Question 1:

Prove $3\sin^{-1}x=\sin^{-1}\left(3x-4x^3\right), x\in\left[-\frac{1}{2},\frac{1}{2}\right]$

Solution 1:

Let $x=\sin\theta$.

$\Rightarrow \sin^{-1}x=\theta.$

We have,

R.H.S = $\sin^{-1}\left(3x-4x^3\right)$

$=\sin^{-1}\left(3\sin\theta-4\sin^3\theta\right)$

$=\sin^{-1}\left(\sin 3\theta\right)$

$=3\theta$

$=3\sin^{-1}x$

L.H.S

Question 2:

Prove $3\cos^{-1}x=\cos^{-1}\left(4x^3-3x\right), x\in\left[\frac{1}{2},1\right]$

Solution 2:

Let $x=\cos\theta$.

$\Rightarrow \cos^{-1}x=\theta.$

We have,

$R.H.S=\cos^{-1}\left(4x^3-3x\right)$

$=\cos^{-1}\left(4\cos^3\theta-3\cos\theta\right)$

$=\cos^{-1}\left(\cos 3\theta\right)$

$=3\theta$

$=3\cos^{-1}x$

$=L.H.S$

Question 3:

Prove $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$

Solution 3:

$$L.H.S.=\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}$$

$$=\tan^{-1}\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}.\frac{7}{24}}$$

$$=\tan^{-1}\frac{\frac{48+77}{11x24}}{\frac{11x24-14}{11x24}}$$

$$=\tan^{-1}\frac{48+77}{264-14}$$

$$=\tan^{-1}\frac{125}{250}$$

$$=\tan^{-1}\frac{1}{2}$$

$$=\text{R.H.S.}$$

Question 4:

Prove $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

Solution 4:

$$\text{L.H.S} = 2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{2.\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{1}{\left(\frac{3}{4}\right)}+\tan^{-1}\frac{1}{7}$$

$$= \tan^{-1} \frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}.\frac{1}{7}}$$

$$= \tan^{-1}\left(\frac{\frac{28+3}{21}}{\frac{21-4}{21}}\right)$$

$$= \tan^{-1}\frac{31}{17}$$

= R.H.S.

Question 5:

Write the function in the simplest form: $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}, x \neq 0$

Solution 5:

Given, $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$$\therefore \tan^{-1}\frac{\sqrt{1+x^2-1}}{x} = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2}\tan^{-1} x$$

Question 6:

Write the function in the simplest form: $\tan^{-1}\frac{1}{\sqrt{x^2-1}},|x|>1$

Solution 6:

Given, $\tan^{-1}\frac{1}{\sqrt{x^2-1}},|x|>1$

Put $x=\operatorname{cosec}\theta \Rightarrow \theta=\operatorname{cosec}^{-1}x$

$\therefore \tan^{-1}\frac{1}{\sqrt{x^2-1}}=\tan^{-1}\frac{1}{\sqrt{\operatorname{cos}ec^2\theta-1}}$

$=\tan^{-1}\left(\frac{1}{\cot\theta}\right)$

$=\tan^{-1}(\tan\theta)$

$=\theta$

$=\operatorname{cosec}^{-1}x$

$=\frac{\pi}{2}-\sec^{-1}x$

Question 7:

Write the function in the simplest form: $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right),x<\pi$

Solution 7:

Given, $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right),x<\pi$

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$

$$=\tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$

$$=\tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$$

$$=\tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$=\frac{x}{2}$$

Question 8:

Write the function in the simplest form: $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right), 0<x<\pi$

Solution 8:

Given, $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$

$$=\tan^{-1}\left(\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right)$$

$$=\tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$=\tan^{-1}(1)-\tan^{-1}(\tan x)$$

$$=\frac{\pi}{4}-x$$

Question 9:

Write the function in the simplest form: $\tan^{-1}\dfrac{x}{\sqrt{a^2-x^2}}, |x|<a$

Solution 9:

Given, $\tan^{-1}\dfrac{x}{\sqrt{a^2-x^2}}$

Put $x=a\sin\theta$

$$\Rightarrow \frac{x}{a}=\sin\theta \Rightarrow \sin^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \tan^{-1}\frac{x}{\sqrt{a^2-x^2}}=\tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$$

$$=\tan^{-1}(\tan\theta)$$

$$=\theta$$

$$=\sin^{-1}\frac{x}{a}$$

Question 10:

Write the function in the simplest form: $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right), a>0; \dfrac{-a}{\sqrt{3}}\le x\le\dfrac{a}{\sqrt{3}}$

Solution 10:

Given, $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right)$

Put $x=a\tan\theta$

$$\Rightarrow \frac{x}{a}=\tan\theta$$

$$\Rightarrow \theta=\tan^{-1}\frac{x}{a}$$

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)=\tan^{-1}\left(\frac{3a^2.a\tan\theta-a^3\tan^3\theta}{a^3-3a.a^2\tan^2\theta}\right)$$

$$=\tan^{-1}\left(\frac{3a^3\tan\theta-a^3\tan^3\theta}{a^3-3a^3\tan^2\theta}\right)$$

$$=\tan^{-1}(\tan 3\theta)$$

$$=3\theta$$

$$=3\tan^{-1}\frac{x}{a}$$

Question 11:

Find the value of $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

Solution 11:

Let $\sin^{-1}\frac{1}{2}=x.$

$$\Rightarrow \sin x=\frac{1}{2}=\sin\left(\frac{\pi}{6}\right).$$

$$\therefore \sin^{-1}\frac{1}{2}=\frac{\pi}{6}$$

$$\therefore \tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]=\tan^{-1}\left[2\cos\left(2\times\frac{\pi}{6}\right)\right]$$

$$=\tan^{-1}\left[2\cos\frac{\pi}{3}\right]$$

$$=\tan^{-1}\left[2\times\frac{1}{2}\right]$$

$$=\tan^{-1}1$$

$$=\frac{\pi}{4}$$

Question 12:

Find the value of $\cot\left(\tan^{-1}a+\cot^{-1}a\right)$

Solution 12:

Given, $\cot\left(\tan^{-1}a+\cot^{-1}a\right)$

$$=\cot\left(\frac{\pi}{2}\right)$$

$$=0$$

Question 13:

Find the value of $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right], |x|<1, y>0$ and $xy<1$

Solution 13:

Let $x=\tan\theta.$

$$\Rightarrow \theta=\tan^{-1}x.$$

$$\therefore \sin^{-1}\frac{2x}{1+x^2}=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$=\sin^{-1}\left(\sin 2\theta\right)$$

$$=2\theta$$

$$=2\tan^{-1}x$$

Let $y=\tan\phi.$

$$\Rightarrow \phi=\tan^{-1}y.$$

$$\therefore \cos^{-1}\frac{1-y^2}{1+y^2}=\cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)$$

$$=\cos^{-1}\left(\cos 2\phi\right)$$

$$=2\phi$$

$$=2\tan^{-1}y$$

$$\therefore \tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right]$$

$$=\tan\frac{1}{2}\left[2\tan^{-1}x+2\tan^{-1}y\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \frac{x+y}{1-xy}$$

Question 14:

If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x.

Solution 14:

Given, $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$

$$\Rightarrow \sin\left(\sin^{-1}\frac{1}{5}\right)\cos\left(\cos^{-1}x\right)+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\Rightarrow \frac{1}{5}\times x+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\Rightarrow \frac{x}{5}+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1 \qquad .....(1)$$

Now, let $\sin^{-1}\frac{1}{5}=y$

$$\Rightarrow \sin^{-1}\frac{1}{5}=y$$

$$\Rightarrow \sin y=\frac{1}{5}$$

$$\Rightarrow \cos y=\sqrt{1-\left(\frac{1}{5}\right)^2}=\frac{2\sqrt{6}}{5}$$

$$\Rightarrow y=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right)$$

$$\therefore \sin^{-1}\frac{1}{5}=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right) \qquad ......(2)$$

Let $\cos^{-1}x=z.$

$\Rightarrow \cos z = x$

$\Rightarrow \sin z = \sqrt{1-x^2}$.

$\Rightarrow z = \sin^{-1}\left(\sqrt{1-x^2}\right)$

$\therefore \cos^{-1} x = \sin^{-1}\left(\sqrt{1-x^2}\right)$ ……..(3)

From (1),(2),and (3) we have:

$$\frac{x}{5} + \cos\left(\cos^{-1}\frac{2\sqrt{6}}{5}\right)\sin\left(\sin^{-1}\sqrt{1-x^2}\right) = 1$$

$$\Rightarrow \frac{x}{5} + \frac{2\sqrt{6}}{5}\sqrt{1-x^2} = 1$$

$$\Rightarrow x + 2\sqrt{6}\sqrt{1-x^2} = 5$$

On squaring both sides, we get:

$$(4)(6)\left(1-x^2\right) = 25 + x^2 - 10x$$

$$\Rightarrow 24 - 24x^2 = 25 + x^2 - 10x$$

$$\Rightarrow 25x^2 - 10x + 1 = 0$$

$$\Rightarrow (5x-1)^2 = 0$$

$$\Rightarrow (5x-1) = 0$$

$$\Rightarrow x = \frac{1}{5}$$

Question 15:

If $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$, then find the value of x.

Solution 15:

Given, $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1-\left(\frac{x-1}{x+2}\right)\left(\frac{x+1}{x+2}\right)}\right] = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{(x-1)(x+2)+(x+1)(x-2)}{(x+2)(x-2)-(x-1)(x+1)}\right] = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x^2-4}{-3}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan\left[\tan^{-1}\frac{4-2x^2}{3}\right]=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{4-2x^2}{3}=1$$

$$\Rightarrow 4-2x^2=3$$

$$\Rightarrow 2x^2=4-3=1$$

$$\Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

Question 16:

Find the values of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

Solution 16:

Given, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$$\Rightarrow \sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\sin^{-1}\left[\sin\left(\pi-\frac{2\pi}{3}\right)\right]$$

$$=\sin^{-1}\left(\sin\frac{\pi}{3}\right)$$

$$=\frac{\pi}{3}$$

Question 17:

Find the values of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

Solution 17:

Given, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

$$\Rightarrow \tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left[-\tan\left(\frac{-3\pi}{4}\right)\right]$$

$$=\tan^{-1}\left[-\tan\left(\pi-\frac{\pi}{4}\right)\right]$$

$$=\tan^{-1}\left(-\tan\frac{\pi}{4}\right)$$

$$=\tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right]$$

$$=\frac{-\pi}{4}$$

Question 18:

Find the values of $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

Solution 18:

Let $\sin^{-1}\frac{3}{5}=x$.

$$\Rightarrow \sin x=\frac{3}{5}$$

$$\Rightarrow \cos x=\sqrt{1-\sin^2 x}=\frac{4}{5}$$

$$\Rightarrow \sec x=\frac{5}{4}.$$

$$\therefore \tan x=\sqrt{\sec^2 x-1}=\sqrt{\frac{25}{16}-1}=\frac{3}{4}$$

$$\therefore x=\tan^{-1}\frac{3}{4}$$

$$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4} \qquad \text{……(i)}$$

Now, $\cot^{-1}\frac{3}{2}=\tan^{-1}\frac{2}{3}$ …….(ii)

Thus, $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

$=\tan\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)$ [using (i) and (ii)]

$=\tan\left[\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4},\frac{2}{3}}\right]$

$=\tan\left(\tan^{-1}\frac{9+8}{12-6}\right)$

$=\tan\left(\tan^{-1}\frac{17}{6}\right)=\frac{17}{6}$

Question 19:

Find the values of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

(A) $\frac{7\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

Solution 19:

$\cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left(\cos\frac{-7\pi}{6}\right)$

$=\cos^{-1}\left[\cos\left(2\pi-\frac{7\pi}{6}\right)\right]$

$=\cos^{-1}\left[\cos\frac{5\pi}{6}\right]$

$=\frac{5\pi}{6}$

Question 20:

Find the values of $\sin\left(\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) 1

Solution 20:

Let $\sin^{-1}\left(\frac{-1}{2}\right)=x$.

$$\Rightarrow \sin x=\frac{-1}{2}$$
$$=-\sin\frac{\pi}{6}$$
$$=\sin\left(\frac{-\pi}{6}\right).$$

Range of the principal value of $\sin^{-1}x$ is $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$.

$$\sin^{-1}\left(\frac{-1}{2}\right)=\frac{\pi}{6}$$

$$\therefore \sin\left(\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right)=\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)$$
$$=\sin\left(\frac{3\pi}{6}\right)$$
$$=\sin\left(\frac{\pi}{2}\right)=1$$