Question 1: Show that the function f: $\mathbf{R}_*$ → $\mathbf{R}_*$ defined by $(x) = \frac{1}{x}$ is one-one and onto, where $\mathbf{R}_*$ is the set of all non-zero real numbers. Is the result true, if the domain $\mathbf{R}_*$ is replaced by N with co-domain being same as $\mathbf{R}_*$?

Solution:

f: R* → R_* is by f(x) = $\frac{1}{x}$

For one – one:

x, y ∈ R∗ such that f(x) = f(y)

⇒ $\frac{1}{x} = \frac{1}{y}$

⇒ x = y

∴ f is one – one.

For onto:

for y ∈ R∗, there exists x = $\frac{1}{y}$ ∈ R∗ [as y ≠ 0] such that

f(x) = $\frac{1}{\left(\frac{1}{y}\right)}$ = y

∴ f is onto.

given function f is one – one and onto.

consider function g: N → R_* defined by g(x) = $\frac{1}{x}$

We have, g(x_1) = $g(x_2)$ ⇒ = $\frac{1}{x_1} = \frac{1}{x_2}$ ⇒ $x_1 = x_2$

∴ g is one – one.

g is not onto as for 1.2 ∈ = R_* there does not exit any x in N such that g(x) = $\frac{1}{1.2}$.

function g is one-one but not onto.

Question 2: Check the injectivity and surjectivity of the following functions:

(i) f: N → N given by f(x) = x^2

(ii) f: Z → Z given by f(x) = x^2

(iii)f: R → R given by f(x) = x^2

(iv) f: N → N given by f(x) = x^3

(v) f: Z → Z given by f(x) = x^3

Solution:

(i) f: N → N is f(x) = x^2

for x, y ∈ N, f(x) = f(y) ⇒ $x^2 = y^2$ ⇒ x = y.

∴ f is injective.

2 ∈ N. But, there does not exist any x in N such that f(x) = x^2 = 2.

∴ f is not surjective.

function f is injective but not surjective.

(ii) f: Z → Z is given by f(x) = x^2

f(−1) = f(1) = 1, but −1 ≠ 1.

∴ f is not injective.

, −2 ∈ Z. But, there does not exist any element x ∈ Z such that

f(x) = −2 or x^2 = −2.

∴ f is not surjective.

function f is neither injective nor surjective.

(iii)f: R → R is given by f(x) = x2

f(−1) = f(1) = 1, but −1 ≠ 1.

∴ f is not injective.

−2 ∈ R. But, there does not exist any element x ∈ R such that f(x) = −2

or $x^2 = -2$.

∴ f is not surjective.

function f is neither injective nor surjective.

(iv) f: N → N given by $f(x) = x^3$

for x, y ∈ N, f(x) = f(y) ⇒ $x^3 = y^3$ ⇒ x = y.

∴ f is injective.

2 ∈ N. But, there does not exist any element x ∈ N such that

f(x) = 2 or $x^3 = 2$.

∴ f is not surjective

function f is injective but not surjective.

(v) f: Z → Z is given by $f(x) = x^3$

for x, y ∈ Z, f(x) = f(y) ⇒ $x^3 = y^3$ ⇒ x = y.

∴ f is injective.

2 ∈ Z. But, there does not exist any element x ∈ Z such that

f(x) = 2 or $x^3 = 2$.

∴ f is not surjective.

function f is injective but not surjective.

Question 3: Prove that the Greatest Integer Function f: R → R given by f(x) = [x], is neither one – one nor onto, where [x] denotes the greatest integer less than or equal to x.

Solution: f: R → R is, f(x) = [x]

f(1.2) = [1.2] = 1, f(1.9) = [1.9] = 1.

∴ f(1.2) = f(1.9), but 1.2 ≠ 1.9.

∴ f is not one – one.

consider 0.7 ∈ R.

f(x) = [x] is an integer. there does not exist any element x ∈ R such that f(x) = 0.7.

∴ f is not onto.

the greatest integer function is neither one – one nor onto.

Question 4: In Show that the Modulus Function f: R → R given by (x) = $|x|$, is neither one – one nor onto, where $|x|$ is x, if x is positive or 0 and |X| is − x, if x is negative.

Solution: f: R → R is f(x) = |x| = { $\begin{cases} X & \text{if } X \geq 0 \\ -X & \text{if } X < 0 \end{cases}$

f(-1) = |-1| = 1 and f(1) = |1| = 1

∴ f(−1) = f(1), but −1 ≠ 1.

∴ f is not one – one.

consider −1 ∈ R.

f(x) = $|x|$ is non-negative. there does not exist any element x in domain R such that f(x) = $|x|$ = −1.

∴ f is not onto.

the modulus function is neither one-one nor onto.

Question 5: Show that the Signum Function f: R → R, given by f(x) = $\begin{cases} 1 & \text{if X>0} \\ 0, & \text{if } X = 0 \\ -1, & \text{if } X < 0 \end{cases}$

is neither one-one nor onto.

Solution: f: R → R is f(x) == $\begin{cases} 1 & \text{if X>0} \\ 0, & \text{if } X = 0 \\ -1, & \text{if } X < 0 \end{cases}$

f(1) = f(2) = 1, but 1 ≠ 2.

∴ f is not one – one.

f(x) takes only 3 values (1, 0, or −1) for the element −2 in co-domain

R, there does not exist any x in domain R such that f(x) = −2.

∴ f is not onto.

the Signum function is neither one – one nor onto

Question 6: Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that f is one – one.

Solution: It is given that A = {1, 2, 3}, B = {4, 5, 6, 7}.

f: A → B is defined as f = {(1, 4), (2, 5), (3, 6)}.

∴ f (1) = 4, f (2) = 5, f (3) = 6

It is seen that the images of distinct elements of A under f are distinct.

function f is one – one.

Question 7: In each of the following cases, state whether the function is one – one, onto or bijective. Justify your answer.

(i) f: R → R defined by f(x) = 3 − 4x

(ii) f: R → R defined by f(x) = 1 + x2

Solution: (i) f: R → R is f(x) = 3 − 4x.

$x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow 3-4x_1 = 3-4x_2$

$\Rightarrow -4x_1 = -4x_2$

$\Rightarrow x_1 = x_2$

∴ f is one – one.

For any real number (y) in R, there exists $\frac{3-y}{4}$ in R such that f $(\frac{3-y}{4})$ = 3-4 $(\frac{3-y}{4})$ = y

∴ f is onto.

f is bijective.

(ii) f: R → R is defined as $f(x) = 1 + x^2$

x^1, $x^2 \in R$ such that $f(x1) = f(x^2)$

$\Rightarrow 1 + X_1^2 = 1 + x_2^2$

$\Rightarrow + X_1^2 = X_2^2$

$\Rightarrow x_1 = \pm x_2$

∴ $f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$

f (1) = f (-1) = 2

∴ f is not one – one.

Consider an element −2 in co-domain R.

$f(x) = 1 + x^2$ is positive for all $x \in R$.

there does not exist any x in domain R such that f(x) = −2.

∴ f is not onto.

f is neither one – one nor onto.

Question 8: Let A and B be sets. Show that f: A × B → B × A such that (a, b) = (b, a) is bijective function.

Solution: f: A × B → B × A is defined as f(a, b) = (b, a).

$(a_1, b_1), (a_2, b_2) \in A \times B$ such that $f(a_1, b_1) = f(a_2, b_2)$

$\Rightarrow (b_1, a_1) = (b_2, a_2)$

$\Rightarrow b_1 = b_2$ and $a_1 = a_2$

$\Rightarrow (a_1, b_1) = (a_2, b_2)$

∴ f is one – one.

$(b, a) \in B \times A$.

there exists $(a, b) \in A \times B$ such that f(a, b) = (b, a)

∴ f is onto.

f is bijective.

Question 9: Let f: N → N be defined by f(n) = $\begin{cases} \frac{n+1}{2}, & \text{if n is odd} \\ \frac{n}{2}, & \text{if n is even} \end{cases}$ for all n ∈ N

State whether the function f is bijective. Justify your answer.

Solution: f: N → N is defined as f(n) = $\begin{cases} \frac{n+1}{2}, & \text{if n is odd} \\ \frac{n}{2}, & \text{if n is even} \end{cases}$ for all n ∈ N

f(1) = $\frac{1+1}{2}$ = 1 and f(2) = $\frac{2}{2}$ = 1

f(1) = f(2), where 1 ≠ 2

∴ f is not one-one.

Consider a natural number (n) in co-domain **N**.

Case **I**: n is odd

∴ n = 2r + 1 for some r ∈ N. there exists 4r + 1∈ N such that

f(4r + 1) = $\frac{4r+1+1}{2}$ = 2r + 1

Case II: n is even

∴ n = 2r for some r ∈ N. there exists 4r ∈ N such that

f(4r) = $\frac{4r}{2}$ = 2r.

∴ f is onto.

f is not a bijective function.

Question 10: Let A = R − {3} and B = R − {1}. Consider the function f: A → B defined by $(x) = \left(\frac{x-2}{x-3}\right)$. Is f one-one and onto? Justify your answer.

Solution: A = R − {3}, B = R − {1} and f: A → B defined by f(x) = $\left(\frac{x-2}{x-3}\right)$

x, y ∈ A such that f(x) = f(y)

$\Rightarrow \frac{X-2}{X-3} = \frac{Y-2}{Y-3}$

⇒ (x – 2)(y – 3) = (y – 2)(x – 3)

⇒ xy – 3x – 2y + 6 = xy – 2x – 3y + 6

⇒ – 3x – 2y = – 2x – 3y ⇒ x = y

∴ f is one-one.

y ∈ B = R − {1}. Then, y ≠ 1.

The function f is onto if there exists x ∈ A such that f(x) = y.

f(x) = y

$\Rightarrow \frac{X-2}{X-3} = y$

⇒ x – 2 = xy – 3y ⇒ x(1 – y) = – 3y + 2

$\Rightarrow x = \frac{2-3y}{1-y} \in A$ [y ≠ 1]

for any y ∈ B, there exists $\frac{2-3y}{1-y}$ ∈ A such that

$f\left(\frac{2-3y}{1-y}\right) = \frac{\left(\frac{2-3y}{1-y}\right)-2}{\left(\frac{2-3y}{1-y}\right)-3} = \frac{2-3y-2+2y}{2-3y-3+3y} = \frac{-y}{-1} = y$

∴ f is onto.

function f is one – one and onto.

Question 11: Let f: **R → R** be defined as $f(x) = x^4$. Choose the correct answer.

(A) f is one-one onto

(B) f is many-one onto

(C) f is one-one but not onto

(D) f is neither one-one nor onto

Solution: f: **R → R** is defined as $f(x) = x^4$.

x, y ∈ R such that f(x) = f(y).

$\Rightarrow x^4 = y^4$

$\Rightarrow x = \pm y$

∴ f(x) = f(y) does not imply that x = y.

For example f(1) = f(–1) = 1

∴ f is not one-one.

Consider an element 2 in co-domain **R** there does not exist any x in domain R such that f(x) = 2.

∴ f is not onto.

function f is neither one – one nor onto.

The correct answer is D.

Question 12: Let f: R → R be defined as f(x) = 3x. Choose the correct answer.

(A) f is one – one onto

(B) f is many – one onto

(C) f is one – one but not onto

(D) f is neither one – one nor onto

Solution: f: R → R is defined as f(x) = 3x.

x, y ∈ R such that f(x) = f(y).

⇒ 3x = 3y

⇒ x = y

∴f is one-one.

for any real number (y) in co-domain R, there exists $y3$ in R such that f$\left(\frac{y}{3}\right)$ = 3$\left(\frac{y}{3}\right)$ = y

∴ f is onto.

function f is one – one and onto.

The correct answer is A.