Exercise 3.4

Using elementary transformations, find the inverse of each of the matrices, if it exists in Exercises 1 to 6.

1. $\begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$.

Sol. Let A = $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

We shall find A^{-1}, if it exists; by elementary **(Row)** transformations (only)

So we must write **A = IA only and not A = AI**

$$\therefore \qquad \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

(Here I is I_2 because A is 2 × 2)

We shall reduce the matrix on left side to I_2.

Here a_{11} = 1

Operate $R_2 \to R_2 - 2R_1$ to make a_{21} = 0

$$\begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

$$\begin{array}{rrr} R_2 \to & 2 & 3 \\ 2R_1 \to & 2 & -2 \\ - & - & + \\ \hline \therefore \quad R_2 - 2R_1 = & 0 & 5 \\ R_2 \to & 0 & 1 \\ 2R_1 \to & 2 & 0 \\ - & - & - \\ \hline \therefore \quad R_2 - 2R_1 = & -2 & 1 \end{array}$$

Operate $R_2 \to \frac{1}{5} R_2$ to make a_{22} = 1

$$\therefore \qquad \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{-2}{5} & \frac{1}{5} \end{bmatrix} A$$

Now operate $R_1 \to R_1 + R_2$ to make a_{12} = 0

$$\Rightarrow \begin{bmatrix} 1+0 & -1+1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\frac{2}{5} & 0+\frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-2}{5} & \frac{1}{5} \end{bmatrix} A$$

$\therefore$ By definition of inverse of a matrix, $A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-2}{5} & \frac{1}{5} \end{bmatrix}$

Note. Any row operation done on left hand side matrix must also be done on the prefactor I_2 of right hand side matrix.

Note. Definition of inverse of a square matrix. A square matrix B is said to be inverse of a square matrix A if **AB = I** and **BA = I**. Then $B = A^{-1}$.

Remark. If the student is interested in finding A^{-1} by elementary column transformations, then he or she should start with A = AI and apply only column operations.

2. $\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

We know that $A = I_2A \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Operate $R_1 \leftrightarrow R_2$ (to make $a_{11} = 1$)

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 2R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2-2 & 1-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1-0 & 0-2 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} A$$

Operate $R_2 \to (-1) R_2$ (to make $a_{22} = 1$)

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} A$$

Operate $R_1 \to R_1 - R_2$ (to make $a_{12} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} A$$

$\therefore$ By definition of inverse of a square matrix, $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

3. $\begin{bmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{2} & \mathbf{7} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

We know that $A = I_2A \Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Here $a_{11} = 1$. To make $a_{21} = 0$, let us operate $R_2 \to R_2 - 2R_1$.

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

$R_2 \to$	2	7
$2R_1 \to$	2	6
$-$	$-$	$-$
$\therefore R_2 - 2R_1 =$	0	1
$R_2 \to$	0	1
$2R_1 \to$	2	0
$-$	$-$	$-$
$\therefore R_2 - 2R_1 =$	-2	1

Now $a_{22} = 1$. To make a_{12} as zero, operate $R_1 \to R_1 - 3R_2$.

$$\Rightarrow \begin{bmatrix} 1-0 & 3-3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+6 & 0-3 \\ -2 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} A$$

$\therefore$ By definition, $A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$.

4. $\begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{5} & \mathbf{7} \end{bmatrix}$

Sol. Set $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$

We know that $A = I_2A \Rightarrow \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Let us try to make $a_{11} = 1$. Operate $R_2 \to R_2 - 2R_1$

$$\Rightarrow \begin{bmatrix} 2 & 3 \\ 5-4 & 7-6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0-2 & 1-0 \end{bmatrix} A \quad \Rightarrow \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

Now operate $R_1 \leftrightarrow R_2$ to make $a_{11} = 1$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} A$$

Operate $R_2 \leftrightarrow R_2 - 2R_1$ to make $a_{21} = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2-2 & 3-2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1+4 & 0-2 \end{bmatrix} A \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix} A$$

Operate $R_1 \to R_1 - R_2$ to make $a_{12} = 0$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} -2-5 & 1+2 \\ 5 & -2 \end{bmatrix}$$

$$\Rightarrow \quad I_2 = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} A \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$$

Remark. In the above solution to make $a_{11} = 1$, we could also operate $R_1 \to \frac{1}{2} R_1$. But for the sake of convenience and to avoid lengthy computations, we should avoid multiplying by fractions.

5. $\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{7} & \mathbf{4} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$

We know that $A = I_2A \quad \Rightarrow \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Let us try to make $a_{11} = 1$. Operate $R_2 \to R_2 - 3R_1$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 7-6 & 4-3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0-3 & 1-0 \end{bmatrix} A \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} A$$

Operate $R_1 \to R_1 - R_2$ to make $a_{11} = 1$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} A$$

Now Operate $R_2 \to R_2 - R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} A$$

Now $a_{12} = 0$ and $a_{22} = 1$.

or $\quad I_2 = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} A$

$\therefore$ By definition of inverse of a square matrix, $A^{-1} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$.

6. $\begin{bmatrix} \mathbf{2} & \mathbf{5} \\ \mathbf{1} & \mathbf{3} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

We know that A = I_2A $\Rightarrow$ $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}A$

Operate $R_1 \leftrightarrow R_2$ to make $a_{11} = 1$;

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}A$$

Operate $R_2 \to R_2 - 2R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2-2 & 5-6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1-0 & 0-2 \end{bmatrix}A$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}A$$

Operate $R_2 \to (-1)R_2$ to make $a_{22} = 1$;

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}A$$

Operate $R_1 \to R_1 - 3R_2$ (to make $a_{12} = 0$)

$$\Rightarrow \begin{bmatrix} 1-0 & 3-3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0+3 & 1-6 \\ -1 & 2 \end{bmatrix}A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}A$$

$\therefore$ By Definition, $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Using elementary transformations, find the inverse of each of the matrices, if it exists, in Exercises 7 to 14.

7. $\begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{5} & \mathbf{2} \end{bmatrix}$.

Sol. Let A = $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

We know that A = I_2A $\Rightarrow$ $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}A$

Let us try to make $a_{11} = 1$.

Operate $R_1 \to 2R_1$ $\Rightarrow$ $\begin{bmatrix} 6 & 2 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}A$

Operate $R_1 \to R_1 - R_2$ (to make $a_{11} = 1$)

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}A$$

Operate $R_2 \to R_2 - 5R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0-10 & 1+5 \end{bmatrix}A \quad \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -10 & 6 \end{bmatrix}A$$

Operate $R_2 \to \frac{1}{2} R_2$ (to make $a_{22} = 1$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} A$$

Now a_{12} has already become zero. Therefore,

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}.$$

8. $\begin{bmatrix} \mathbf{4} & \mathbf{5} \\ \mathbf{3} & \mathbf{4} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$

We know that $A = I_2A \quad \Rightarrow \quad \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Operate $R_1 \to R_1 - R_2$ (to make $a_{11} = 1$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 3R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 3-3 & 4-3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0-3 & 1+3 \end{bmatrix} A \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} A$$

Now a_{22} has already become 1.

Operate $R_1 \to R_1 - R_2$ (to make $a_{12} = 0$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} 1+3 & -1-4 \\ -3 & 4 \end{bmatrix} A$$

$$\Rightarrow \quad I_2 = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} A. \text{ Therefore, } A^{-1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}.$$

9. $\begin{bmatrix} \mathbf{3} & \mathbf{10} \\ \mathbf{2} & \mathbf{7} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$

We know that $A = I_2A \quad \Rightarrow \quad \begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Operate $R_1 \to R_1 - R_2$ (to make $a_{11} = 1$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 2R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2-2 & 7-6 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0-2 & 1+2 \end{bmatrix} A \Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} A$$

Now $a_{22} = 1$. Operate $R_1 \to R_1 - 3R_2$ (to make $a_{12} = 0$)

$$\Rightarrow \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+6 & -1-9 \\ -2 & 3 \end{bmatrix} A$$

$$\Rightarrow \qquad I_2 = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix} A \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$$

10. $\begin{bmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{-4} & \mathbf{2} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$

We know that $A = I_2 A \Rightarrow \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Let us try to make $a_{11} = 1$

Operate $R_1 \to R_1 + R_2$.

$$\Rightarrow \begin{bmatrix} 3-4 & -1+2 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+1 \\ 0 & 1 \end{bmatrix} A \Rightarrow \begin{bmatrix} -1 & 1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A$$

Operate $R_1 \to (-1)\, R_1$

$$\Rightarrow \qquad \begin{bmatrix} 1 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 + 4R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \qquad \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -4 & -3 \end{bmatrix} A$$

Operate $R_2 \to \left(-\frac{1}{2}\right) R_2$ (to make $a_{22} = 1$)

$$\Rightarrow \qquad \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & \frac{3}{2} \end{bmatrix} A$$

Operate $R_1 \to R_1 + R_2$ (to make $a_{12} = 0$)

$$\Rightarrow \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix} A$$

$\therefore$ By definition of inverse of a matrix; $A^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$.

11. $\begin{bmatrix} \mathbf{2} & \mathbf{-6} \\ \mathbf{1} & \mathbf{-2} \end{bmatrix}$.

Sol. Let A = $\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

We know that A = I_2A $\Rightarrow$ $\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$A

Operate $R_1 \leftrightarrow R_2$ (to make a_{11} = 1)

$$\Rightarrow \quad \begin{bmatrix} 1 & -2 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 2R_1$ (to make a_{21} = 0)

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 2-2 & -6+4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1-0 & 0-2 \end{bmatrix} A \quad \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} A$$

Operate $R_2 \to \left(-\frac{1}{2}\right) R_2$ (to make a_{22} = 1)

$$\Rightarrow \quad \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & 1 \end{bmatrix} A$$

Operate $R_1 \to R_1 + 2R_2$ (to make a_{12} = 0)

$$\Rightarrow \quad \begin{bmatrix} 1+0 & -2+2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0-1 & 1+2 \\ \frac{-1}{2} & 1 \end{bmatrix} A$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} A \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}.$$

12. $\begin{bmatrix} \mathbf{6} & \mathbf{-3} \\ \mathbf{-2} & \mathbf{1} \end{bmatrix}$.

Sol. Let A = $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$.

Here, A is a 2 × 2 matrix. So, we start with A = I_2 A

$$\text{or} \quad \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Operating $R_1 \to 1/6\ R_1$ to make a_{11} = 1,

$$\text{we have} \quad \begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & 1 \end{bmatrix} A$$

Operating $R_2 \to R_2 + 2R_1$ to make non-diagonal entry a_{21} below a_{11} as zero,

$$\text{we have} \quad \begin{bmatrix} 1 & \frac{-1}{2} \\ -2+2 & 1-\frac{2}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0+\frac{2}{6} & 1+0 \end{bmatrix} A$$

or $$\begin{bmatrix} 1 & \frac{-1}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ \frac{1}{3} & 1 \end{bmatrix} A$$

Here, all entries in second row of left side matrix are zero.

$\therefore$ A^{-1} does not exist.

Note. If after doing one or more elementary row operations, we obtain all 0's in one or more rows of the left hand matrix A, then A^{-1} does not exist and we say A is not invertible.

13. $\begin{bmatrix} \mathbf{2} & \mathbf{-3} \\ \mathbf{-1} & \mathbf{2} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

We know that $A = I_2A \Rightarrow \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Operate $R_1 \to R_1 + R_2$ (to make $a_{11} = 1$)

$$\Rightarrow \begin{bmatrix} 2-1 & -3+2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+1 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 + R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} A$$

Now $a_{22} = 1$. Operate $R_1 \to R_1 + R_2$ (to make $a_{12} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} A$$

$\therefore$ By definition; $A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.

14. $\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{4} & \mathbf{2} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

We know that $A = I_2A \Rightarrow \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Operate $R_1 \to \frac{1}{2} R_1$ (to make $a_{11} = 1$)

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 4R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 4-4 & 2-2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0-2 & 1-0 \end{bmatrix} A \Rightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -2 & 1 \end{bmatrix} A$$

Here one row (namely second row) of the matrix on L.H.S. contains zeros only.

Hence, A^{-1} does not exist.

Using elementary transformations, find the inverse of each of the matrices, if it exists, in Exercises 15 to 17.

15. $\begin{bmatrix} \mathbf{2} & \mathbf{-3} & \mathbf{3} \\ \mathbf{2} & \mathbf{2} & \mathbf{3} \\ \mathbf{3} & \mathbf{-2} & \mathbf{2} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

We know that $A = I_3A$ (we have taken I_3 because matrix A is of order 3×3)

$$\Rightarrow \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Let us try to make $a_{11} = 1$

Operate $R_1 \to R_1 - R_3$

$$\Rightarrow \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_1 \to (-1) R_1$ to make $a_{11} = 1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$ (to make $a_{21} = 0$ and $a_{31} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 2-2 & 2-2 & 3+2 \\ 3-3 & -2-3 & 2+3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0+2 & 1-0 & 0-2 \\ 0+3 & 0-0 & 1-3 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 5 \\ 0 & -5 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 0 & -2 \end{bmatrix} A$$

Operate $R_2 \leftrightarrow R_3$ (to make a_{22} non-zero)

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 5 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -2 \\ 2 & 1 & -2 \end{bmatrix} A$$

Operate $R_2 \to \left(-\frac{1}{5}\right) R_2$ to make $a_{22} = 1$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -\frac{3}{5} & 0 & \frac{2}{5} \\ 2 & 1 & -2 \end{bmatrix} A$$

Operate $R_1 \to R_1 - R_2$ (to make $a_{12} = 0$). Here a_{32} is already zero.

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -1+\frac{3}{5} & 0-0 & 1-\frac{2}{5} \\ -\frac{3}{5} & 0 & \frac{2}{5} \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 0 & \frac{2}{5} \\ 2 & 1 & -2 \end{bmatrix} A$$

Operate $R_3 \to \frac{1}{5} R_3$ (to make $a_{33} = 1$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix} A$$

Operate $R_2 \to R_2 + R_3$ (to make $a_{23} = 0$). Here a_{13} is already zero.

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (= I_3) = \begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix} A$$

$$\therefore \quad \text{By definition} \quad A^{-1} = \begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix}.$$

16. $\begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{-2} \\ \mathbf{-3} & \mathbf{0} & \mathbf{-5} \\ \mathbf{2} & \mathbf{5} & \mathbf{0} \end{bmatrix}$.

Sol. Let A = $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$

We know that A = I_3A

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Here a_{11} is already 1.

Operate $R_2 \to R_2 + 3R_1$ and $R_3 \to R_3 - 2R_1$ (to make $a_{21} = 0$ and $a_{31} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ -3+3 & 0+9 & -5-6 \\ 2-2 & 5-6 & 0+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0+3 & 1+0 & 0+0 \\ 0-2 & 0-0 & 1-0 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -11 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 \leftrightarrow R_3$ to make a_{22} simpler entry

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & -1 & 4 \\ 0 & 9 & -11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} A$$

Operate $R_2 \to (-1) R_2$ to make $a_{22} = 1$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -4 \\ 0 & 9 & -11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix} A$$

Operate $R_1 \to R_1 - 3R_2$ to make $a_{12} = 0$ and $R_3 \to R_3 - 9R_2$ (to make $a_{32} = 0$)

$$\Rightarrow \begin{bmatrix} 1-0 & 3-3 & -2+12 \\ 0 & 1 & -4 \\ 0 & 9-9 & -11+36 \end{bmatrix} = \begin{bmatrix} 1-6 & 0-0 & 0+3 \\ 2 & 0 & -1 \\ 3-18 & 1-0 & 0+9 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -4 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ 2 & 0 & -1 \\ -15 & 1 & 9 \end{bmatrix} A$$

Operate $R_3 \to \dfrac{1}{25} R_3$ to make $a_{33} = 1$.

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ 2 & 0 & -1 \\ -\frac{15}{25} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A$$

Operate $R_1 \to R_1 - 10R_3$, (to make $a_{13} = 0$) and $R_2 \to R_2 + 4R_3$ (to make $a_{23} = 0$).

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (= I_3) = \begin{bmatrix} -5 + \frac{150}{25} & 0 - \frac{10}{25} & 3 - \frac{90}{25} \\ 2 - \frac{60}{25} & 0 + \frac{4}{25} & -1 + \frac{36}{25} \\ \frac{-15}{25} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A$$

$$\Rightarrow \quad I_3 = \begin{bmatrix} 1 & \frac{-2}{5} & \frac{-3}{5} \\ \frac{-2}{5} & \frac{4}{25} & \frac{11}{25} \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A$$

$$\therefore \quad \text{By Definition,} \quad A^{-1} = \begin{bmatrix} 1 & \frac{-2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix}.$$

17. $\begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{5} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{3} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

We know that $A = I_3 \; A \Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Let us try to make $a_{11} = 1$

Operate $R_2 \to R_2 - 2R_1$

$$\Rightarrow \quad \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_1 \leftrightarrow R_2$ (to make $a_{11} = 1$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 2R_1$ to make $a_{21} = 0$. Here a_{31} is already 0

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 5 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 \leftrightarrow R_3$ (to make $a_{22} = 1$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -5 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -2 & 0 \end{bmatrix} A$$

Operate $R_1 \to R_1 - R_2$ to make $a_{12} = 0$ and $R_3 \to R_3 + 2R_2$ to make $a_{32} = 0$.

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 1 \\ 5 & -2 & 2 \end{bmatrix} A$$

Now $a_{33} = 1$

Operate $R_1 \to R_1 + R_3$ (to make $a_{13} = 0$) and $R_2 \to R_2 - 3R_3$ (to make $a_{23} = 0$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (= I_3) = \begin{bmatrix} -2+5 & 1-2 & -1+2 \\ 0-15 & 0+6 & 1-6 \\ 5 & -2 & 2 \end{bmatrix} A$$

$$\text{or} \quad I_3 = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$

$$\therefore \quad \text{By definition,} \quad A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}.$$

18. Matrices A and B will be inverse of each other only if

(A) AB = BA **(B) AB = BA = 0**

(C) AB = 0, BA = I **(D) AB = BA = I.**

Sol. Option (D) *i.e.,* **AB = BA = I** is correct answer by definition of inverse of a square matrix.