Question 1: Let $f: \{1, 3, 4\} \rightarrow \{1, 2, 5\}$ and $g: \{1, 2, 5\} \rightarrow \{1, 3\}$ be given by $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Write down gof.

Solution: The functions $f: \{1, 3, 4\} \rightarrow \{1, 2, 5\}$ and $g: \{1, 2, 5\} \rightarrow \{1, 3\}$ are

$f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$.

$gof(1) = g[f(1)] = g(2) = 3$ [as $f(1) = 2$ and $g(2) = 3$]

$gof(3) = g[f(3)] = g(5) = 1$ [as $f(3) = 5$ and $g(5) = 1$]

$gof(4) = g[f(4)] = g(1) = 3$ [as $f(4) = 1$ and $g(1) = 3$]

$\therefore gof = \{(1, 3), (3, 1), (4, 3)\}$

Question 2: Let f, g and h be functions from **R to R**. Show that

$(f + g)oh = foh + goh$

$(f.g)oh = (foh).(goh)$

Solution: To prove: $(f + g)oh = foh + goh$

LHS $= [(f + g)oh](x)$

$= (f + g)[h(x)] = f[h(x)] + g[h(x)]$

$= (foh)(x) + (goh)(x)$

$= \{(foh)(x) + (goh)\}(x) =$ RHS

$\therefore \{(f + g)oh\}(x) = \{(foh)(x) + (goh)\}(x)$ for all $x \in R$

Hence, $(f+g)oh = foh + goh$

To Prove: $(f.g)oh = (foh).(goh)$

LHS $= [(f.g)oh](x)$

$= (f.g)[h(x)] = f[h(x)] \, . \, g[h(x)]$

$= (foh)(x) \, . \, (goh)(x)$

$= \{(foh).(goh)\}(x) =$ RHS

$\therefore [(f.g)oh](x) = \{(foh).(goh)\}(x)$ for all $x \in R$

Hence, $(f.g)oh = (foh).(goh)$

Question 3: Find gof and fog, if

(i) (x) = $|x|$ and (x) = $|5x-2|$

(ii) (x) = $8x^3$ and (x) = $x^{\frac{1}{3}}$

Solution: (i) f(x) = |x| and g(x) = |5x-2|

∴ go f(x) = g(f(x)) = g(|x|) = |5|x|-2|

fog(x) = f(g(x)) = f(|5x-2|) = ||5x-2|| = |5x-2|

(ii) f(x) = 8x³ and g(x) = $x^{\frac{1}{3}}$

∴ gof(x) = g(f(x)) = g(8x³) = $(8X^3)^{\frac{1}{3}}$ = 2x

fog(x) = f(g(x)) = f($x^{\frac{1}{3}}$)³ = 8($x^{\frac{1}{3}}$)³ = 8x

Question 4: If (x) = (4x+3) (6x−4), $x \neq \frac{2}{3}$, show that fof(x) = x, for all $x \neq \frac{2}{3}$. What is the inverse of f?

Solution: It is given that f(x) = $\frac{(4x+3)}{(6x-4)}$, $x \neq \frac{2}{3}$

(fof)(x) = f(f(x)) = f ($\frac{(4x+3)}{(6x-4)}$ = $\frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}$

= $\frac{16x+12+18x-12}{24x+18-24s+16} = \frac{34x}{34}$

∴fof(x) = x, for all x ≠ $\frac{2}{3}$.

⇒ fof = I_x

the given function f is invertible and the inverse of f is f itself.

Question 5: State with reason whether following functions have inverse

(i) f: {1, 2, 3, 4} → {10} with

f = {(1, 10), (2, 10), (3, 10), (4, 10)}

(ii) g: {5, 6, 7, 8} → {1, 2, 3, 4} with

g = {(5, 4), (6, 3), (7, 4), (8, 2)}

(iii) h: {2, 3, 4, 5} → {7, 9, 11, 13} with

h = {(2, 7), (3, 9), (4, 11), (5, 13)}

Solution: (i) f: {1, 2, 3, 4} → {10} defined as f = {(1, 10), (2, 10), (3, 10), (4, 10)}

f is a many one function as

f(1) = f(2) = f(3) = f(4) = 10

∴f is not one – one.

function f does not have an inverse.

(ii) g: {5, 6, 7, 8} → {1, 2, 3, 4} defined as

g = {(5, 4), (6, 3), (7, 4), (8, 2)}

g is a many one function as g(5) = g(7) = 4.

∴ g is not one – one.

g does not have an inverse.

(iii) h: {2, 3, 4, 5} → {7, 9, 11, 13} defined as

h = {(2, 7), (3, 9), (4, 11), (5, 13)}

all distinct elements of the set {2, 3, 4, 5} have distinct images under h.

∴ Function h is one – one.

h is onto since for every element y of the set {7, 9, 11, 13}, there exists an element x in the set {2, 3, 4, 5}, such that h(x) = y.

h is a one – one and onto function.

h has an inverse.

Question 6: Show that f: [−1, 1] → R, given by(x) = $X(X$+2) is one – one. Find the inverse of the function f: [−1, 1] → Range f.

(Hint: For y ∈ Range f, y = (x) = $(X$+2)) , for some x in [−1, 1], i.e., x = 2y(1−y))

Solution: f: [−1, 1] → R is given as (x) = $X(X$+2)

For one – one

f(x) = f(y)

⇒ $(X$+2) = $(Y$+2)

⇒ xy +2x = xy +2y

⇒ 2x = 2y

⇒ x = y

∴ f is a one – one function.

f: [−1, 1] → Range f is onto.

∴ f: [−1, 1] → Range f is one – one and onto and therefore, the inverse of the function f: [−1, 1] → Range f exists.

g: Range f → [−1, 1] be the inverse of f.

y be an arbitrary element of range f.

Since f: [−1, 1] → Range f is onto,

y = f(x) for some x ∈ [−1, 1]

$$\Rightarrow y = \frac{x}{(x+2)}$$

$$\Rightarrow xy + 2y = x$$

$$\Rightarrow x(1-y) = 2y$$

$$\Rightarrow x - \frac{2y}{1-y}, y \neq 1$$

define g: Range f → [−1, 1] as

$$g(y)=\frac{2y}{1-y}, y\neq 1$$

$$(gof)(x)=g(f(x))=g(\frac{x}{(x+2)})=2(\frac{2(\frac{x}{x+2})}{1-(\frac{x}{x+2})})=\frac{2x}{x+2-x}=\frac{2x}{2}=\text{x}$$

and

$$(fog)(y)=f(g(y))=f(\frac{2y}{1-y})=\frac{\frac{2y}{1-y}}{\frac{2y}{1-y}+2}=\frac{2y}{2y+2-2y}=\frac{2y}{2}=$$

∴ $gof = \text{x} = I_{[-1,1]}$ and fog = y = $I_{\text{Range f}}$

∴ f^{-1} = g

$$\Rightarrow f^{-1}(y)=\frac{2y}{1-y}, y\neq 1$$

Question 7: Consider f: R → R given by f(x) = 4x + 3. Show that f is invertible. Find the inverse of f.

Solution: f: R → R is given by, f(x) = 4x + 3

For one – one

f(x) = f(y)

⇒ 4x + 3 = 4y + 3

⇒ 4x = 4y

⇒ x = y

∴ f is a one – one function

For onto

y ∈ R, let y = 4x + 3.

⇒ x = $\frac{y-3}{4}$ ∈ R

for any y ∈ R, there exists x = $\frac{y-3}{4}$ ∈ R, such that

f(x) = f$\left(\frac{y-3}{4}\right)$ = 4$\left(\frac{y-3}{4}\right)$ + 3 = y.

∴ f is onto.

f is one – one and onto and therefore, f^{-1} exists.

define g: R → R by g(x) = $\frac{y-3}{4}$

(gof)(x) = g(f(x)) = g(4x + 3) = $\frac{(4x+3)-3}{4} = \frac{4x}{4}$ = x

and

(fog)(y) = f(g(y)) = f ($\frac{y-3}{4}$) = 4 ($\frac{y-3}{4}$) + 3 = y-3 + 3 = y

∴ gof = fog = I_R

f is invertible and the inverse of f is given by f^{-1} (y) = g(y) = $\frac{y-3}{4}$.

Question 8: Consider f: $\boldsymbol{R}+$ → [4, ∞) given by f(x) = x^2 + 4. Show that f is invertible with the inverse $f-1$ of given f by$f^{-1}(y)$ = $\sqrt{Y-4}$, where $\boldsymbol{R}+$ is the set of all non-negative real numbers.

Solution: f: $\boldsymbol{R}_+$ → [4, ∞) is given as f(x) = x^2 + 4.

For one – one

f(x) = f(y)

⇒ x^2 + 4 = y^2 + 4

⇒ x^2 = y^2

⇒ x = y [$as\ x = y \in \boldsymbol{R}_+$]

∴ f is a one – one function.

For onto

$y \in [4, \infty)$, let $y = x^2 + 4$

$\Rightarrow x^2 = y - 4 \geq 0$ [as $y \geq 4$]

$\Rightarrow x = \sqrt{Y-4} \geq 0$

for any $y \in [4, \infty)$, there exists $x = \sqrt{Y-4} \in R_+$, such that

$f(x) = f(\sqrt{Y-4}) = (\sqrt{Y-4})^2 + 4 = y - 4 + 4 = y$

∴ f is onto.

f is one – one and onto and therefore, f^{-1} exists.

Let us define g: $[4, \infty) \to R+$ by $g(y) = \sqrt{Y-4}$

$(gof)(x) = g(f(x)) = g(x^2 + 4) = \sqrt{(x^2+4)-4} = \sqrt{\sqrt{X^2}} = x$

and

$(fog)(y) = f(g(y)) = f(\sqrt{Y-4}) = (\sqrt{Y-4})^2 + 4 = y - 4 + 4 = y$

∴ $gof = fog = I_R$

f is invertible and the inverse of f is given by $f^{-1}(y) = g(y) = \sqrt{Y-4}$

Question 9: Consider f: $\boldsymbol{R}_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{\left(\sqrt{y+6}\right)-1}{3}\right)$

Solution: f: $R_+ \to [-5, \infty)$ is given as $f(x) = 9x^2 + 6x - 5$.

y be an arbitrary element of $[-5, \infty)$.

$y = 9x^2 + 6x - 5$

$\Rightarrow y = (3x + 1)^2 - 1 - 5 = (3x + 1)^2 - 6$

$\Rightarrow y + 6 = (3x + 1)^2$

$\Rightarrow 3x + 1 = \sqrt{Y+6}$ [as $y \geq -5 \Longrightarrow y + 6 > 0$]

$\Rightarrow$ x = $\left(\dfrac{\left(\sqrt{y+6}\right)-1}{3}\right)$

∴ f is onto, range f = [−5, ∞).

define g: [−5, ∞) → R+ as g(y) = $\left(\dfrac{\left(\sqrt{y+6}\right)-1}{3}\right)$

(gof)(x) = g(f(x)) = g($9x^2$ + 6x-5) = g($(3x + 1)^2$ -6)

= $\sqrt{(3x+1)^2-6+6}-1$

= $\dfrac{3x+1-1}{3} = \dfrac{3x}{3} = X$

and

(fog)(y) = f(g(y)) = $\left(\dfrac{\sqrt{Y+6}-1}{3}\right) = \left[3\left(\dfrac{\sqrt{Y+6}-1}{3}\right)+1^2\right]$ – 6

= $\left(\sqrt{Y+6}\right)^2-6$ = + 6 – 6 = y

∴ gof = x = I_R and fog = y = $l_{Range\ f}$

f is invertible and the inverse of f is given by

$f^{-1}(y)$ = g(y) = $\left(\dfrac{\sqrt{Y+6}-1}{3}\right)$

Question 10: Let f: X → Y be an invertible function. Show that f has unique inverse.

(Hint: suppose g_1 and g_2 are two inverses of f. Then for all y ∈ Y, $fog_1(y) = I_Y(y) = fog_2(y)$. Use one – one ness of f).

Solution: Let f: X → Y be an invertible function.

suppose f has two inverses (g_1 and g_2)

for all y ∈Y,

$fog1(y) = I_Y(y) = fog2(y)$

⇒ $f(g_1 (y)) = f(g_2 (y))$

⇒ $g_1 (y) = g_2 (y)$ [as f is invertible ⇒ f is one – one]

⇒ $g_1 = g_2$ [as g is one – one]

f has a unique inverse.

Question 11: Consider f: {1, 2, 3} → {a, b, c} given by f(1) = a, f(2) = b and f(3) = c. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

Solution: Function f: {1, 2, 3} → {a, b, c} is given by f(1) = a, f(2) = b, and f(3) = c

If we define g: {a, b, c} → {1, 2, 3} as g(a) = 1, g(b) = 2, g(c) = 3.

We have

(fog)(a) = f(g(a)) = f(1) = a

(fog)(b) = f(g(b)) = f(2) = b

(fog)(c) = f(g(c)) = f(3) = c

and

(gof)(1) = g(f(1)) = f(a) = 1

(gof)(2) = g(f(2)) = f(b) = 2

(gof)(3) = g(f(3)) = f(c) = 3

∴ gof = I_X and fog = I_Y, where X = {1, 2, 3} and Y= {a, b, c}.

inverse of f exists and f−1 = g.

∴ f^{-1} :{ a, b, c} → {1, 2, 3} is given by f^{-1} (a) = 1, f^{-1} (b) = 2, f^{-1} (c) = 3

If we define h: {1, 2, 3} → {a, b, c} as h (1) = a, h (2) = b, h (3) = c

(goh)(1) = g(h(1)) = g(a) = 1

(goh)(2) = g(h(2)) = g(b) = 2

(goh)(3) = g(h(3)) = g(c) = 3

and

(hog)(a) = h(g(a)) = h(1) = a

(hog)(b) = h(g(b)) = h(2) = b

(hog)(c) = h(g(c)) = h(3) = c

∴ goh = I_X and hog = I_Y, where X = {1, 2, 3} and Y = {a, b, c}.

the inverse of g exists and $g^{-1} = h \Rightarrow (f^{-1})^{-1} = h$.

h = f.

Hence, $(f^{-1})^{-1} = f$

Question 12: Let f: X → Y be an invertible function. Show that the inverse of f^{-1} is f,

i.e., $(f^{-1})^{-1} = f$.

Solution: Let f: X → Y be an invertible function.

there exists a function g: Y → X such that gof = I_X and fog = I_Y.

$f^{-1} = g$.

gof = I_X and fog = I_Y

⇒ f^{-1} of = I_X and fof−1 = I_Y

f^{-1} : Y → X is invertible and f is the inverse of f^{-1} i.e., $(f^{-1})^{-1} = f$

Question 13: If f: R → R be given by $(x) = (3-x^3)^{\frac{1}{3}}$, then fof (x) is

(A) $x^{\frac{1}{3}}$

(B) x^3

(C) X

(D) $(3 - x^3)$

Solution: f: R → R as $f(x) = (3-x^3)^{\frac{1}{3}}$

$\therefore$ fof(x) = f(f(x)) = $f(3-x^3)^{\frac{1}{3}}$ = $[3-((3-x^3)^{\frac{1}{3}})^3]^{\frac{1}{3}}$

= $[3-(3-x^3)]^{\frac{1}{3}}$ =$(x^3)^{\frac{1}{3}}$

$\therefore$ fof(x) = x

The correct answer is C.

Question 14: : Let f: R- {- $\frac{4}{3}$ } → R be a function as f(x) = $\frac{4x}{3x+4}$. The inverse of f is map g: Range f → R- {- $\frac{4}{3}$ } given by

(A) g(y) = $\frac{3y}{3-4y}$

(B) g(y) = $\frac{4y}{4-3y}$

(C) g(y) = $\frac{4y}{4-3y}$

(D) g(y) = $\frac{3y}{4-3y}$

Solution: $f:\boldsymbol{R}-$ {- $\frac{4}{3}$ } → $\boldsymbol{R}$ be a function as f(x) = $\frac{4x}{3x+4}$

y be an arbitrary element of Range f.

there exists x ∈ R- {- $\frac{4}{3}$ } such that y = f(x)

⇒ y = $\frac{4x}{3x+4}$

⇒ 3xy + 4y = 4x

⇒ x (4-3y) = 4y

$\Rightarrow$ x = $\frac{4y}{4-3y}$

define g: Range f → R− {- $\frac{4}{3}$ } as g(y) = $\frac{4y}{4-3y}$

gof(x) = $g(f(x))$ = $g(\frac{4x}{3x+4}) = \frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4x}{3x+4}\right)}$

= $\frac{16x}{12x+16-12x} = \frac{16x}{16} = x$

and

fo(y) = $f(g(y))$ = $f(\frac{4y}{4-3y}) = \frac{4\left(\frac{4y}{4-3y}\right)}{3\left(\frac{4y}{4-3y}\right)+4}$

= $\frac{16y}{12y+16-12x} = \frac{16y}{16} = y$

∴ gof = $I_{R-\left\{\frac{4}{3}\right\}}$ and fog = $I_{Range\ f}$

g is the inverse of f i.e., f − 1 = g.

the inverse of f is the map g: Range f → R- $\left\{-\frac{4}{3}\right\}$, which is given by g(y) = $\frac{4y}{4-3y}$.

The correct answer is B.