NCERT Class 12 Maths

Solutions

Chapter - 11

Three Dimensional Geometry

Exercise 11.3

Note: Formula for question numbers 1 and 2.
If p is the length of perpendicular from the origin to a plane and $\hat{n}$ is a unit normal vector to the plane, then equation of the plane is $\vec{r} \cdot \hat{n} = p$ (where of course p being length is > 0).

1. In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(a) $z = 2$ **(b) $x + y + z = 1$**

(c) $2x + 3y - z = 5$ **(d) $5y + 8 = 0$**

Sol. (a) **Given:** Equation of the plane is $z = 2$

Let us first reduce it to vector form $\vec{r} \cdot \vec{n} = d$ where $d > 0$

or $0x + 0y + 1z = 2$ (Here $d = 2 > 0$)

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (0\hat{i} + 0\hat{j} + \hat{k}) = 2$

$(\because \; a_1a_2 + b_1b_2 + c_1c_2 = (a_1\hat{i} + b_1\hat{j} + c_1\hat{k}) \cdot (a_2\hat{i} + b_2\hat{j} + c_2\hat{k}))$

$\Rightarrow \vec{r} \cdot \vec{n} = 2$ where we know that

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ = (Position vector of point P(x, y, z))

and here $\vec{n} = 0\hat{i} + 0\hat{j} + \hat{k}$

Now let us reduce $\vec{r} \cdot \vec{n} = d$ to $\vec{r} \cdot \hat{n} = p$

Dividing both sides by $|\vec{n}|$, $\dfrac{\vec{r} \cdot \vec{n}}{|\vec{n}|} = 2$

i.e., $\vec{r} \cdot \hat{n} = 2 = p$ where $\hat{n} = \dfrac{\vec{n}}{|\vec{n}|} = \dfrac{0\hat{i} + 0\hat{j} + \hat{k}}{\sqrt{0+0+1} = 1}$

i.e., $\hat{n} = 0\hat{i} + 0\hat{j} + \hat{k}$ and $p = 2$

$\therefore$ By definition, direction cosines of normal to the plane are coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\hat{n}$ *i.e.,* 0, 0, 1 and length of perpendicular from the origin to the plane is $p = 2$.

(b) **Given:** Equation of the plane is $x + y + z = 1$

$\Rightarrow 1x + 1y + 1z = 1$ (Here $d = 1 > 0$)

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$

i.e., $\vec{r} \cdot \vec{n} = 1$ where $\vec{n} = \hat{i} + \hat{j} + \hat{k}$

Dividing both sides by $|\vec{n}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$, we have

$$\vec{r} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{1}{|\vec{n}|}$$

i.e., $\vec{r} \cdot \hat{n} = \frac{1}{\sqrt{3}} = p$ where $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\hat{i}+\hat{j}+\hat{k}}{|\vec{n}| = \sqrt{3}}$

i.e., $\hat{n} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$ and $p = \frac{1}{\sqrt{3}}$

$\therefore$ By definition, direction cosines of the normal to the plane are the coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ in $\hat{n}$ i.e., $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and length of perpendicular from the origin to the plane is $p = \frac{1}{\sqrt{3}}$.

(*c*) **Given:** Equation of the plane is $2x + 3y - z = 5$

$\Rightarrow$ $2x + 3y + (-1)z = 5$ (Here $d = 5 > 0$)

$\Rightarrow$ $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 5$

i.e., $\vec{r} \cdot \vec{n} = 5$ where $\vec{n} = 2\hat{i} + 3\hat{j} - \hat{k}$

Dividing both sides by $|\vec{n}| = \sqrt{4+9+1} = \sqrt{14}$,

we have $\vec{r} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{5}{|\vec{n}|}$

i.e., $\vec{r} \cdot \hat{n} = \frac{5}{\sqrt{14}} = p$ where $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i}+3\hat{j}-\hat{k}}{\sqrt{4+9+1} = \sqrt{14}}$

i.e., $\hat{n} = \frac{2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} - \frac{1}{\sqrt{14}}\hat{k}$

$\therefore$ By definition, direction cosines of the normal to the plane are coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ in $\hat{n}$ i.e., $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}$ and length of perpendicular from the origin to the plane is $\frac{5}{\sqrt{14}}$.

(*d*) **Given:** Equation of the plane is

$$5y + 8 = 0 \quad \text{or} \quad 5y = -8$$

Dividing both sides by -1 to make R.H.S. (= d) as positive,

$-5y = 8$ or $0x + (-5)y + 0z = 8$ | Now $d = 8 > 0$

$\Rightarrow$ $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (0\hat{i} - 5\hat{j} + 0\hat{k}) = 8$

i.e., $\vec{r} \cdot \vec{n} = 8$ where $\vec{n} = 0\hat{i} - 5\hat{j} + 0\hat{k}$

Dividing both sides by $|\vec{n}| = \sqrt{0^2 + (-5)^2 + 0^2}$

i.e., $|\vec{n}| = \sqrt{25} = 5$

we have $\vec{r} \cdot \dfrac{\vec{n}}{|\vec{n}|} = \dfrac{8}{5}$ *i.e.,* $\vec{r} \cdot \hat{n} = \dfrac{8}{5} = p$

where $\hat{n} = \dfrac{\vec{n}}{|\vec{n}|} = \dfrac{0\hat{i} - 5\hat{j} + 0\hat{k}}{5}$

$= \dfrac{0}{5}\hat{i} - \dfrac{5}{5}\hat{j} + \dfrac{0}{5}\hat{k} = 0\hat{i} - \hat{j} + 0\hat{k}$ and $p = \dfrac{8}{5}$.

∴ By definition, direction cosines of the normal to the plane are coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ in $\hat{n}$ *i.e.,* 0, – 1, 0 and length of perpendicular from the origin to the plane is $\dfrac{8}{5}$.

2. Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector

$$\mathbf{3\hat{i} + 5\hat{j} - 6\hat{k}}.$$

Sol. Here $\vec{n} = 3\hat{i} + 5\hat{j} - 6\hat{k}$

∴ The unit vector perpendicular to plane is

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{(3)^2 + (5)^2 + (-6)^2}} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}$$

Also $p = 7$ (given)

Hence, the equation of the required plane is $\vec{r} \cdot \hat{n} = p$

i.e., $\vec{r} \cdot \dfrac{(3\hat{i} + 5\hat{j} - 6\hat{k})}{\sqrt{70}} = 7$

or $\vec{r} \cdot (3\hat{i} + 5\hat{j} - 6\hat{k}) = 7\sqrt{70}$.

3. Find the Cartesian equation of the following planes:

(*a*) $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$ **(*b*)** $\vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$

(*c*) $\vec{r} \cdot [(s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}] = 15$.

Sol. (*a*) Vector equation of the plane is

$$\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2 \quad ...(i)$$

Putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in (*i*) (we know that in 3-D, $\vec{r}$ is the position vector of any point, P(*x*, *y*, *z*)), Cartesian equation of the plane is

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$$

$\Rightarrow \quad x(1) + y(1) + z(-1) = 2 \quad \Rightarrow \quad x + y - z = 2.$

(*b*) We know that $\vec{r}$ is the position vector of any arbitrary point P(*x*, *y*, *z*) on the plane.

$\therefore \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k},$

$\therefore \quad \vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$ (given)

$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$

$\Rightarrow \qquad 2x + 3y - 4z = 1$

which is the required Cartesian equation of the plane.

(*c*) Vector equation of the plane is

$$\vec{r} \cdot [(s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}] = 15 \qquad ...(i)$$

We know that $\vec{r}$ is the position vector of any point P(*x*, *y*, *z*) on plane (*i*).

$\therefore \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in (*i*), Cartesian equation of the required plane is

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot [(s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}] = 15$$

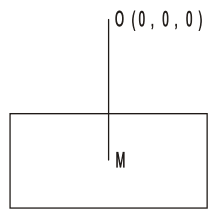
i.e., $\qquad x(s - 2t) + y(3 - t) + z(2s + t) = 15.$

4. In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.

(*a*) $2x + 3y + 4z - 12 = 0$ **(*b*) $3y + 4z - 6 = 0$**

(*c*) $x + y + z = 1$

(*d*) $5y + 8 = 0$.



Sol. (*a*) **Given:** Equation of the plane is

$$2x + 3y + 4z - 12 = 0 \qquad ...(i)$$

Given point is O(0, 0, 0)

Let M be the foot of perpendicular drawn from the origin (0, 0, 0) to plane (*i*).

$\therefore$ By definition, direction ratios of perpendicular OM to plane (*i*) are coefficients of *x*, *y*, *z* in (*i*) *i.e.,* 2, 3, 4 = *a*, *b*, *c*.

$\therefore$ Equations of perpendicular OM are

$$\frac{x-0}{2} = \frac{y-0}{3} = \frac{z-0}{4} = \lambda(\text{say}) \quad \left| \ \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}\right.$$

$\Rightarrow \quad \frac{x}{2} = \frac{y}{3} = \frac{z}{4} = \lambda \qquad \Rightarrow \quad \frac{x}{2} = \lambda, \ \frac{y}{3} = \lambda$ and $\frac{z}{4} = \lambda$

$\Rightarrow \qquad x = 2\lambda,\ y = 3\lambda,\ z = 4\lambda$

$\therefore$ Point M of this line OM is M(2λ, 3λ, 4λ) ...(*ii*)

for some real λ.

But point M lies on plane (*i*)

Putting $x = 2\lambda$, $y = 3\lambda$, $z = 4\lambda$ in (*i*), we have

$$2(2\lambda) + 3(3\lambda) + 4(4\lambda) - 12 = 0$$

$$\Rightarrow \quad 4\lambda + 9\lambda + 16\lambda = 12 \quad \Rightarrow \quad 29\lambda = 12$$

$$\Rightarrow \quad \lambda = \frac{12}{29}$$

Putting $\lambda = \frac{12}{29}$ in (*i*), foot of perpendicular $M\left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$.

(*b*) For figure, see figure of part (*a*).

Given: Equation of the plane is $3y + 4z - 6 = 0$ ...(*i*)

Given point is O(0, 0, 0)

Let M be the foot of perpendicular drawn from the origin to plane (*i*).

$\therefore$ By definition direction ratios of perpendicular OM to plane (*i*) are coefficients of x, y, z in (*i*) *i.e.,* $0, 3, 4 = a, b, c$.

$\therefore$ Equations of perpendicular OM are

$$\frac{x-0}{0} = \frac{y-0}{3} = \frac{z-0}{4} = \lambda(\text{say}) \qquad \left| \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \right.$$

$$\Rightarrow \frac{x}{0} = \frac{y}{3} = \frac{z}{4} = \lambda(\text{say}) \qquad \Rightarrow \frac{x}{0} = \lambda, \ \frac{y}{3} = \lambda \text{ and } \frac{z}{4} = \lambda$$

$$\Rightarrow \quad x = 0, \ y = 3\lambda, \ z = 4\lambda$$

$\therefore$ Point M of this line OM is $M(0, 3\lambda, 4\lambda)$ ...(*ii*)

for some real λ.

But point M lies on plane (*i*)

Putting $x = 0$, $y = 3\lambda$, $z = 4\lambda$ in (*i*), we have

$$3(3\lambda) + 4(4\lambda) - 6 = 0 \quad \text{or} \quad 9\lambda + 16\lambda = 6$$

$$\Rightarrow \quad 25\lambda = 6 \quad \Rightarrow \quad \lambda = \frac{6}{25}$$

Putting $\lambda = \frac{6}{25}$ in (*ii*), the required foot M of perpendicular is $\left(0, \frac{18}{25}, \frac{24}{25}\right)$.

(*c*) For figure, see figure of part (*a*).

Given: Equation of the plane is

$$x + y + z = 1 \qquad \text{...}(i)$$

Given point is O(0, 0, 0)

Let M be the foot of perpendicular drawn from the origin (0, 0, 0) to plane (*i*).

$\therefore$ By definition direction ratios of perpendicular OM to plane (*i*) are coefficients of x, y, z in (*i*) *i.e.,* $1, 1, 1 = a, b, c$.

$\therefore$ Equations of perpendicular OM are

$$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{1} \qquad \left| \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \right.$$

i.e., $\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = \lambda(\text{say})$ $\therefore$ $\frac{x}{1} = \lambda, \frac{y}{1} = \lambda$ and $\frac{z}{1} = \lambda$

$\Rightarrow$ $x = \lambda, y = \lambda, z = \lambda$

$\therefore$ Point M of line OM is M($\lambda, \lambda, \lambda$) ...(*ii*)

for some real λ.

But point M lies on plane (*i*)

Putting $x = \lambda, y = \lambda, z = \lambda$ in (*i*), we have

$$\lambda + \lambda + \lambda = 1 \Rightarrow 3\lambda = 1 \Rightarrow \lambda = \frac{1}{3}$$

Putting $\lambda = \frac{1}{3}$ in (*ii*), required foot M of perpendicular is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

(*d*) For figure, see figure of part (*a*).

Given: Equation of the plane is

$$5y + 8 = 0 \quad ...(i)$$

Given point is O(0, 0, 0)

Let M be the foot of perpendicular drawn from the origin (0, 0, 0) to plane (*i*).

$\therefore$ By definition, direction ratios of perpendicular OM to plane (*i*) are coefficients of x, y, z in (*i*) *i.e.,* 0, 5, 0 = a, b, c.

$\therefore$ Equations of perpendicular OM are

$$\frac{x-0}{0} = \frac{y-0}{5} = \frac{z-0}{0} \quad \left| \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \right.$$

i.e., $\frac{x}{0} = \frac{y}{5} = \frac{z}{0} = \lambda(\text{say})$ $\therefore$ $\frac{x}{0} = \lambda, \frac{y}{5} = \lambda$ and $\frac{z}{0} = \lambda$

$\Rightarrow$ $x = 0, y = 5\lambda, z = 0$

$\therefore$ Point M of line OM is M(0, 5λ, 0) ...(*ii*)

for some real λ.

But point M lies on plane (*i*)

Putting $x = 0, y = 5\lambda$ and $z = 0$ in (*i*), we have

$$5(5\lambda) + 8 = 0 \quad \text{or} \quad 25\lambda = -8$$

$$\Rightarrow \quad \lambda = -\frac{8}{25}$$

Putting $\lambda = -\frac{8}{25}$ in (*i*), required foot M of perpendicular is $\left(0, \frac{-40}{25}, 0\right) = \left(0, \frac{-8}{5}, 0\right)$.

5. Find the vector and cartesian equations of the planes

(*a*) that passes through the point (1, 0, – 2) and the normal to the plane is $\hat{i} + \hat{j} - \hat{k}$.

(*b*) that passes through the point (1, 4, 6) and the normal vector to the plane is $\hat{i} - 2\hat{j} + \hat{k}$.

Sol. (*a*) *Vector form of equation of the plane*

The given point on the plane is (1, 0, – 2)

∴ The position vector of the given point is

$$\vec{a} = (1, 0, -2) = \hat{i} + 0\hat{j} - 2\hat{k} = \hat{i} - 2\hat{k}$$

Also Given: Normal vector to the plane is

$$\vec{n} = \hat{i} + \hat{j} - \hat{k}$$

∴ Vector equation of the required plane is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0 \quad i.e., \quad \vec{r} \cdot \vec{n} - \vec{a} \cdot \vec{n} = 0$$

i.e., $$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

Putting values of $\vec{a}$ and $\vec{n}$,

$$\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = (\hat{i} - 2\hat{k}) \cdot (\hat{i} + \hat{j} - \hat{k})$$

i.e., $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 1(1) + 0(1) + (-2)(-1) = 1 + 2 = 3$

i.e., $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 3$

Cartesian form of equation of the plane

The plane passes through the point $(1, 0, -2) = (x_1, y_1, z_1)$

Normal vector to the plane is $\vec{n} = \hat{i} + \hat{j} - \hat{k}$

∴ Direction ratios of normal to the plane are coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{n}$ *i.e.*, 1, 1, – 1.

∴ Cartesian equation of the required plane is

$$\boldsymbol{a(x - x_1) + b(y - y_1) + c(z - z_1) = 0}$$

or $$1(x - 1) + 1(y - 0) - (z + 2) = 0$$

i.e., $$x - 1 + y - z - 2 = 0$$

i.e., $$x + y - z = 3.$$

(*b*) **Vector form of the equation of the plane**

The given point on the plane is (1, 4, 6).

∴ The position vector of the given point is

$$\vec{a} = (1, 4, 6) = \hat{i} + 4\hat{j} + 6\hat{k}$$

Also **Given:** normal vector to the plane is $\vec{n} = \hat{i} - 2\hat{j} + \hat{k}$.

∴ Equation of the plane is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$

or $\vec{r} \cdot \vec{n} - \vec{a} \cdot \vec{n} = 0$ *i.e.*, $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

Putting values of $\vec{a}$ and $\vec{n}$

$$\vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) = (\hat{i} + 4\hat{j} + 6\hat{k}) \cdot (\hat{i} - 2\hat{j} + \hat{k})$$
$$= 1 - 8 + 6 = -1 \quad ...(i)$$

Cartesian Form

The plane passes through the point $(1, 4, 6) = (x_1, y_1, z_1)$.

Normal vector to the plane is $\vec{n} = \hat{i} - 2\hat{j} + \hat{k}$.

$\therefore$ D.R.'s of the normal to the plane are coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ in $\vec{n}$

i.e., $1, -2, 1 = a, b, c$

$\therefore$ Equation of the required plane is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

or $$1(x - 1) - 2(y - 4) + 1(z - 6) = 0$$

or $$x - 1 - 2y + 8 + z - 6 = 0$$

or $$x - 2y + z + 1 = 0$$

Alternatively for Cartesian form

From eqn. (*i*), $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} - 2\hat{j} + \hat{k}) = -1$

or $x - 2y + z = -1$ or $x - 2y + z + 1 = 0$.

6. Find the equations of the planes that passes through three points:

(*a*) (1, 1, – 1), (6, 4, – 5), (– 4, – 2, 3)

(*b*) (1, 1, 0), (1, 2, 1), (– 2, 2, – 1)

Sol. We know that through three collinear points A, B, C *i.e.,* through a straight line, we can pass an infinite number of planes.

(*a*) The three given points are

$$A(1, 1, -1), B(6, 4, -5), C(-4, -2, 3)$$

Let us examine whether these points are collinear.

Direction ratios of line AB are

$$6 - 1, 4 - 1, -5 + 1 \quad | \; x_2 - x_1, y_2 - y_1, z_2 - z_1$$

$$= 5, 3, -4 = a_1, b_1, c_1$$

Again direction ratios of line BC are

$$-4 - 6, -2 - 4, 3 - (-5) = -10, -6, 8 = a_2, b_2, c_2$$

Here $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{5}{-10} = \frac{3}{-6} = -\frac{4}{8}$

$\Rightarrow -\frac{1}{2} = -\frac{1}{2} = -\frac{1}{2}$ which is true.

$\therefore$ Lines AB and BC are parallel.

But B is their common point.

$\therefore$ Points A, B and C are collinear and hence an infinite number of planes can be drawn through the three given collinear points.

(*b*) The three given points are

$$A(1, 1, 0) = (x_1, y_1, z_1), B(1, 2, 1) = (x_2, y_2, z_2)$$

and $C(-2, 2, -1) = (x_3, y_3, z_3)$

Let us examine whether these points are collinear.

Direction ratios of line AB are

$$1 - 1, 2 - 1, 1 - 0 \quad | \; x_2 - x_1, y_2 - y_1, z_2 - z_1$$

i.e., $0, 1, 1 = a_1, b_1, c_1$

Direction ratios of line BC are

$$-2 - 1, 2 - 2, -1 - 1 = -3, 0, -2 = a_2, b_2, c_2$$

Here $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{0}{-3} = \frac{1}{0} = \frac{1}{-2}$

which is not true.

∴ Points A, B, C are not collinear.

∴ Equation of the unique plane passing through these three points A, B, C is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z-0 \\ 1-1 & 2-1 & 1-0 \\ -2-1 & 2-1 & -1-0 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

Expanding along first row,

$$(x - 1)(-1 - 1) - (y - 1)(0 + 3) + z(0 + 3) = 0$$

$\Rightarrow$ $-2(x - 1) - 3(y - 1) + 3z = 0$

$\Rightarrow$ $-2x + 2 - 3y + 3 + 3z = 0$

$\Rightarrow$ $-2x - 3y + 3z + 5 = 0$

$\Rightarrow$ $2x + 3y - 3z - 5 = 0$

or $2x + 3y - 3z = 5$

which is the equation of required plane.

7. Find the intercepts cut off by the plane $2x + y - z = 5$.

Sol. Equation of the plane is $2x + y - z = 5$

Dividing every term by 5, (to make R.H.S. 1)

$$\frac{2x}{5} + \frac{y}{5} - \frac{z}{5} = 1 \text{ or } \frac{x}{\left(\frac{5}{2}\right)} + \frac{y}{5} + \frac{z}{-5} = 1$$

Comparing with intercept form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, we have

$a = \frac{5}{2}$, $b = 5$, $c = -5$ which are the intercepts cut off by the plane on x-axis, y-axis and z-axis respectively.

8. Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOX plane.

Sol. We know that equation of ZOX plane is $y = 0$.

∴ Equation of any plane parallel to ZOX plane is $y = k$ ...(*i*)

(∵ Equation of any plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$ *i.e.,* change only the constant term)

To find *k*. Plane (*i*) makes an intercept 3 on the y-axis ($\Rightarrow$ $x = 0$ and $z = 0$) *i.e.,* plane (*i*) passes through (0, 3, 0).

Putting $x = 0$, $y = 3$ and $z = 0$ in (*i*), $3 = k$.

Putting $k = 3$ in (*i*), equation of required plane is $y = 3$.

9. Find the equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and the point (2, 2, 1).

Sol. Equations of the given planes are

$$3x - y + 2z - 4 = 0 \quad \text{and} \quad x + y + z - 2 = 0$$

(Here R.H.S. of each equation is already zero)

We know that equation of any plane through the intersection of these two planes is

L.H.S. of plane I + λ(L.H.S. of plane II) = 0

i.e., $3x - y + 2z - 4 + \lambda(x + y + z - 2) = 0$ ...(*i*)

To find λ. Given: Required plane (*i*) passes through the point (2, 2, 1).

Putting $x = 2$, $y = 2$ and $z = 1$ in (*i*),

$$6 - 2 + 2 - 4 + \lambda(2 + 2 + 1 - 2) = 0$$

or $2 + 3\lambda = 0 \Rightarrow 3\lambda = -2 \Rightarrow \lambda = -\frac{2}{3}$

Putting $\lambda = -\frac{2}{3}$ in (*i*), equation of required plane is

$$3x - y + 2z - 4 - \frac{2}{3}(x + y + z - 2) = 0$$

$\Rightarrow$ $9x - 3y + 6z - 12 - 2x - 2y - 2z + 4 = 0$

$\Rightarrow$ $7x - 5y + 4z - 8 = 0.$

10. Find the vector equation of the plane passing through the intersection of the planes $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$, $\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and through the point (2, 1, 3).

Sol. Vector equation of first plane is

$\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$ i.e $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$

i.e. $2x + 2y - 3z - 7 = 0$ (making R.H.S. zero) ...(*i*)

Vector equation of second plane is

$\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ i.e $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$

i.e. $2x + 5y + 3z - 9 = 0$ (making R.H.S. zero) ...(*ii*)

We know that equation of any plane passing through the line of intersection of planes (*i*) and (*ii*) is

L.H.S of (*i*) + λ L.H.S of (*ii*) = 0

i.e. $2x + 2y - 3z - 7 + \lambda\ (2x + 5y + 3z - 9) = 0$

i.e. $2x + 2y - 3z - 7 + 2\lambda x + 5\lambda y + 3\lambda z - 9\lambda = 0$

i.e. $(2 + 2\lambda)\, x + (2 + 5\lambda)\, y + (-3 + 3\lambda)\, z = 7 + 9\lambda$ ...(*iii*)

To find λ : Given plane (*iii*) passes through the point (2,1,3) putting $x = 2$, $y = 1$, $z = 3$ in (*iii*),

$(2 + 2\lambda)\, 2 + (2 + 5\lambda)\, 1 + (-3 + 3\lambda)\, 3 = 7 + 9\lambda$

or $4 + 4\lambda + 2 + 5\lambda - 9 + 9\lambda = 7 + 9\lambda$

$9\lambda - 3 = 7 \Rightarrow 9\lambda = 10 \Rightarrow \lambda = \frac{10}{9}$

Putting $\lambda = \frac{10}{9}$ in (*iii*), equation of required plane is

$$\left(2 + \frac{20}{9}\right)x + \left(2 + \frac{50}{9}\right)y + \left(-3 + \frac{30}{9}\right)z = 7 + 10$$

or $\frac{38}{9}x + \frac{68}{9}y + \frac{3}{9}z = 17$

Multiplying by L.C.M. = 9, $38x + 68y + 3z = 153$

or $x\ (38) + y\ (68) + z\ (3) = 153$

or $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (38\hat{i} + 68\hat{j} + 3\hat{k}) = 153$

i.e. $\vec{r} \cdot (38\hat{i} + 68\hat{j} + 3\hat{k}) = 153$

which is the required vector equation of the plane.

11. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$.

Sol. Equations of the given planes are

$x + y + z = 1$ and $2x + 3y + 4z = 5$

Making R.H.S. zero, equations of the planes are

$x + y + z - 1 = 0$ and $2x + 3y + 4z - 5 = 0$.

We know that equation of any plane through the intersection of the two planes is

(L.H.S. of I) + λ(L.H.S. of II) = 0

i.e., $x + y + z - 1 + \lambda(2x + 3y + 4z - 5) = 0$ ...(*i*)

i.e., $x + y + z - 1 + 2\lambda x + 3\lambda y + 4\lambda z - 5\lambda = 0$

i.e., $(1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z - 1 - 5\lambda = 0$

Given: This plane is perpendicular to the plane

$x - y + z = 0$

$\therefore$ $\boldsymbol{a_1a_2 + b_1b_2 + c_1c_2 = 0}$

i.e., Product of coefficients of $x + ... = 0$

$\therefore$ $(1 + 2\lambda) - (1 + 3\lambda) + 1 + 4\lambda = 0$

$\Rightarrow$ $1 + 2\lambda - 1 - 3\lambda + 1 + 4\lambda = 0 \Rightarrow 3\lambda + 1 = 0 \Rightarrow 3\lambda = -1$

$\Rightarrow$ $\lambda = \frac{-1}{3}$

Putting $\lambda = \frac{-1}{3}$ in (*i*), equation of required plane is

$$x + y + z - 1 - \frac{1}{3}(2x + 3y + 4z - 5) = 0$$

Multiplying by L.C.M. = 3,

$3x + 3y + 3z - 3 - 2x - 3y - 4z + 5 = 0 \Rightarrow x - z + 2 = 0.$

12. Find the angle between the planes whose vector equations are

$\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5$ and $\vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3$.

Sol. Equation of one plane is

$$\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5 \quad ...(i)$$

Comparing (*i*) with $\vec{r} \cdot \vec{n_1} = d_1$, we have

normal vector to plane (*i*) is $\vec{n_1} = 2\hat{i} + 2\hat{j} - 3\hat{k}$

Equation of second plane is $\vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3 \quad ...(ii)$

Comparing (*ii*) with $\vec{r} \cdot \vec{n_2} = d_2$, we have

normal vector to plane (*ii*) is $\vec{n_2} = 3\hat{i} - 3\hat{j} + 5\hat{k}$

Let θ be the **acute** angle between planes (*i*) and (*ii*).

∴ By definition, angle between normals $\vec{n_1}$ and $\vec{n_2}$ to planes (*i*) and (*ii*) is also θ.

$$\therefore \quad \cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}||\vec{n_2}|} = \frac{|2(3) + 2(-3) + (-3)5|}{\sqrt{4+4+9}\sqrt{9+9+25}}$$

$$= \frac{|6 - 6 - 15|}{\sqrt{17}\sqrt{43}} = \frac{|-15|}{\sqrt{17 \times 43}} = \frac{15}{\sqrt{731}} \therefore \theta = \cos^{-1}\frac{15}{\sqrt{731}}.$$

13. In the following cases, determine whether the given planes are parallel or perpendicular and in case they are neither, find the angle between them.

(*a*) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$
(*b*) $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$
(*c*) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$
(*d*) $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$
(*e*) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$.

Sol. (*a*) Equations of the given planes are

$$7x + 5y + 6z + 30 = 0$$

$(a_1x + b_1y + c_1z + d_1 = 0)$

and $3x - y - 10z + 4 = 0 \quad (a_2x + b_2y + c_2z + d_2 = 0)$

Here $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ becomes $\frac{7}{3} = \frac{5}{-1} = \frac{6}{-10}$ which is not true.

∴ The two planes are not parallel.

Again $a_1a_2 + b_1b_2 + c_1c_2 = 21 - 5 - 60 = 21 - 65 = -44 \neq 0$

∴ Planes are not perpendicular.

Now let θ be the angle between the two planes.

$$\therefore \cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{|7(3) + 5(-1) + 6(-10)|}{\sqrt{(7)^2 + (5)^2 + (6)^2}\sqrt{(3)^2 + (-1)^2 + (-10)^2}}$$

$$= \frac{|21 - 5 - 60|}{\sqrt{49 + 25 + 36}\,\sqrt{9 + 1 + 100}} = \frac{|-44|}{\sqrt{110}\,\sqrt{110}}$$

$$= \frac{|-44|}{110} = \frac{44}{110} = \frac{2}{5} \quad \therefore \quad \theta = \cos^{-1}\left(\frac{2}{5}\right).$$

(*b*) Equations of the given planes are

$$2x + y + 3z - 2 = 0 \quad (a_1x + b_1y + c_1z + d_1 = 0)$$

and $x - 2y + 5 = 0$ *i.e.,* $x - 2y + 0.z + 5 = 0$

$$(a_2x + b_2y + c_2z + d_2 = 0)$$

Are these planes parallel?

Here $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{2}{1} = \frac{1}{-2} = \frac{3}{0}$ which is not true.

$\downarrow$

(Ratio of coefficients of x in equations of two planes)

$\therefore$ The given planes are not parallel.

Are these planes perpendicular?

$a_1a_2 + b_1b_2 + c_1c_2 = 2(1) + 1(-2) + 3(0) = 2 - 2 + 0 = 0$

$\downarrow$

(Product of coefficients of x)

$\therefore$ Given planes are perpendicular.

(*c*) Equations of the given planes are

$$2x - 2y + 4z + 5 = 0 \quad (a_1x + b_1y + c_1z + d_1 = 0)$$

and $3x - 3y + 6z - 1 = 0 \quad (a_2x + b_2y + c_2z + d_2 = 0)$

Are these planes parallel?

Here $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{2}{3} = \frac{-2}{-3} = \frac{4}{6} \Rightarrow \frac{2}{3} = \frac{2}{3} = \frac{2}{3}$

which is true.

$\therefore$ The given planes are parallel.

(*d*) Equations of the given planes are

$$2x - y + 3z - 1 = 0 \quad (a_1x + b_1y + c_1z + d_1 = 0)$$

and $2x - y + 3z + 3 = 0 \quad (a_2x + b_2y + c_2z + d_2 = 0)$

Are these planes parallel?

Here $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{2}{2} = \frac{-1}{-1} = \frac{3}{3} \Rightarrow 1 = 1 = 1$

which is true.

$\therefore$ The given planes are parallel.

(*e*) Equations of the given planes are

$$4x + 8y + z - 8 = 0 \quad (a_1x + b_1y + c_1z + d_1 = 0)$$

and $y + z - 4 = 0$ *i.e.,* $0x + y + z - 4 = 0$

$$(a_2x + b_2y + c_2z + d_2 = 0)$$

Are these planes parallel?

Here $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{4}{0} = \frac{8}{1} = \frac{1}{1}$ which is not true.

$\therefore$ The given planes are not parallel.

Are these planes perpendicular?

Here $a_1a_2 + b_1b_2 + c_1c_2 = 4(0) + 8(1) + 1(1)$

$= 0 + 8 + 1 = 9 \neq 0$ ...(*i*)

$\therefore$ The given planes are not perpendicular.

To find the **(acute)** angle θ between the given planes.

$$\therefore \quad \cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}$$

$$= \frac{|4(0) + 8(1) + 1(1)|}{\sqrt{16 + 64 + 1}\sqrt{0^2 + 1^2 + 1^2}} = \frac{|8 + 1|}{\sqrt{81}\sqrt{2}}$$

$$= \frac{9}{9\sqrt{2}} = \frac{1}{\sqrt{2}} = \cos 45° \qquad \therefore \theta = 45°.$$

14. In the following cases find the distances of each of the given points from the corresponding given plane.

Point	**Plane**
(*a*) (0, 0, 0)	**$3x - 4y + 12z = 3$**
(*b*) (3, – 2, 1)	**$2x - y + 2z + 3 = 0$**
(*c*) (2, 3, – 5)	**$x + 2y - 2z = 9$**
(*d*) (– 6, 0, 0)	**$2x - 3y + 6z - 2 = 0$.**

Sol. (*a*) Distance (of course perpendicular) of the point (0, 0, 0) from the plane $3x - 4y + 12z = 3$ or $3x - 4y + 12z - 3 = 0$ (Making R.H.S. zero) is

$$\frac{|\boldsymbol{ax_1 + by_1 + cz_1 + d}|}{\sqrt{\boldsymbol{a^2 + b^2 + c^2}}} = \frac{|3(0) - 4(0) + 12(0) - 3|}{\sqrt{(3)^2 + (-4)^2 + (12)^2}}$$

$$= \frac{|-3|}{\sqrt{9 + 16 + 144}} = \frac{3}{\sqrt{169}} = \frac{3}{13}.$$

(*b*) Length of perpendicular from the point (3, – 2, 1) on the plane $2x - y + 2z + 3 = 0$

(Substitute the point for x, y, z in L.H.S. of Eqn. of plane and divide by $\sqrt{a^2 + b^2 + c^2}$)

$$= \frac{|2(3) - (-2) + 2(1) + 3|}{\sqrt{(2)^2 + (-1)^2 + (2)^2}} = \frac{|6 + 2 + 2 + 3|}{\sqrt{4 + 1 + 4} = \sqrt{9}} = \frac{13}{3}$$

(*c*) Length of perpendicular from the point (2, 3, – 5) on the plane

$x + 2y - 2z = 9$ or $x + 2y - 2z - 9 = 0$ (Making R.H.S. zero)

$$= \frac{|2 + 2(3) - 2(-5) - 9|}{\sqrt{(1)^2 + (2)^2 + (-2)^2}} = \frac{|2 + 6 + 10 - 9|}{\sqrt{1 + 4 + 4}} = \frac{9}{\sqrt{9}} = \frac{9}{3} = 3.$$

(*d*) Distance of the point (– 6, 0, 0) from the plane

$2x - 3y + 6z - 2 = 0$

(Here R.H.S. is already zero)

$$= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|2(-6) - 3(0) + 6(0) - 2|}{\sqrt{(2)^2 + (-3)^2 + (6)^2}}$$

$$= \frac{|-12-2|}{\sqrt{4+9+36}} = \frac{|-14|}{\sqrt{49}} = \frac{14}{7} = 2.$$