x^2(y-z)$$
$$= (y-z)[-yz + xy + xz - x^2]$$
$$= (y-z)[-y(z-x) + x(z-x)]$$
$$= (y-z)(z-x)(-y+x) = (x-y)(y-z)(z-x)$$

Remark. The above method of finding the value of Δ is equivalent to expanding Δ along third column.

5. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ **and** A_{ij} **is Cofactor of** a_{ij}**, then value of** Δ **is given by**

(A) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$

(B) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$

(C) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$

(D) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$.

Sol. Option (D) is correct answer as given in Result I of Note after solution of Q. No. 2 and used in the solution of Q. No. 3 and 4 above.

Remark. The values of expressions given in options (A) and (C) are each equal to zero as given in Result II of Note after solution of Q. No. 2.