Exercise 3.2

1. Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$.

Find each of the following:

(*i*) A + B **(*ii*) A – B** **(*iii*) 3A – C**

(*iv*) AB **(*v*) BA.**

Sol. (*i*) $A + B = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2+1 & 4+3 \\ 3-2 & 2+5 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}$

(*ii*) $A - B = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2-1 & 4-3 \\ 3+2 & 2-5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}$

(*iii*) $3A - C = 3\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} - C = \begin{bmatrix} 3\times 2 & 3\times 4 \\ 3\times 3 & 3\times 2 \end{bmatrix} - C$

$$= \begin{bmatrix} 6 & 12 \\ 9 & 6 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6+2 & 12-5 \\ 9-3 & 6-4 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$$

(*iv*) $AB = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$

Performing row by column multiplication,

$$= \begin{bmatrix} 2(1)+4(-2) & 2(3)+4(5) \\ 3(1)+2(-2) & 3(3)+2(5) \end{bmatrix} = \begin{bmatrix} 2-8 & 6+20 \\ 3-4 & 9+10 \end{bmatrix} = \begin{bmatrix} -6 & 26 \\ -1 & 19 \end{bmatrix}$$

(*v*) $BA = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$

Performing row by column multiplication,

$$= \begin{bmatrix} 1(2)+3(3) & 1(4)+3(2) \\ (-2)2+5(3) & (-2)(4)+5(2) \end{bmatrix} = \begin{bmatrix} 2+9 & 4+6 \\ -4+15 & -8+10 \end{bmatrix} = \begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}$$

Note. From solutions of part (*iv*) and (*v*), we can easily observe that AB need not be equal to BA *i.e.,* matrix multiplication need not be commutative.

2. Compute the following:

(*i*) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

(*ii*) $\begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$

(*iii*) $\begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$

(iv) $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$.

Sol. (*i*) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a+a & b+b \\ -b+b & a+a \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & 2a \end{bmatrix}$

(*ii*) $\begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$

$= \begin{bmatrix} a^2+b^2+2ab & b^2+c^2+2bc \\ a^2+c^2-2ac & a^2+b^2-2ab \end{bmatrix} = \begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}$

(*iii*) $\begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} -1+12 & 4+7 & -6+6 \\ 8+8 & 5+0 & 16+5 \\ 2+3 & 8+2 & 5+4 \end{bmatrix} = \begin{bmatrix} 11 & 11 & 0 \\ 16 & 5 & 21 \\ 5 & 10 & 9 \end{bmatrix}$

(*iv*) $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$

$= \begin{bmatrix} \cos^2 x + \sin^2 x & \sin^2 x + \cos^2 x \\ \sin^2 x + \cos^2 x & \cos^2 x + \sin^2 x \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

3. Compute the indicated products:

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [2 \quad 3 \quad 4]$

(iii) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$ **(vi)** $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$

Sol. (*i*) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is defined because the pre-matrix has 2 columns which is equal to the number of rows of the post-matrix.

Performing row by column multiplication,

$$= \begin{bmatrix} a(a)+b(b) & a(-b)+b(a) \\ (-b)a+a(b) & (-b)(-b)+a(a) \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & b^2+a^2 \end{bmatrix}$$

(*ii*) $\begin{bmatrix}1\\2\\3\end{bmatrix}_{3\times 1}\begin{bmatrix}2 & 3 & 4\end{bmatrix}_{1\times 3}$ is defined because the pre-matrix has one column which is equal to the number of rows of the post-matrix.

Performing row by column multiplication,

$$= \begin{bmatrix}1(2) & 1(3) & 1(4)\\2(2) & 2(3) & 2(4)\\3(2) & 3(3) & 3(4)\end{bmatrix} = \begin{bmatrix}2 & 3 & 4\\4 & 6 & 8\\6 & 9 & 12\end{bmatrix}_{3\times 3}$$

(*iii*) $\begin{bmatrix}1 & -2\\2 & 3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\end{bmatrix}$

$$= \begin{bmatrix}1(1)+(-2)2 & 1(2)+(-2)3 & 1(3)+(-2)1\\2(1)+3(2) & 2(2)+3(3) & 2(3)+3(1)\end{bmatrix}$$

(Row by column multiplication)

$$= \begin{bmatrix}1-4 & 2-6 & 3-2\\2+6 & 4+9 & 6+3\end{bmatrix} = \begin{bmatrix}-3 & -4 & 1\\8 & 13 & 9\end{bmatrix}$$

(*iv*) $\begin{bmatrix}2 & 3 & 4\\3 & 4 & 5\\4 & 5 & 6\end{bmatrix}\begin{bmatrix}1 & -3 & 5\\0 & 2 & 4\\3 & 0 & 5\end{bmatrix}$

Performing row by column multiplication

$$= \begin{bmatrix}2(1)+3(0)+4(3) & 2(-3)+3(2)+4(0) & 2(5)+3(4)+4(5)\\3(1)+4(0)+5(3) & 3(-3)+4(2)+5(0) & 3(5)+4(4)+5(5)\\4(1)+5(0)+6(3) & 4(-3)+5(2)+6(0) & 4(5)+5(4)+6(5)\end{bmatrix}$$

$$= \begin{bmatrix}2+0+12 & -6+6+0 & 10+12+20\\3+0+15 & -9+8+0 & 15+16+25\\4+0+18 & -12+10+0 & 20+20+30\end{bmatrix} = \begin{bmatrix}14 & 0 & 42\\18 & -1 & 56\\22 & -2 & 70\end{bmatrix}$$

(*v*) $\begin{bmatrix}2 & 1\\3 & 2\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\-1 & 2 & 1\end{bmatrix}$ is defined because the pre-matrix has 2 columns which is equal to the number of rows of the post-matrix.

Performing row by column multiplication,

$$= \begin{bmatrix}2(1)+1(-1) & 2(0)+1(2) & 2(1)+1(1)\\3(1)+2(-1) & 3(0)+2(2) & 3(1)+2(1)\\(-1)1+1(-1) & (-1)0+1(2) & (-1)1+1(1)\end{bmatrix}$$

$$= \begin{bmatrix}2-1 & 0+2 & 2+1\\3-2 & 0+4 & 3+2\\-1-1 & 0+2 & -1+1\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\\1 & 4 & 5\\-2 & 2 & 0\end{bmatrix}.$$

(*vi*) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6-1+9 & -9-0+3 \\ -2+0+6 & 3+0+2 \end{bmatrix}$

(Row by column multiplication)

$= \begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$.

4. If A $= \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, B $= \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$ and C $= \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$, then compute (A + B) and (B – C). Also, verify that A + (B – C) = (A + B) – C.

Sol. $A + B = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1+3 & 2-1 & -3+2 \\ 5+4 & 0+2 & 2+5 \\ 1+2 & -1+0 & 1+3 \end{bmatrix}$

$\Rightarrow \quad A + B = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$ ...(*i*)

Again $\quad B - C = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3-4 & -1-1 & 2-2 \\ 4-0 & 2-3 & 5-2 \\ 2-1 & 0+2 & 3-3 \end{bmatrix}$

$\Rightarrow \quad B - C = \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$ ...(*ii*)

Putting the value of (B – C) from (*ii*) in L.H.S.

$= A + (B - C)$

$= \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1-1 & 2-2 & -3+0 \\ 5+4 & 0-1 & 2+3 \\ 1+1 & -1+2 & 1+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$ ...(*iii*)

Putting the value of (A + B) from (*i*) in R.H.S. = (A + B) – C

$= \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 4-4 & 1-1 & -1-2 \\ 9-0 & 2-3 & 7-2 \\ 3-1 & -1+2 & 4-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix} \quad ...(iv)$$

From (*iii*) and (*iv*), we have L.H.S. = R.H.S.

5. If $A = \begin{bmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{bmatrix}$ **and** $B = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{bmatrix}$**, then compute 3A – 5B.**

Sol. $3A - 5B = 3 \begin{bmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{bmatrix} - 5 \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{bmatrix}$

Multiplying each entry of first matrix by 3 and each entry of second matrix by 5

$$= \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \\ 7 & 6 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \\ 7 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 2-2 & 3-3 & 5-5 \\ 1-1 & 2-2 & 4-4 \\ 7-7 & 6-6 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Remark. Here answer is a zero matrix.

6. Simplify $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$.

Sol. $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

Multiplying each entry of first matrix by $\cos\theta$ and each entry of second matrix by $\sin\theta$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Remark. The answer matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ of this question is identity (unit) matrix I_2.

7. Find X and Y if

(i) $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ **and** $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(ii) $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ **and** $3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$

Sol. (*i*) **Given:** $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ ...(*i*)

and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ ...(*ii*)

Adding eqns. (*i*) and (*ii*), we have

$$2X = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$$

$$\therefore \quad X = \frac{1}{2}\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} \frac{10}{2} & \frac{0}{2} \\ \frac{2}{2} & \frac{8}{2} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}.$$

Eqn. (*i*) – eqn. (*ii*) gives

$$2Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 7-3 & 0-0 \\ 2-0 & 5-3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$$

$$\therefore \quad Y = \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{2} & \frac{0}{2} \\ \frac{2}{2} & \frac{2}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

(*ii*) **Given:** $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ ...(*i*)

and $3X + 2Y = \begin{bmatrix} -2 & -2 \\ -1 & 5 \end{bmatrix}$ ...(*ii*)

Multiplying equation (*i*) by 2, we have

$$4X + 6Y = 2\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 0 \end{bmatrix} \quad ...(iii)$$

Multiplying equation (*ii*) by 3, we have

$$9X + 6Y = 3\begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ -3 & 15 \end{bmatrix} \quad ...(iv)$$

Equation (*iv*) – equation (*iii*) gives

$$5X = \begin{bmatrix} 6 & -6 \\ -3 & 15 \end{bmatrix} - \begin{bmatrix} 4 & 6 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} 6-4 & -6-6 \\ -3-8 & 15-0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -12 \\ -11 & 15 \end{bmatrix}$$

$$\therefore \quad X = \frac{1}{5}\begin{bmatrix} 2 & -12 \\ -11 & 15 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{12}{5} \\ -\frac{11}{5} & 3 \end{bmatrix}.$$

Now from equation (*i*),

$$3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - 2X$$

$$= \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - 2\begin{bmatrix} \frac{2}{5} & \frac{-12}{5} \\ \frac{-11}{5} & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} \frac{4}{5} & \frac{-24}{5} \\ \frac{-22}{5} & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - \frac{4}{5} & 3 + \frac{24}{5} \\ 4 + \frac{22}{5} & 0 - 6 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & \frac{39}{5} \\ \frac{42}{5} & -6 \end{bmatrix}$$

$$\Rightarrow Y = \frac{1}{3}\begin{bmatrix} \frac{6}{5} & \frac{39}{5} \\ \frac{42}{5} & -6 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix}$$

8. Find X if Y = $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and 2X + Y = $\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$.

Sol. $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \Rightarrow 2X = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - Y$

$$\Rightarrow \quad 2X = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1-3 & 0-2 \\ -3-1 & 2-4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix}$$

$$\Rightarrow \quad X = \frac{1}{2}\begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}.$$

9. Find *x* and *y*, if $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$.

Sol. Given: $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2 + y & 6 \\ 1 & 2x + 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Equating corresponding entries, we have

$2 + y = 5$ and $2x + 2 = 8$

$\Rightarrow \quad y = 5 - 2 = 3$ and $2x = 8 - 2 = 6 \Rightarrow x = 3$

$\therefore \quad x = 3, y = 3.$

10. Solve the equation for *x*, *y*, *z* and *t* if

$$2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

Sol. Given: $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2x & 2z \\ 2y & 2t \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+3 & 2z-3 \\ 2y+0 & 2t+6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

Since the two matrices are equal, so the corresponding elements are equal.

Thus, $2x + 3 = 9$

$\Rightarrow$ $2x = 9 - 3 = 6$ $\Rightarrow$ $x = 3$

Also $2z - 3 = 15$ $\Rightarrow$ $2z = 18$ $\Rightarrow$ $z = 9$

Also $2y = 12$ $\Rightarrow$ $y = 6$

and $2t + 6 = 18$ and $2t = 12$ $\Rightarrow$ $t = 6$

$\therefore$ $x = 3$, $y = 6$, $z = 9$ and $t = 6$.

11. If $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$**, find the values of *x* and *y*.**

Sol. Given: $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2x \\ 3x \end{bmatrix} + \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x-y \\ 3x+y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

Equating corresponding entries, we have

$2x - y = 10$ ...(*i*)

and $3x + y = 5$ ...(*ii*)

Adding eqns. (*i*) and (*ii*) we have $5x = 15$

or $x = \frac{15}{5} = 3$

Putting $x = 3$ in (*ii*), $9 + y = 5$ $\Rightarrow$ $y = 5 - 9 = -4$

$\therefore$ $x = 3$, $y = -4$.

12. Given: $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$**; find the values of *x*, *y*, *z* and *w*.**

Sol. Given: $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}$$

Equating corresponding entries, we have

$3x = x + 4$ $\Rightarrow$ $2x = 4$ $\Rightarrow$ $x = 2$ ...(*i*)

and $3y = 6 + x + y$ $\Rightarrow$ $2y = 6 + x = 6 + 2$ (By (*i*))

$\Rightarrow$ $2y = 8$ $\Rightarrow$ $y = 4$ ...(*ii*) and $3z = -1 + z + w \Rightarrow 2z - w = -1$ ...(*iii*)

and $3w = 2w + 3 \Rightarrow w = 3.$

Putting $w = 3$ in eqn. (*iii*),

$2z - 3 = -1 \Rightarrow 2z = 2 \Rightarrow z = 1$

$\therefore$ $x = 2,\quad y = 4,\quad z = 1,\quad w = 3.$

13. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that F(*x*) F(*y*) = F(*x* + *y*).

Sol. Given: $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ...(*i*)

Changing *x* to *y* in (*i*), $F(y) = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\text{L.H.S.} = F(x)\,F(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing row by column multiplication,

$$= \begin{bmatrix} \cos x \cos y - \sin x \sin y + 0 & -\cos x \sin y - \sin x \cos y + 0 & 0 - 0 + 0 \\ \sin x \cos y + \cos x \sin y + 0 & -\sin x \sin y + \cos x \cos y + 0 & 0 + 0 + 0 \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos (x+y) & -\sin (x+y) & 0 \\ \sin (x+y) & \cos (x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ $[\because\ -\cos x \sin y - \sin x \cos y = -(\cos x \sin y + \sin x \cos y) = -\sin (x + y)]$

Now, changing *x* to *x* + *y* in (*i*), we get

$$F(x + y) = \begin{bmatrix} \cos (x+y) & -\sin (x+y) & 0 \\ \sin (x+y) & \cos (x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ Thus, L.H.S. = R.H.S.

14. Show that:

(*i*) $\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$

(*ii*) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

Sol. (*i*) L.H.S. $= \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5(2) + (-1)3 & 5(1) + (-1)4 \\ 6(2) + 7(3) & 6(1) + 7(4) \end{bmatrix}$

$$= \begin{bmatrix} 10-3 & 5-4 \\ 12+21 & 6+28 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 33 & 34 \end{bmatrix} \qquad ...(i)$$

$$\text{R.H.S.} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 2(5)+1(6) & 2(-1)+1(7) \\ 3(5)+4(6) & 3(-1)+4(7) \end{bmatrix}$$

$$= \begin{bmatrix} 10+6 & -2+7 \\ 15+24 & -3+28 \end{bmatrix} = \begin{bmatrix} 16 & 5 \\ 39 & 25 \end{bmatrix} \qquad ...(ii)$$

From (*i*) and (*ii*), we can say that L.H.S. ≠ R.H.S.

(Because corresponding entries of matrices $\begin{bmatrix} 7 & 1 \\ 33 & 34 \end{bmatrix}$ and $\begin{bmatrix} 16 & 5 \\ 39 & 25 \end{bmatrix}$ are not same).

(*ii*) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

Here, matrices A and B are both of order 3 × 3 respectively, therefore AB and BA are both of same order 3 × 3.

$$\text{Now, } AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

Performing row by column multiplication,

$$= \begin{bmatrix} 1(-1)+2(0)+3(2) & 1(1)+2(-1)+3(3) & 1(0)+2(1)+3(4) \\ 0(-1)+1(0)+0(2) & 0(1)+1(-1)+0(3) & 0(0)+1(1)+0(4) \\ 1(-1)+1(0)+0(2) & 1(1)+1(-1)+0(3) & 1(0)+1(1)+0(4) \end{bmatrix}$$

$$\text{or } AB = \begin{bmatrix} -1+6 & 1-2+9 & 2+12 \\ 0 & -1 & 1 \\ -1 & 1-1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 14 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \qquad ...(i)$$

$$\text{Again, } BA = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Performing row by column multiplication,

$$= \begin{bmatrix} (-1)1+1(0)+0(1) & (-1)2+1(1)+0(1) & (-1)3+1(0)+0(0) \\ 0(1)+(-1)0+1(1) & 0(2)+(-1)1+1(1) & 0(3)+(-1)0+1(0) \\ 2(1)+3(0)+4(1) & 2(2)+3(1)+4(1) & 2(3)+3(0)+4(0) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2+1 & -3 \\ 1 & -1+1 & 0 \\ 2+4 & 4+3+4 & 6 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 \\ 1 & 0 & 0 \\ 6 & 11 & 6 \end{bmatrix} \qquad ...(ii)$$

From (i) and (ii), AB ≠ BA because corresponding entries of matrices AB and BA are not same.

Remark. From both questions (i), (ii) we can learn that matrix multiplication is not commutative.

15. Find $A^2 - 5A + 6I$ if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$.

Sol. $A^2 = A \,.\, A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Performing row by column multiplication,

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1-0 & 1-3+0 \end{bmatrix} \text{ or } A^2 = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

∴ $A^2 - 5A + 6I = A^2 - 5A + 6I_3$ (Here I is I_3 because matrices A and A^2 are of order 3 × 3)

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5 \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5-10+6 & -1-0+0 & 2-5+0 \\ 9-10+0 & -2-5+6 & 5-15+0 \\ 0-5+0 & -1+5+0 & -2-0+6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

Remark. The above question can also be stated as:

If $f(x) = x^2 - 5x + 6$ and $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$; then find f(A).

16. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, prove that $A^3 - 6A^2 + 7A + 2I = 0$.

Sol. Given: $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ ∴ $A^2 = A \,.\, A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1+0+4 & 0+0+0 & 2+0+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+0+6 & 0+0+0 & 4+0+9 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$\therefore \quad A^3 = A^2 \,.\, A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5+0+16 & 0+0+0 & 10+0+24 \\ 2+0+10 & 0+8+0 & 4+4+15 \\ 8+0+26 & 0+0+0 & 16+0+39 \end{bmatrix} \text{or } A^3 = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

L.H.S. $= A^3 - 6A^2 + 7A + 2I$

$= A^3 - 6A^2 + 7A + 2I_3$

[Here I is I_3 because A, A^2, A^3 are matrices of order 3×3]

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6 \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 0 & -14 \\ 0 & -16 & -7 \\ -14 & 0 & -23 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 14 \\ 0 & 16 & 7 \\ 14 & 0 & 23 \end{bmatrix}$$

$$= \begin{bmatrix} -9+9 & 0+0 & -14+14 \\ 0+0 & -16+16 & -7+7 \\ -14+14 & 0+0 & -23+23 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

= (zero matrix) O = R.H.S.

17. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find k so that $A^2 = kA - 2I$.

Sol. Given: $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$

$$\therefore \quad A^2 = A \,.\, A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$$

Putting values of A^2, A and I in the given equation $A^2 = kA - 2I$, we have

$$\begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = k\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3k & -2k \\ 4k & -2k \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{bmatrix}$$

Equating corresponding entries, we have

$3k - 2 = 1 \Rightarrow 3k = 3 \Rightarrow k = 1$ and $-2 = -2k \Rightarrow k = 1$

and $4k = 4 \Rightarrow k = 1$ and $-4 = -2k - 2 \Rightarrow 2k = -2 + 4 = 2$

$\Rightarrow \quad k = 1$

Therefore, value of $k = 1$ and is same from all the four equations. Therefore, k exists and = 1.

18. If A = $\begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2, show that I + A = (I – A) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

Sol. A = $\begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2

i.e., $\quad I = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\text{L.H.S.} = I + A = I_2 + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix} \quad ...(i)$$

Again, $\quad I - A = I_2 - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}$$

$$\text{R.H.S.} = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Performing row by column multiplication,

$$= \begin{bmatrix} \cos\alpha + \sin\alpha\tan\frac{\alpha}{2} & -\sin\alpha + \cos\alpha\tan\frac{\alpha}{2} \\ -\cos\alpha\tan\frac{\alpha}{2} + \sin\alpha & \sin\alpha\tan\frac{\alpha}{2} + \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha + \sin\alpha\dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & -\sin\alpha + \cos\alpha\dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \\ -\cos\alpha\dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} + \sin\alpha & \sin\alpha\dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} + \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\cos\alpha\cos\frac{\alpha}{2} + \sin\alpha\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & \dfrac{-\sin\alpha\cos\frac{\alpha}{2} + \cos\alpha\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \\ \dfrac{-\cos\alpha\sin\frac{\alpha}{2} + \sin\alpha\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & \dfrac{\sin\alpha\sin\frac{\alpha}{2} + \cos\alpha\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \end{bmatrix}$$

Numerator of a_{12} is $= -\left(\sin\alpha\cos\frac{\alpha}{2} - \cos\alpha\sin\frac{\alpha}{2}\right)$

$$= \begin{bmatrix} \dfrac{\cos\left(\alpha - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}} & \dfrac{-\sin\left(\alpha - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}} \\ \dfrac{\sin\left(\alpha - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}} & \dfrac{\cos\left(\alpha - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}} \end{bmatrix} = \begin{bmatrix} \dfrac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & \dfrac{-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \\ \dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & \dfrac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \end{bmatrix}$$

[$\because$ cos A cos B + sin A sin B = cos (A – B)
and sin A cos B – cos A sin B = sin (A – B)]

$$= \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix} \qquad ...(ii)$$

From equations (*i*) and (*ii*), we have L.H.S. = R.H.S.

i.e., $\quad I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

19. A trust fund has ₹ 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹ 30,000 in two types of bonds, if the trust fund must obtain an annual interest of

(*a*) ₹ 1800 **(*b*) ₹ 2000.**

Sol. Let the investment in first bond be ₹ x,

then the investment in second bond = ₹ $(30,000 - x)$

Interest paid by first bond = 5% = $\dfrac{5}{100}$ per rupee

Interest paid by second bond = 7% = $\dfrac{7}{100}$ per rupee

Matrix of investment is $A = [x \quad 30000 - x]_{1 \times 2}$

Matrix of annual interest per rupee is $B = \begin{bmatrix} \dfrac{5}{100} \\ \dfrac{7}{100} \end{bmatrix}_{2\times 1}$

Matrix of total annual interest is

$$AB = [x \qquad 30000 - x]\begin{bmatrix} \dfrac{5}{100} \\ \dfrac{7}{100} \end{bmatrix} = \left[\dfrac{5x}{100} + \dfrac{7(30000 - x)}{100}\right]$$

$$= \left[\dfrac{5x + 210000 - 7x}{100}\right] = \left[\dfrac{210000 - 2x}{100}\right]$$

$\therefore$ Total annual interest = ₹ $\dfrac{2,10,000 - 2x}{100}$

(a) total annual interest is given to be ₹ 1,800

$$\therefore \qquad \frac{2,10,000 - 2x}{100} = 1,800$$

$\Rightarrow \qquad 2,10,000 - 2x = 1,80,000 \quad \therefore \quad x = 15,000$

Hence, investment in first bond = ₹ 15,000

and investment in second bond = ₹ $(30,000 - x)$

= ₹ $(30,000 - 15,000)$ = ₹ 15,000.

(b) Total annual interest is given to be ₹ 2,000

$$\therefore \qquad \frac{2,10,000 - 2x}{100} = 2,000$$

$\Rightarrow \quad 2,10,000 - 2x = 2,00,000 \qquad \therefore \quad x = 5,000$

Hence, investment in first bond = ₹ 5,000 and investment in second bond = ₹ $(30,000 - x)$ = ₹ $(30,000 - 5,000)$ **= ₹ 25,000.**

20. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are ₹ 80, ₹ 60 and ₹ 40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

Sol. Let us represent the number of books as a 1×3 row matrix

$$B = \begin{bmatrix} \text{10 dozen} & \text{8 dozen} & \text{10 dozen} \\ 10 \times 12 = 120 & 8 \times 12 = 96 & 10 \times 12 = 120 \end{bmatrix}$$

Let us represent the selling prices of each book as a 3×1 column

matrix $S = \begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix}$

$\therefore$ [Total amount received by selling all books]$_{1 \times 1}$

$$= BS = [120 \quad 96 \quad 120]_{1 \times 3} \begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix}_{3 \times 1}$$

$$= [120(80) + 96(60) + 120(40)]_{1 \times 1}$$
$$= [9600 + 5760 + 4800] = [20160]$$

Equating corresponding entries,

Total amount received by selling all the books = ₹ 20160.

Assume X, Y, Z, W and P are matrices of order $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$ and $p \times k$ respectively. Choose the correct answer in Exercises 21 and 22.

21. The restriction on n, k and p so that PY + WY will be defined are:

(A) $k = 3$, $p = n$ **(B) k is arbitrary, $p = 2$**

(C) p is arbitrary, $k = 3$ **(D) $k = 2$, $p = 3$.**

Sol. Given: Matrix PY + WY is defined ($\Rightarrow$ possible).

Matrix P is of order $p \times k$ and matrix Y is of order $3 \times k$ and matrix W is of order $n \times 3$.

Now PY + WY = (P + W) Y ... (*i*)

We know that sum P + W is defined if two matrices

$$\underset{p \times k}{\text{P}} + \underset{n \times 3}{\text{W}}$$

P and W are of same order. Therefore $p = n$ and $k = 3$ and order of P + W is $n \times 3$ (or $p \times k$)

Therefore from (1), PY + WY = (P + W) Y is defined as

$$\underset{n \times 3}{(\text{P} + \text{W})} \; \underset{3 \times k}{\text{Y}}$$

Number of columns in P + W is same as number of rows in Y.

$\therefore$ $p = n$ and $k = 3$

$\therefore$ Option (A) is the correct answer *i.e.*, $k = 3$ and $p = n$.

22. If $n = p$, then order of the matrix 7X – 5Z is

(A) $p \times 2$ (B) $2 \times n$ (C) $n \times 3$ (D) $p \times n$.

Sol. Since $n = p$ (given), the order of matrices X and Z are equal.

$\therefore$ 7X – 5Z is well defined and the order of 7X – 5Z is same as the order of X and Z.

$\therefore$ The order of 7X – 5Z is either equal to $2 \times n$ or $2 \times p$

($\because$ $n = p$)

$\therefore$ The correct option is (B), *i.e.*, the order of 7X – 5Z is $2 \times n$.