NCERT Class 12 Maths Solutions

Chapter - 4

Determinants

Exercise 4.2

Using the properties of determinants and without expanding in Exercises 1 to 5, prove that:

1. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0.$

Sol. On $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$, operate $C_1 \to C_1 + C_2$

$$= \begin{vmatrix} x+a & a & x+a \\ y+b & b & y+b \\ z+c & c & z+c \end{vmatrix} = 0. \quad (\because \; C_1 \text{ and } C_3 \text{ are identical})$$

2. $\begin{vmatrix} \boldsymbol{a-b} & \boldsymbol{b-c} & \boldsymbol{c-a} \\ \boldsymbol{b-c} & \boldsymbol{c-a} & \boldsymbol{a-b} \\ \boldsymbol{c-a} & \boldsymbol{a-b} & \boldsymbol{b-c} \end{vmatrix} = \mathbf{0.}$

Sol. On $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$, operate $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix} = \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0.$$

($\because$ All entries of one column here first are zero)

Note: The reader can do the above problem by operating $R_1 \to R_1 + R_2 + R_3$ also.

3. $\begin{vmatrix} \mathbf{2} & \mathbf{7} & \mathbf{65} \\ \mathbf{3} & \mathbf{8} & \mathbf{75} \\ \mathbf{5} & \mathbf{9} & \mathbf{86} \end{vmatrix} = \mathbf{0.}$

Sol. On $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$, operate $C_3 \to C_3 - C_1$ $= \begin{vmatrix} 2 & 7 & 63 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix}$

Taking 9 common from third column $= 9 \begin{vmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{vmatrix} = 9(0) = 0.$

[Because two columns (one and three) are identical]

4. $\begin{vmatrix} \mathbf{1} & \boldsymbol{bc} & \boldsymbol{a(b+c)} \\ \mathbf{1} & \boldsymbol{ca} & \boldsymbol{b(c+a)} \\ \mathbf{1} & \boldsymbol{ab} & \boldsymbol{c(a+b)} \end{vmatrix} = \mathbf{0}$

Sol. The given determinant is $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = \begin{vmatrix} 1 & bc & ab+ac \\ 1 & ca & bc+ba \\ 1 & ab & ac+bc \end{vmatrix}$

Operate $C_3 \to C_3 + C_2$, $= \begin{vmatrix} 1 & bc & ab+bc+ac \\ 1 & ca & ab+bc+ac \\ 1 & ab & ab+bc+ac \end{vmatrix}$

Taking $(ab + bc + ac)$ common from C_3,

$$= (ab + bc + ac) \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$$

$= (ab + bc + ac)\ 0 = 0.$ $(\because\ C_1$ and C_3 are identical)

5. $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}.$

Sol. L.H.S. $= \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$

Operate $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} b+c+c+a+a+b & q+r+r+p+p+q & y+z+z+x+x+y \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & 2(p+q+r) & 2(x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Taking 2 common from R_1

$$= 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Operate $R_1 \to R_1 - R_2$ (to get single letter entries as required in the determinant on R.H.S.)

$$= 2\begin{vmatrix} b & q & y \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Now operate $R_3 \to R_3 - R_1$ (to get single letter entries as required in the determinant on R.H.S.)

$$= 2\begin{vmatrix} b & q & y \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix}$$

Now operate $R_2 \to R_2 - R_3$ (objective being same as in the above two operations)

$$= 2\begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix}$$

Interchanging R_2 and R_3, $= -2\begin{vmatrix} b & q & y \\ a & p & x \\ c & r & z \end{vmatrix}$

Interchanging R_1 and R_2,

$$= -(-2)\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \text{R.H.S.}$$

By using properties of determinants, in Exercise 6 to 14, show that:

6. $$\begin{vmatrix} \mathbf{0} & \boldsymbol{a} & -\boldsymbol{b} \\ -\boldsymbol{a} & \mathbf{0} & -\boldsymbol{c} \\ \boldsymbol{b} & \boldsymbol{c} & \mathbf{0} \end{vmatrix} = \mathbf{0}.$$

Sol. Let $\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$ ...(*i*)

Taking (–1) common from each row, we have

$$\Delta = (-1)^3\begin{vmatrix} 0 & -a & b \\ a & 0 & c \\ -b & -c & 0 \end{vmatrix}$$

Interchanging rows and columns in the determinant on R.H.S.,

$$\Delta = -\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} \qquad (\because \ (-1)^3 = -1)$$

$\Rightarrow \quad \Delta = -\Delta$ (By (*i*))

Shifting $-\Delta$ from R.H.S. to L.H.S., $\Delta + \Delta = 0$ or $2\Delta = 0$

$\therefore \quad \Delta = \frac{0}{2} = 0.$

Note. 1. We can also do this question by taking (– 1) common from each column.

2. When you are asked to prove that a determinant is equal to zero or two determinants are equal, then it is to be proved so only without expanding.

3. It may be remarked that the determinant of Q. No. 6 above is determinant of *a* skew symmetric matrix of order 3.

7. $$\begin{vmatrix} -\boldsymbol{a}^2 & \boldsymbol{ab} & \boldsymbol{ac} \\ \boldsymbol{ba} & -\boldsymbol{b}^2 & \boldsymbol{bc} \\ \boldsymbol{ca} & \boldsymbol{cb} & -\boldsymbol{c}^2 \end{vmatrix} = \mathbf{4}\boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{c}^2.$$

Sol. L.H.S. $= \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$

Taking a, b, c common from R_1, R_2, R_3 respectively,

$$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Operate $R_1 \to R_1 + R_2$ (to create two zeros in a line (here first row))

$$= abc \begin{vmatrix} 0 & 0 & 2c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Expanding along first row ($\because$ There are two zeros in it)

$$= abc \;.\; 2c \begin{vmatrix} a & -b \\ a & b \end{vmatrix} = abc \;.\; 2c\,(ab + ab)$$

$$= abc \;.\; 2c \;.\; 2ab = 4a^2b^2c^2 = \text{R.H.S.}$$

Note. Whenever we are asked to find the value of a determinant by using "Properties of Determinants", we must create two zeros in a line (Row or Column).

8. (*i*) $\begin{vmatrix} \mathbf{1} & \boldsymbol{a} & \boldsymbol{a^2} \\ \mathbf{1} & \boldsymbol{b} & \boldsymbol{b^2} \\ \mathbf{1} & \boldsymbol{c} & \boldsymbol{c^2} \end{vmatrix} = \boldsymbol{(a - b)(b - c)(c - a)}$

(*ii*) $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{a^3} & \boldsymbol{b^3} & \boldsymbol{c^3} \end{vmatrix} = \boldsymbol{(a - b)(b - c)(c - a)(a + b + c)}.$

Sol. (*i*) L.H.S. $= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

Expanding along first column

$$= 1 \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} (b-a) & (b-a)(b+a) \\ (c-a) & (c-a)(c+a) \end{vmatrix}$$

Taking out $(b - a)$ common from first row and $(c - a)$ common from second row

$$= (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$= (b-a)(c-a)(c+a-b-a) = (b-a)(c-a)(c-b)$
$= -(a-b)(c-a)(-(b-c)) = (a-b)(b-c)(c-a).$

Remark. For expanding a determinant of order 3 we should make all entries except one entry of a row or column as zeros (*i.e.,* we should make two entries as zeros) and then expand the determinant along this row or column. For doing so, the ideal situation is that all entries of a row or column are 1 each.

If each entry of a **column** is 1, then, to create two zeros, subtract first **row** from each of the remaining two **rows.**

If each entry of a **row** is 1, then to create two zeros, subtract first **column** from each of the remaining two columns.

(*ii*) L.H.S. $= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

Here all entries of a row are 1 each.

So operate $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ (to creat two zeros in a line (here first row))

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$$

Expanding along first row, $= 1\begin{vmatrix} b-a & c-a \\ b^3-a^3 & c^3-a^3 \end{vmatrix}$

(Forming factors)

$$= \begin{vmatrix} (b-a) & (c-a) \\ (b-a)(b^2+a^2+ab) & (c-a)(c^2+a^2+ac) \end{vmatrix}$$

Taking $(b-a)$ common from C_1 and $(c-a)$ common from C_2,

$$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b^2+a^2+ab & c^2+a^2+ac \end{vmatrix}$$

$= (b-a)(c-a)\ (c^2+a^2+ac-b^2-a^2-ab)$
$= (b-a)(c-a)(c^2-b^2+ac-ab)$
$= (b-a)(c-a)\ [(c-b)(c+b)+a(c-b)]$
$= (b-a)(c-a)(c-b)(c+b+a)$
$= -(a-b(c-a)\ [-(b-c)]\ (a+b+c)$
$= (a-b)(b-c)(c-a)(a+b+c) =$ R.H.S.

9. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx).$

Sol. L.H.S. = $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$

Multiplying R_1, R_2, R_3 by x, y, z respectively (to make each entry of third column same here (xyz))

$$= \frac{1}{xyz}\begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix}$$

Taking xyz common from C_3, $= \dfrac{xyz}{xyz}\begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix} = \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix}$

Now all entries of a **column** are same. So operate $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ to create two zeros in a column.

$$= \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 - x^2 & y^3 - x^3 & 0 \\ z^2 - x^2 & z^3 - x^3 & 0 \end{vmatrix}$$

Expanding along third column $= 1\begin{vmatrix} y^2 - x^2 & y^3 - x^3 \\ z^2 - x^2 & z^3 - x^3 \end{vmatrix}$

(Forming factors) $= \begin{vmatrix} (y-x)(y+x) & (y-x)(y^2+x^2+xy) \\ (z-x)(z+x) & (z-x)(z^2+x^2+zx) \end{vmatrix}$

Taking $(y - x)$ common from R_1 and $(z - x)$ common from R_2

$= (y - x)(z - x)\begin{vmatrix} y+x & y^2+x^2+xy \\ z+x & z^2+x^2+zx \end{vmatrix}$

$= (y - x)(z - x)\,[(y + x)(z^2 + x^2 + zx) - (z + x)(y^2 + x^2 + xy)]$

$= (y - x)(z - x)\,[yz^2 + yx^2 + xyz + xz^2 + x^3 + x^2z$
$\qquad - zy^2 - zx^2 - xyz - xy^2 - x^3 - x^2y]$

$= (y - x)(z - x)\,[yz^2 - zy^2 + xz^2 - xy^2]$

$= (y - x)(z - x)\,\mathbf{[yz(z - y) + x(z^2 - y^2)]}$

$= (y - x)(z - x)\,[yz(z - y) + x(z - y)(z + y)]$

$= (y - x)(z - x)(z - y)\,[yz + x(z + y)]$

$= -(x - y)(z - x)\,[-(y - z)]\,(yz + xz + xy)$

$= (x - y)(y - z)(z - x)(xy + yz + zx) =$ R.H.S.

10. **(*i*)** $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x + 4)(4 - x)^2.$

(*ii*) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y + k).$

Sol. (*i*) L.H.S. $= \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

Here sum of entries of each **column** is same (= $5x + 4$), so let us operate $\mathbf{R_1 \to R_1 + R_2 + R_3}$ to make all entries of first row equal (= $5x + 4$).

$$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Taking $(5x + 4)$ common from R_1,

$$= (5x + 4) \begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Now each entry of one (here first) row is 1, so let us operate $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ to create two zeros in a zero.

$$= (5x + 4) \begin{vmatrix} 1 & 0 & 0 \\ 2x & 4-x & 0 \\ 2x & 0 & 4-x \end{vmatrix}$$

Expanding along first row

$$= (5x + 4) \,.\, 1 \begin{vmatrix} 4-x & 0 \\ 0 & 4-x \end{vmatrix}$$

$$= (5x + 4)(4 - x)^2 = \text{R.H.S.}$$

Remark. We could also start here by operating $C_1 \to C_1 + C_2 + C_3$.

(*ii*) L.H.S. $= \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$

Here sum of entries of each row is same (= $3y + k$), so let us operate $\mathbf{C_1 \to C_1 + C_2 + C_3}$ to make all entries of first column equal (= $3y + k$)

$$= \begin{vmatrix} 3y+k & y & y \\ 3y+k & y+k & y \\ 3y+k & y & y+k \end{vmatrix}$$

Taking $(3y + k)$ common from C_1,

$$= (3y + k) \begin{vmatrix} 1 & y & y \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix}$$

Now each entry of one (here first) column is 1, so let us operate $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ to create two zeros in a column,

$$= (3y + k) \begin{vmatrix} 1 & y & y \\ 0 & k & 0 \\ 0 & 0 & k \end{vmatrix}$$

Expanding along first column, $= (3y + k) \cdot 1 \begin{vmatrix} k & 0 \\ 0 & k \end{vmatrix}$

$= (3y + k)k^2 = k^2(3y + k) =$ R.H.S.

Remark. We could also start here by operating

$R_1 \to R_1 + R_2 + R_3$.

11. ***(i)*** $\begin{vmatrix} \boldsymbol{a-b-c} & \boldsymbol{2a} & \boldsymbol{2a} \\ \boldsymbol{2b} & \boldsymbol{b-c-a} & \boldsymbol{2b} \\ \boldsymbol{2c} & \boldsymbol{2c} & \boldsymbol{c-a-b} \end{vmatrix} = \boldsymbol{(a + b + c)^3}$.

(ii) $\begin{vmatrix} \boldsymbol{x+y+2z} & \boldsymbol{x} & \boldsymbol{y} \\ \boldsymbol{z} & \boldsymbol{y+z+2x} & \boldsymbol{y} \\ \boldsymbol{z} & \boldsymbol{x} & \boldsymbol{z+x+2y} \end{vmatrix} = \boldsymbol{2(x + y + z)^3}$.

Sol. (*i*) L.H.S. $= \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Here sum of entries of each **column** is same $(= a + b + c)$, so let us operate $\mathbf{R_1 \to R_1 + R_2 + R_3}$ to make all entries of first row equal $(= a + b + c)$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking $(a + b + c)$ common from R_1,

$$= (a + b + c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Now each entry of one (here first) row is 1, so let us operate $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ to create two zeros in a row,

$$= (a + b + c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

Expanding along first row

$$= (a + b + c) \,.\, 1 \,.\, \begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix}$$

$$= (a + b + c) [(- b - c - a) (- c - a - b)]$$
$$= (a + b + c) (-) (b + c + a) (-) (c + a + b)$$
$$= (a + b + c)^3 = \text{R.H.S.}$$

Remark. Here we can't operate $C_1 \to C_1 + C_2 + C_3$ because sum of entries of each row is not same.

(*ii*) $$\text{L.H.S.} = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

Here sum of entries of each **row** is same ($= 2x + 2y + 2z = 2(x + y + z)$), so let us operate $\mathbf{C_1 \to C_1 + C_2 + C_3}$ to make all entries of first column equal ($= 2(x + y + z)$)

$$= \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$$

Taking $2(x + y + z)$ common from C_1,

$$= 2(x + y + z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

Now each entry of one (here first) column is 1, so let us operate $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ to create two zeros in a column

$$= 2(x + y + z) \begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$

Expanding along first column

$$= 2(x + y + z) \,.\, 1 \,.\, \begin{vmatrix} x+y+z & 0 \\ 0 & x+y+z \end{vmatrix}$$

$$= 2(x + y + z) [(x + y + z)^2 - 0]$$
$$= 2(x + y + z)^3 = \text{R.H.S.}$$

12. $$\begin{vmatrix} \mathbf{1} & \boldsymbol{x} & \boldsymbol{x^2} \\ \boldsymbol{x^2} & \mathbf{1} & \boldsymbol{x} \\ \boldsymbol{x} & \boldsymbol{x^2} & \mathbf{1} \end{vmatrix} = \mathbf{(1 - x^3)^2}.$$

Sol. L.H.S. $= \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$

Here sum of entries of each **column** is same $(= 1 + x + x^2)$, so let us operate $\mathbf{R_1 \to R_1 + R_2 + R_3}$ to make all entries of first **row** equal $(= 1 + x + x^2)$

$$= \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Taking $(1 + x + x^2)$ common from R_1,

$$= (1 + x + x^2) \begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Now each entry of one (here first) row is 1, so let us operate $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ to create two zeros in a row.

$$= (1 + x + x^2) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix}$$

Expanding along first row

$$= (1 + x + x^2) \cdot 1 \begin{vmatrix} 1-x^2 & x-x^2 \\ x^2-x & 1-x \end{vmatrix}$$

$$= (1 + x + x^2) \begin{vmatrix} (1-x)(1+x) & x(1-x) \\ -x(1-x) & (1-x) \end{vmatrix}$$

$= (1 + x + x^2) [(1 - x)^2 (1 + x) + x^2(1 - x)^2]$

$= (1 + x + x^2) (1 - x)^2 (1 + x + x^2) = (1 + x + x^2)^2 (1 - x)^2$

$= [(1 + x + x^2) (1 - x)]^2$ $\qquad (\because \ A^2B^2 = (AB)^2)$

$= (1 - x + x - x^2 + x^2 - x^3)^2 = (1 - x^3)^2 =$ R.H.S.

Remark. For the above question, we could also operate $C_1 \to C_1 + C_2 + C_3$ because sum of entries of each row is also same and $(= 1 + x + x^2)$.

13. $\begin{vmatrix} \mathbf{1+a^2-b^2} & \mathbf{2ab} & \mathbf{-2b} \\ \mathbf{2ab} & \mathbf{1-a^2+b^2} & \mathbf{2a} \\ \mathbf{2b} & \mathbf{-2a} & \mathbf{1-a^2-b^2} \end{vmatrix} = \mathbf{(1 + a^2 + b^2)^3}$**.**

Sol. Operating $C_1 \to C_1 - b\, C_3, C_2 \to C_2 + a\, C_3$ in L.H.S. as suggested by the factor $(1 + a^2 + b^2)$ in R.H.S.

$$\text{L.H.S.} = \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

[$\because$ $2b - b(1 - a^2 - b^2) = 2b - b + a^2b + b^3$
$= b + a^2b + b^3 = b(1 + a^2 + b^2)$]

Taking out $(1 + a^2 + b^2)$ common from C_1 and C_2

$$= (1 + a^2 + b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

Operating $R_3 \to R_3 - bR_1$ (to create another zero in first column)

$$= (1 + a^2 + b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & -a & 1-a^2+b^2 \end{vmatrix}$$

Expanding along C_1

$$= (1 + a^2 + b^2)^2 \,.\, 1 \begin{vmatrix} 1 & 2a \\ -a & 1-a^2+b^2 \end{vmatrix}$$

$$= (1 + a^2 + b^2)^2 \,(1 - a^2 + b^2 + 2a^2) = (1 + a^2 + b^2)^3.$$

14. $$\begin{vmatrix} \boldsymbol{a^2+1} & \boldsymbol{ab} & \boldsymbol{ac} \\ \boldsymbol{ab} & \boldsymbol{b^2+1} & \boldsymbol{bc} \\ \boldsymbol{ca} & \boldsymbol{cb} & \boldsymbol{c^2+1} \end{vmatrix} = \boldsymbol{1 + a^2 + b^2 + c^2}.$$

Sol. Multiplying C_1, C_2, C_3 by a, b, c respectively and in return dividing the determinant by abc,

$$\Delta = \frac{1}{abc} \begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix}$$

Taking out a, b, c common from R_1, R_2, R_3 respectively,

$$= \frac{abc}{abc} \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$ Operating $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

Taking out $(1 + a^2 + b^2 + c^2)$ common from C_1,

$$= (1 + a^2 + b^2 + c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$,

$$= (1 + a^2 + b^2 + c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along first column,

$$= (1 + a^2 + b^2 + c^2) \times 1\ (1 - 0) = 1 + a^2 + b^2 + c^2.$$

Choose the correct answer in Exercises 15 and 16:

15. Let A be a square matrix of order 3 × 3, then | kA | is equal to

(A) k | A | **(B) k^2 | A |**

(C) k^3 | A | **(D) $3k$ | A |.**

Sol. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3×3. ...(*i*)

$\therefore$ By definition of scalar multiplication of a matrix,

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix} \quad \therefore \ | kA | = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix}$$

Taking k common from each row,

$$= k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$= k^3$ | A | [By (*i*)]

Remark. In general, if A is a square matrix of order $n \times n$; then we can prove that **| kA | = k^n | A |.**

$\therefore$ Option (C) is the correct answer.

16. Which of the following is correct:

(A) Determinant is a square matrix.

(B) Determinant is a number associated to a matrix.

(C) Determinant is a number associated to a square matrix.

(D) None of these.

Sol. Option (C) is the correct answer.

i.e., Determinant is a number associated to a square matrix.