Chapter 8
Binomial Theorem

Exercise 8.1

Question 1:

Expand the expression $(1-2x)^5$

Solution 1:

By using Binomial Theorem, the expression $(1-2x)^5$ can be expanded as $(1-2x)^5$

$= {}^5C_0(1)^5 - {}^5C_1(1)^4(2x) + {}^5C_2(1)^3(2x)^2 - {}^5C_3(1)^2(2x)^3 + {}^5C_4(1)^1(2x)^4 - {}^5C_5(2x)^5$

$= 1 - 5(2x) + 10(4x)^2 - 10(8x^3) + 5(16x^4) - (32x^5)$

$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

Question 2:

Expand the expression $\left(\frac{2}{x}-\frac{x}{2}\right)^5$

Solution 2:

By using Binomial Theorem, the expression $\left(\frac{2}{x}-\frac{x}{2}\right)^5$ can be expanded as

$\left(\frac{2}{x}-\frac{x}{2}\right)^5 = {}^5C_0\left(\frac{2}{x}\right)^5 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2 - {}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5$

$= \frac{32}{x^3} - 5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right) + 10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right) - 10\left(\frac{4}{x^2}\right)\left(\frac{x^3}{8}\right) + 5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right) - \frac{x^5}{32}$

$= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32}$

Question 3:

Expand the expression $(2x-3)^6$

Solution 3:

By using Binomial Theorem, the expression $(2x-3)^6$ can be expanded as

$(2x-3)^6 = {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_2(2x)^4(3)^2 - {}^6C_3(2x)^3(3)^3 + {}^6C_4(2x)^2(3)^4 - {}^6C_5(2x)(3)^5 + {}^6C_6(3)^6$

$= 64x^6 - 6(32x^5)(3) + 15(16x^4)(9) - 20(8x^3)(27) + 15(4x^2)(81) - 6(2x)(243) + 729$

$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$

Question 4:

Expand the expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

Solution 4:

By using Binomial Theorem, the expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$ can be expanded as

$$\left(\frac{x}{3}+\frac{1}{x}\right)^5 = {}^5C_0\left(\frac{x}{3}\right)^5 + {}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + {}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2 + {}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3 + {}^5C_4\left(\frac{x}{3}\right)\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{1}{x}\right)^5$$

$$= \frac{x^5}{243} + 5\left(\frac{x^4}{81}\right)\left(\frac{1}{x}\right) + 10\left(\frac{x^3}{27}\right)\left(\frac{1}{x^2}\right) + 10\left(\frac{x^2}{9}\right)\left(\frac{1}{x^3}\right) + 5\left(\frac{x}{3}\right)\left(\frac{1}{x^4}\right) + \frac{1}{x^5}$$

$$= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}$$

Question 5:

Expand $\left(x+\frac{1}{x}\right)^6$

Solution 5:

By using Binomial Theorem, the expression $\left(x+\frac{1}{x}\right)^6$ can be expanded as

$$\left(x+\frac{1}{x}\right)^6 = {}^6C_0(x)^6 + {}^6C_1(x)^5\left(\frac{1}{x}\right) + {}^6C_2(x)^4\left(\frac{1}{x}\right)^2 + {}^6C_3(x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + {}^6C_5(x)\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6$$

$$= x^6 + 6(x)^5\left(\frac{1}{x}\right) + 15(x)^4\left(\frac{1}{x^2}\right) + 20(x)^3\left(\frac{1}{x^3}\right) + 15(x)^2\left(\frac{1}{x^4}\right) + 6(x)\left(\frac{1}{x^5}\right) + \frac{1}{x^6}$$

$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

Question 6:

Using Binomial Theorem, evaluate $(96)^3$

Solution 6:

96 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, $96 = 100 - 4$

$$\therefore (96)^3 = (100-4)^3$$

$$= {}^3C_0(100)^3 - {}^3C_1(100)^2(4) + {}^3C_2(100)(4)^2 - 3C_3(4)^3$$

$$= (100)^3 - 3(100)^2(4) + 3(100)(4)^2 - (4)^3$$

$$= 1000000 - 120000 + 4800 - 64$$

$$= 884736$$

Question 7:

Using Binomial Theorem, evaluate $(102)^5$

Solution 7:

102 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, $102 = 100 + 2$

$\therefore (102)^5 = (100+2)^5$

$= {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3 + {}^5C_4(100)(2)^4 + {}^5C_5(2)^5$

$= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32$

$= 11040808032$

Question 8:

Using Binomial Theorem, evaluate $(101)^4$

Solution 8:

101 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, $101 = 100 + 1$

$\therefore (101)^4 = (100+1)^4$

$= {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)(1)^3 + {}^4C_4(1)^4$

$= (100)^4 + 4(100)^3 + 6(100)^2 + 4(100) + (1)^4$

$= 100000000 + 4000000 + 60000 + 400 + 1$

$= 104060401$

Question 9:

Using Binomial Theorem, evaluate $(99)^5$

Solution 9:

99 can be written as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, $99 = 100 - 1$

$\therefore (99)^5 = (100-1)^5$

$= {}^5C_0(100)^5 - {}^5C_1(100)^4(1) + {}^5C_2(100)^3(1)^2 - {}^5C_3(100)^2(1)^3 + {}^5C_4(100)(1)^4 - {}^5C_5(1)^5$

$= (100)^5 - 5(100)^4 + 10(100)^3 - 10(100)^2 + 5(100) - 1$

$= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1$

$= 10010000500 - 500100001$

$= 9509900499$

Question 10:

Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

Solution 10:

By splitting 1.1 and then applying Binomial Theorem, the first few terms of $(1.1)^{10000}$ be obtained as

$$(1.1)^{10000} = (1+0.1)^{10000}$$
$$= {}^{10000}C_0 + {}^{10000}C_1(1.1) + \text{Other positive terms}$$
$$= 1 + 10000 \times 1.1 + \text{Other positive terms}$$
$$= 1 + 11000 + \text{Other positive terms}$$
$$> 1000$$

Hence, $(1.1)^{10000} > 1000$.

Question 11:

Find $(a+b)^4 - (a-b)^4$. Hence, evaluate. $\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4$

Solution 11:

Using Binomial Theorem, the expressions, $(a+b)^4$ and $(a-b)^4$, can be expanded as

$$(a+b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4$$
$$(a-b)^4 = {}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4$$
$$\therefore (a+b)^4 - (a-b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4 - \left[{}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4\right]$$
$$= 2\left({}^4C_1a^3b + {}^4C_3ab^3\right) = 2\left(4a^3b + 4ab^3\right)$$
$$= 8ab\left(a^2 + b^2\right)$$

By putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$$\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4 = 8\left(\sqrt{3}\right)\left(\sqrt{2}\right)\left\{\left(\sqrt{3}\right)^2 + \left(\sqrt{2}\right)^2\right\}$$
$$= 8\left(\sqrt{6}\right)\{3+2\} = 40\sqrt{6}$$

Question 12:

Find $(x+1)^6 + (x-1)^6$. Hence or otherwise evaluate. $\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6$

Solution 12:

Using Binomial Theorem, the expression, $(x+1)^6$ and $(x-1)^6$, can be expanded as

$$(x+1)^6 = {}^6C_0x^6 + {}^6C_1x^5 + {}^6C_2x^4 + {}^6C_3x^3 + {}^6C_4x^2 + {}^6C_5x + {}^6C_6$$
$$(x-1)^6 = {}^6C_0x^6 - {}^6C_1x^5 + {}^6C_2x^4 - {}^6C_3x^3 + {}^6C_4x^2 - {}^6C_5x + {}^6C_6$$
$$\therefore (x+1)^6 + (x-1)^6 = 2\left[{}^6C_0x^6 + {}^6C_2x^4 + {}^6C_4x^2 + {}^6C_6\right]$$
$$= 2\left[x^6 + 15x^4 + 15x^2 + 1\right]$$
By putting $x = \sqrt{2}$ we obtain
$$\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6 = 2\left[\left(\sqrt{2}\right)^6 + 15\left(\sqrt{2}\right)^4 + 15\left(\sqrt{2}\right)^2 + 1\right]$$
$$= 2(8 + 15\times4 + 15\times2 + 1)$$
$$= 2(8+60+30+1)$$
$$= 2(99) = 198$$

Question 13:
Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

Solution 13:
In order to show that $9^{n+1} - 8n - 9$ is divisible by 64, it has to be prove that, $9^{n+1} - 8n - 9 = 64k$, where k is some natural number
By Binomial Theorem,
$$(1+a)^m = {}^mC_0 + {}^mC_1a + {}^mC_2a^2 + ... + {}^mC_ma^m$$
For $a = 8$ and $m = n+1$, we obtain
$$(1+8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1(8) + {}^{n+1}C_2(8)^2 + ... + {}^{n+1}C_{n+1}(8)^{n+1}$$
$$\Rightarrow 9^{n+1} = 1 + (n+1)(8) + 8^2\left[{}^{n+1}C_2 + {}^{n+1}C_3\times8 + ... + {}^{n+1}C_{n+1}(8)^{n-1}\right]$$
$$\Rightarrow 9^{n+1} = 9 + 8n + 64\left[{}^{n+1}C_2 + {}^{n+1}C_3\times8 + ... + {}^{n+1}C_{n+1}(8)^{n-1}\right]$$
$\Rightarrow 9^{n+1} - 8n - 9 = 64k$, where $k = {}^{n+1}C_2 + {}^{n+1}C_3\times8 + ... + {}^{n+1}C_{n+1}(8)^{n-1}$ is a natural number
Thus, $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

Question 14:
Prove that $\sum_{r=0}^{n} 3^r\,{}^nC_r = 4^n$

Solution 14:
By Binomial Theorem,
$$\sum_{r=0}^{n} {}^nC_r\,a^{n-r}b^r = (a+b)^n$$
By putting $b = 3$ and $a = 1$ in the above equation, we obtain

$$\sum_{r=0}^{n} {}^nC_r (1)^{n-r} (3)^r = (1+3)^n$$

$$\Rightarrow \sum_{r=0}^{n} 3^r \, {}^nC_r = 4^n$$

Hence proved.

Exercise 8.2

Question 1:

Find the coefficient of x^5 in $(x+3)^8$

Solution 1:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$

Assuming that x^5 occurs in the $(r+1)^{th}$ term of the expansion $(x+3)^8$, we obtain

$T_{r+1} = {}^8C_r (x)^{8-r} (3)^r$

Comparing the indices of x in x^5 in T_{r+1},

We obtain r = 3

Thus, the coefficient of x^5 is ${}^8C_3 (3)^3 = \frac{8!}{3!5!} \times 3^3 = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3 \cdot 2.5!} \cdot 3^3 = 1512$.

Question 2:

Find the coefficient of a^5b^7 in $(a-2b)^{12}$

Solution 2:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$

Assuming that a^5b^7 occurs in the $(r+1)^{th}$ term of the expansion $(a-2b)^{12}$, we obtain

$T_{r+1} = {}^{12}C_r (a)^{12-r} (-2b)^r = {}^{12}C_r (-2)^r (a)^{12-r} (b)^r$

Comparing the indices of a and b in a^5b^7 in T_{r+1},

We obtain r = 7

Thus, the coefficient of a^5b^7 is

$${}^{12}C_7 (-2)^7 = \frac{12!}{7!5!} \cdot 2^7 = \frac{12 \cdot 11.10 \cdot 9 \cdot 8 \cdot 7!}{5 \cdot 4.3 \cdot 2 \cdot 7!} \cdot (-2)^7 = -(792)(128) = -101376.$$

Question 3:

Write the general term in the expansion of $\left(x^2-y\right)^6$

Solution 3:

It is known that the general term T_{r+1} {which is the $(r+1)^{th}$ term} in the binomial expansion of $(a+b)^n$ is given by $T_{r+1}={}^nC_r\, a^{n-r}b^r$.

Thus, the general term in the expansion of $\left(x^2-y^6\right)$ is

$T_{r+1}={}^6C_r\left(x^2\right)^{6-r}(-y)^r=(-1)^r\,{}^6C_r.x^{12-2r}.y^r$

Question 4:

Write the general term in the expansion of $\left(x^2-yx\right)^{12}$, $x\neq 0$

Solution 4:

It is known that the general term T_{r+1} {which is the $(r+1)^{th}$ term} in the binomial expansion of $(a+b)^n$ is given by $T_{r+1}={}^nC_r\, a^{n-r}b^r$.

Thus, the general term in the expansion of $\left(x^2-yx\right)^{12}$ is

$T_{r+1}={}^{12}C_r\left(x^2\right)^{12-r}(-yx)^r=(-1)^r\,{}^{12}C_r.x^{24-2r}.y^r=(-1)^r\,{}^{12}C_r.x^{24-r}.y^r$

Question 5:

Find the 4th term in the expansion of $(x-2y)^{12}$.

Solution 5:

It is known $(r+1)^{th}$ term, T_{r+1}, in the binomial expansion of $(a+b)^n$ is given by $T_{r+1}={}^nC_r\, a^{n-r}b^r$.

Thus, the 4th term in the expansion of $\left(x^2-2y\right)^{12}$ is

$T_4=T_{3+1}={}^{12}C_3\,(x)^{12-3}(-2y)^3=(-1)^3\cdot\dfrac{12!}{3!9!}\cdot x^9\cdot(2)^3\cdot y^3=-\dfrac{12\cdot 11\cdot 10}{3\cdot 2}\cdot(2)^3\,x^9y^3=-1760\,x^9y^3$

Question 6:

Find the 13th term in the expansion of $\left(9x-\dfrac{1}{3\sqrt{x}}\right)^{18}$, $x\neq 0$

Solution 6:

It is known $(r+1)^{th}$ term, T_{r+1}, in the binomial expansion of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r\, a^{n-r} b^r$.

Thus, the 13th term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$ is

$$T_{13} = T_{12+1} = {}^{18}C_{12}(9x)^{18-12}\left(-\frac{1}{3\sqrt{x}}\right)^{12}$$

$$= (-1)^{12}\frac{18!}{12!6!}(9)^6(x)^6\left(\frac{1}{3}\right)^{12}\left(\frac{1}{\sqrt{x}}\right)^{12}$$

$$= \frac{18\cdot17\cdot16\cdot15\cdot14\cdot13.12!}{12!.6\cdot5\cdot4\cdot3\cdot2}\cdot x^6\left(\frac{1}{x^6}\right).3^{12}\left(\frac{1}{3^{12}}\right) \qquad \left[9^6 = \left(3^2\right)^6 = 3^{12}\right]$$

$$= 18564$$

Question 7:

Find the middle terms in the expansions of $\left(3 - \frac{x^3}{6}\right)^7$

Solution 7:

It is known that in the expansion of $(a+b)^n$, in n is odd, then there are two middle terms,

Namely $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term.

Therefore, the middle terms in the expansion $\left(3 - \frac{x^3}{6}\right)^7$ are $\left(\frac{7+1}{2}\right)^{th} = 4^{th}$ and $\left(\frac{7+1}{2}+1\right)^{th} = 5^{th}$ term

$$T_4 = T_{3+1} = {}^7C_3(3)^{7-3}\left(-\frac{x^3}{6}\right)^3 = (-1)^3\frac{7!}{3!4!}\cdot3^4\cdot\frac{x^9}{6^3}$$

$$= -\frac{7\cdot6\cdot5.4!}{3\cdot2.4!}\cdot3^4\cdot\frac{1}{2^3\cdot3^3}\cdot x^9 = -\frac{105}{8}x^9$$

$$T_5 = T_{4+1} = {}^7C_4(3)^{7-4}\left(-\frac{x^3}{6}\right)^4 = (-1)^4\frac{7!}{4!3!}\cdot3^3\cdot\frac{x^{12}}{6^4}$$

$$= \frac{7\cdot6\cdot5.4!}{4!.3\cdot2}\cdot\frac{3^3}{2^4\cdot3^4}\cdot x^{12} = \frac{35}{48}x^{12}$$

Thus, the middle terms in the expansion of $\left(3 - \frac{x^3}{6}\right)^7$ are $-\frac{105}{8}x^9$ and $\frac{35}{48}x^{12}$.

Question 8:

Find the middle terms in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$

Solution 8:

It is known that in the expansion of $(a+b)^n$, in n is even, then the middle term is $\left(\frac{n}{2}+1\right)^{th}$ term.

Therefore, the middle term in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ is $\left(\frac{10}{2}+1\right)^{th}=6^{th}$

$$T_4=T_{5+1}={}^{10}C_5\left(\frac{x}{3}\right)^{10-5}(9y)^5=\frac{10!}{5!5!}\cdot\frac{x^5}{3^5}\cdot 9^5\cdot y^5$$
$$=\frac{10\cdot 9\cdot 8\cdot 7\cdot 6.5!}{5\cdot 4\cdot 3\cdot 2.5!}\cdot\frac{1}{3^5}\cdot 3^{10}\cdot x^5y^5 \qquad \left[9^5=\left(3^2\right)^5=3^{10}\right]$$
$$=252\times 3^5\cdot x^5\cdot y^5=6123x^5y^5$$

Thus, the middle term in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ is $61236x^5y^5$.

Question 9:

In the expansion of $(1+a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

Solution 9:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by

$$T_{r+1}={}^nC_ra^{n-r}b^r$$

Assuming that a^m occurs in the $(r+1)^{th}$ term of the expansion $(1+a)^{m+n}$, we obtain

$$T_{r+1}={}^{m+n}C_r(1)^{m+n-r}(a)^r={}^{m+n}C_ra^r$$

Comparing the indices of a in a^m in T_{r+1},
We obtain r = m
Therefore, the coefficient of a^m is

$${}^{m+n}C_m=\frac{(m+n)!}{m!(m+n-m)!}=\frac{(m+n)!}{m!n!}\ldots\ldots(1)$$

Assuming that a^n occurs in the $(k+1)^{th}$ term of the expansion $(1+a)^{m+n}$, we obtain

$$T_{k+1}={}^{m+n}C_k(1)^{m+n-k}(a)^k={}^{m+n}C_k(a)^k$$

Comparing the indices of a in a^n and in T_{k+1},
We obtain
$k=n$

Therefore, the coefficient of a^n is

$$^{m+n}C_n = \frac{(m+n)!}{n!(m+n-n)!} = \frac{(m+n)!}{n!m!} \text{.......}(2)$$

Thus, from (1) and (2), it can be observed that the coefficients of a^m and a^n in the expansion of $(1+a)^{m+n}$ are equal.

Question 10:

The coefficients of the $(r-1)^{th}$, r^{th} and $(r+1)^{th}$ terms in the expansion of $(x+1)^n$ are in the ratio $1:3:5$. Find n and r.

Solution 10:

It is known that $(k+1)^{th}$ term, (T_{k+1}), in the binomial expansion of $(a+b)^n$ is given by

$T_{k+1} = {}^nC_k a^{n-k} b^k$.

Therefore, $(r-1)^{th}$ term in the expansion of $(x+1)^n$ is

$$T_{r-1} = {}^nC_{r-2}(x)^{n-(r-2)}(1)^{(r-2)} = {}^nC_{r-2}x^{n-r+2}$$

$(r+1)$ term in the expansion of $(x+1)^n$ is

$$T_{r+1} = {}^nC_r(x)^{n-r}(1)^r = {}^nC_r x^{n-r}$$

r^{th} term in the expansion of $(x+1)^n$ is

$$T_r = {}^nC_{r-1}(x)^{n-(r-1)}(1)^{(r-1)} = {}^nC_{r-1}x^{n-r+1}$$

Therefore, the coefficients of the $(r-1)^{th}$, r^{th} and $(r+1)^{th}$ terms in the expansion of $(x+1)^n$ ${}^nC_{r-2}$, ${}^nC_{r-1}$, and ${}^nC_r$ are respectively. Since these coefficients are in the ratio $1:3:5$, we obtain

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3} \text{ and } \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{5}$$

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{n!}{(r-2)!(n-r+2)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{(r-1)(r-2)!(n-r+1)!}{(r-2)!(n-r+2)!(n-r+1)!}$$

$$= \frac{r-1}{n-r+2}$$

$$\therefore \frac{r-1}{n-r+2} = \frac{1}{3}$$

$$\Rightarrow 3r-3 = n-r+2$$

$$\Rightarrow n-4r+5 = 0 \text{ ......}(1)$$

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{n!}{(r-1)!(n-r+1)} \times \frac{r!(n-r)!}{n!} = \frac{r(r-1)!(n-r)!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{r}{n-r+1}$$

$\therefore \frac{r}{n-r+1}=\frac{3}{5}$

$\Rightarrow 5r=3n-3r+3$

$\Rightarrow 3n-8r+3=0 \qquad ....(2)$

Multiplying (1) by 3 and subtracting it from (2), we obtain

$4r-12=0$

$\Rightarrow r=3$

Putting the value of r in (1), we obtain n

$-12+5=0$

$\Rightarrow n=7$

Thus, $n=7$ and $r=3$

Question 11:

Prove that the coefficient of x^n in the expansion of $(1+x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1+x)^{2n-1}$.

Solution 11:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by

$T_{r+1}={}^nC_r a^{n-r} b^r$.

Assuming that x^n occurs in the $(r+1)^{th}$ term of the expansion of $(1+x)^{2n}$, we obtain

$T_{r+1}={}^{2n}C_r (1)^{2n-r}(x)^r={}^{2n}C_r (x)^r$

Comparing the indices of x in x^n and in T_{r+1}, we obtain $r=n$

Therefore, the coefficient of x^n in the expansion of $(1+x)^{2n}$ is

$${}^{2n}C_n=\frac{(2n)!}{n!(2n-n)!}=\frac{(2n)!}{n!n!}=\frac{(2n)!}{(n!)^2} \quad ....(1)$$

Assuming that x^n occurs in the $(k+1)^{th}$ term of the expansion of $(1+x)^{2n-1}$, we obtain

$T_{k+1}={}^{2n}C_k (1)^{2n-1-k}(x)^k={}^{2n}C_k (x)^k$

Comparing the indices of x in x^n and in T_{k+1}, we obtain $k=n$

Therefore, the coefficient of x^n in the expansion of $(1+x)^{2n-1}$ is

$${}^{2n-1}C_n=\frac{(2n-1)!}{n!(2n-1-n)!}=\frac{(2n-1)!}{n!(n-1)!}$$

$$=\frac{2n.(2n-1)!}{2n.n!(n-1)!}=\frac{(2n)!}{2.n!n!}=\frac{1}{2}\left[\frac{(2n)!}{(n!)^2}\right] \quad ....(2)$$

From (1) and (2), it is observed that

$\frac{1}{2}\left({}^{2n}C_n\right)={}^{2n-1}C_n$

$\Rightarrow {}^{2n}C_n = 2\left({}^{2n-1}C_n\right)$

Therefore, the coefficient of x^n expansion of $(1+x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1+x)^{2n-1}$.

Hence proved.

Question 12:

Find a positive value of m for which the coefficient of x^2 in the expansion $(1+x)^m$ is 6.

Solution 12:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Assuming that x^2 occurs in the $(r+1)^{th}$ term of the expansion of $(1+x)^m$, we obtain

$T_{r+1} = {}^mC_r (1)^{m-r} (x)^r = {}^mC_r (x)^r$

Comparing the indices of x in x^2 and in T_{r+1}, we obtain $r = 2$

Therefore, the coefficient of x^2 is ${}^mC_2$

It is given that the coefficient of x^2 in the expansion $(1+x)^m$ is 6.

$\therefore {}^mC_2 = 6$

$\Rightarrow \dfrac{m!}{2!(m-2)!} = 6$

$\Rightarrow \dfrac{m(m-1)(m-2)!}{2\times(m-2)!} = 6$

$\Rightarrow m(m-1) = 12$

$\Rightarrow m^2 - m - 12 = 0$

$\Rightarrow m^2 - 4m + 3m - 12 = 0$

$\Rightarrow m(m-4) + 3(m-4) = 0$

$\Rightarrow (m-4)(m+3) = 0$

$\Rightarrow (m-4) = 0 \ or \ (m+3) = 0$

$\Rightarrow m = 4 \text{ or } m = -3$

Thus, the positive value of m, for which the coefficient of x^2 in the expansion $(1+x)^m$ is 6, is 4.

Miscellaneous Exercise

Question 1:

Find a, b and n in the expansion of $(a+b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.

Solution 1:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

The first three terms of the expansion are given as 729, 7290 and 30375 respectively.
Therefore, we obtain

$$T_1 = {}^nC_0\, a^{n-0}b^0 = a^n = 729 \text{ ......}(1)$$

$$T_2 = {}^nC_1\, a^{n-1}b^1 = na^{n-1}b = 7290 \text{ ......}(2)$$

$$T_2 = {}^nC_1\, a^{n-2}b^2 = \frac{n(n-1)}{2}a^{n-2}b^2 = 30375 \text{ ......}(3)$$

Dividing (2) by (1), we obtain

$$\frac{na^{n-1}b}{a^n} = \frac{7290}{729}$$

$$\Rightarrow \frac{nb}{a} = 10 \qquad \text{.......}(4)$$

Dividing (3) by (2), we obtain

$$\frac{n(n-1)a^{n-2}b^2}{2na^{n-1}b} = \frac{30375}{7290}$$

$$\Rightarrow \frac{(n-1)b}{2a} = \frac{30375}{7290}$$

$$\Rightarrow \frac{(n-1)b}{a} = \frac{30375\times 2}{7290} = \frac{25}{3}$$

$$\Rightarrow \frac{nb}{a} - \frac{b}{a} = \frac{25}{3}$$

$$\Rightarrow 10 - \frac{b}{a} = \frac{25}{3} \qquad [\text{Using}(4)]$$

$$\Rightarrow \frac{b}{a} = 10 - \frac{25}{3} = \frac{5}{3} \qquad \text{.....}(5)$$

From (4) and (5), we obtain

$$n\cdot\frac{5}{3} = 10$$

$$\Rightarrow n = 6$$

Substituting $n = 6$ in equation (1), we obtain a^6
$= 729$

$$\Rightarrow a = \sqrt[6]{729} = 3$$

From (5), we obtain

$$\frac{b}{3}=\frac{5}{3}\Rightarrow b=5$$

Thus, $a=3,\ b=5,$ and $n=6$.

Question 2:

Find a if the coefficients of x^2 and x^3 in the expansion of $(3+ax)^9$ are equal.

Solution 2:

It is known that $(r+1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a+b)^n$ is given by

$T_{r+1}={}^nC_r a^{n-r} b^r$.

Assuming that x^2 occurs in the $(r+1)^{th}$ term in the expansion of $(3+ax)^9$, we obtain

$$T_{r+1}={}^9C_r(3)^{9-r}(ax)^r={}^9C_r(3)^{9-r}a^r x^r$$

Comparing the indices of x in x^2 and in T_{r+1}, we obtain

$r=2$

Thus, the coefficient of x^2 is

$${}^9C_2(3)^{9-2}a^2=\frac{9!}{2!7!}(3)^7a^2=36(3)^7a^2$$

Assuming that x^3 occurs in the $(k+1)^{th}$ term in the expansion of $(3+ax)^9$, we obtain

$$T_{k+1}={}^9C_k(3)^{9-k}(ax)^k={}^9C_k(3)^{9-k}a^k x^k$$

Comparing the indices of x in x^3 and in T_{k+1}, we obtain $k=3$

Thus, the coefficient of x^3 is

$${}^9C_3(3)^{9-3}a^3=\frac{9!}{3!6!}(3)^6a^3=84(3)^6a^3$$

It is given that the coefficient of x^2 and x^3 are the same.

$$84(3)^6a^3=36(3)^7a^2$$

$$\Rightarrow 84a=36\times 3$$

$$\Rightarrow a=\frac{36\times 3}{84}=\frac{104}{84}$$

$$\Rightarrow a=\frac{9}{7}$$

Thus, the required value of a is 9/7.

Question 3:

Find the coefficient of x^5 in the product $(1+2x)^6(1-x)^7$ using binomial theorem.

Solution 3:

Using Binomial Theorem, the expressions, $(1+2x)^6$ and $(1-x)^7$, can be expanded as

$$(1+2x)^6 = {}^6C_0 + {}^6C_1(2x) + {}^6C_2(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 + {}^6C_5(2x)^5 + {}^6C_6(2x)^6$$

$$= 1+6(2x)+15(2x)^2+20(2x)^3+15(2x)^4+6(2x)^5+(2x)^6$$

$$= 1+12x+60x^2+160x^3+240x^4+192x^5+64x^6$$

$$(1-x)^7 = {}^7C_0 - {}^7C_1(x) + {}^7C_2(x)^2 - {}^7C_3(x)^3 + {}^7C_4(x)^4 - {}^7C_5(x)^5 + {}^7C_6(x)^6 - {}^7C_7(x)^7$$

$$= 1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7$$

$$\therefore (1+2x)^6(1-x)^7$$

$$= \left(1+12x+60x^2+160x^3+240x^4+192x^5+64x^6\right)\left(1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7\right)$$

The complete multiplication of the two brackets is not required to be carried out. Only those terms, which involve x^5, are required.

The terms containing x^5 are

$$1\left(-21x^5\right)+(12x)\left(35x^4\right)+\left(60x^2\right)\left(-35x^3\right)+\left(160x^3\right)\left(21x^2\right)+\left(240x^4\right)(-7x)+\left(192x^5\right)(1)$$

$$= 171x^5$$

Thus, the coefficient of x^5 in the given product is 171.

Question 4:

If a and b are distinct integers, prove that $a-b$ is a factor of a^n-b^n, whenever n is a positive integer. [**Hint:** write $a^n=(a-b+b)^n$ and expand]

Solution 4:

In order to prove that $(a-b)$ is a factor of $\left(a^n-b^n\right)$, it has to be proved that

$a^n-b^n=k(a-b)$, where k is some natural number

It can be written that, $a=a-b+b$

$$\therefore a^n=(a-b+b)^n=\left[(a-b)+b\right]^n$$

$$= {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1}b+....+{}^nC_{n-1}(a-b)b^{n-1}+{}^nC_nb^n$$

$$= (a-b)^n + {}^nC_1(a-b)^{n-1}b+...+{}^nC_{n-1}(a-b)b^{n-1}+b^n$$

$$\Rightarrow a^n-b^n=(a-b)\left[(a-b)^{n-1}+{}^nC_1(a-b)^{n-2}b+....+{}^nC_{n-1}b^{n-1}\right]$$

$$\Rightarrow a^n-b^n=k(a-b)$$

Where, $k=\left[(a-b)^{n-1}+{}^nC_1(a-b)^{n-2}b+....+{}^nC_{n-1}b^{n-1}\right]$ is a natural number

This shows that $(a-b)$ is a factor of $\left(a^n-b^n\right)$, where n is a positive integer.

Question 5:

Evaluate $\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6$

Solution 5:

Firstly, the expression $(a+b)^6-(a-b)^6$ is simplified by using Binomial Theorem. This can be done as

$$(a+b)^6={}^6C_0a^6+{}^6C_1a^5b+{}^6C_2a^4b^2+{}^6C_3a^3b^3+{}^6C_4a^2b^4+{}^6C_5a^1b^5+{}^6C_6b^6$$
$$=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$$
$$(a-b)^6={}^6C_0a^6-{}^6C_1a^5b+{}^6C_2a^4b^2-{}^6C_3a^3b^3+{}^6C_4a^2b^4-{}^6C_5a^1b^5+{}^6C_6b^6$$
$$=a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$$
$$\therefore (a+b)^6-(a-b)^6=2\left[6a^5b+20a^3b^3+6ab^5\right]$$

Putting $a=\sqrt{3}$ and $b=\sqrt{2}$, we obtain

$$\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6=2\left[6\left(\sqrt{3}\right)^5\left(\sqrt{2}\right)+20\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3+6\left(\sqrt{3}\right)\left(\sqrt{2}\right)^5\right]$$
$$=2\left[54\sqrt{6}+120\sqrt{6}+24\sqrt{6}\right]$$
$$=2\times198\sqrt{6}$$
$$=396\sqrt{6}$$

Question 6:

Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4+\left(a^2-\sqrt{a^2-1}\right)^4$

Solution 6:

Firstly, the expression $(x+y)^4+(x-y)^4$ is simplified by using Binomial Theorem.
This can be done as

$$(x+y)^4={}^4C_0x^4+{}^4C_1x^3y+{}^4C_2x^2y^2+{}^4C_3xy^3+{}^4C_4y^4$$
$$=x^4+4x^3y+6x^2y^2+4xy^3+y^4$$
$$(x-y)^4={}^4C_0x^4-{}^4C_1x^3y+{}^4C_2x^2y^2-{}^4C_3xy^3+{}^4C_4y^4$$
$$=x^4-4x^3y+6x^2y^2-4xy^3+y^4$$
$$\therefore (x+y)^4+(x-y)^4=2\left(x^4+6x^2y^2+y^4\right)$$

Putting $x=a^2$ and $y=\sqrt{a^2-1}$, we obtain

$$\left(a^2+\sqrt{a^2-1}\right)^4+\left(a^2-\sqrt{a^2-1}\right)^4=2\left[\left(a^2\right)^4+6\left(a^2\right)^2\left(\sqrt{a^2-1}\right)^2+\left(\sqrt{a^2-1}\right)^4\right]$$
$$=2\left[a^8+6a^4\left(a^2-1\right)+\left(a^2-1\right)^2\right]$$
$$=2\left[a^8+6a^6-6a^4+a^4-2a^2+1\right]$$

$=2\left[a^8+6a^6-5a^4-2a^2+1\right]$
$=2a^8+12a^6-10a^4-4a^2+2$

Question 7:

Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Solution 7:

$0.99=1-0.01$
$\therefore (0.99)^5=(1-0.01)^5$
$={}^5C_0(1)^5-{}^5C_1(1)^4(0.01)+{}^5C_2(1)^3(0.01)^2$ [Approximately]
$=1-5(0.01)+10(0.01)^2$
$=1-0.05+0.001$
$=1.001-0.05$
$=0.951$

Thus, the value of $(0.99)^5$ is approximately 0.951.

Question 8:

Find n, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$.

Solution 8:

In the expansion, $(a+b)^n={}^nC_0a^n+{}^nC_1a^{n-1}b+{}^nC_2a^{n-2}b^2+\ldots+{}^nC_{n-1}ab^{n-1}+{}^nC_nb^n$

Fifth term from the beginning $={}^nC_4a^{n-4}b^4$

Fifth term from the end $={}^nC_4a^4b^{n-4}$

Therefore, it is evident that in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ are fifth term from the beginning is ${}^nC_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4$ and the fifth term from the end is ${}^nC_{n-4}\left(\sqrt[4]{2}\right)^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}$

$${}^nC_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4={}^nC_4\frac{\left(\sqrt[4]{2}\right)^n}{\left(\sqrt[4]{2}\right)^4}\cdot\frac{1}{3}=\frac{n!}{6.4!(n-4)!}\left(\sqrt[4]{2}\right)^n \quad .....(1)$$

$${}^nC_{n-4}\left(\sqrt[4]{2}\right)^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}={}^nC_{n-4}\frac{\left(\sqrt[4]{3}\right)^4}{\left(\sqrt[4]{3}\right)^n}={}^nC_{n-4}.2\cdot\frac{3}{\left(\sqrt[4]{3}\right)^n}=\frac{6n!}{(n-4)!4!}\cdot\frac{1}{\left(\sqrt[4]{3}\right)^n} \quad .....(2)$$

It is given that the ratio of the fifth term from the beginning to the fifth term from the end is $\sqrt{6}:1$. Therefore, from (1) and (2), we obtain

$$\frac{n!}{6.4!(n-4)!}\left(\sqrt[4]{2}\right)^n : \frac{6n!}{(n-4)!4!}\cdot\frac{1}{\left(\sqrt[4]{3}\right)^n}=\sqrt{6}:1$$

$$\Rightarrow \frac{\left(\sqrt[4]{2}\right)^n}{6}:\frac{6}{\left(\sqrt[4]{3}\right)^n}=\sqrt{6}:1$$

$$\Rightarrow \frac{\left(\sqrt[4]{2}\right)^n}{6}\times\frac{\left(\sqrt[4]{3}\right)^n}{6}=\sqrt{6}$$

$$\Rightarrow \left(\sqrt[4]{6}\right)^n=36\sqrt{6}$$

$$\Rightarrow 6^{n/4}=6^{5/2}$$

$$\Rightarrow \frac{n}{4}=\frac{5}{2}$$

$$\Rightarrow n=4\times\frac{5}{2}=10$$

Thus, the value of n is 10.

Question 9:

Expand using Binomial Theorem $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4$, $x\neq 0$

Solution 9:

$$\left(1+\frac{x}{2}-\frac{2}{x}\right)^4$$

$$={}^nC_0\left(1+\frac{x}{2}\right)^4-{}^nC_1\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right)+{}^nC_2\left(1+\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2-{}^nC_3\left(1+\frac{x}{2}\right)\left(\frac{2}{x}\right)^3+{}^nC_4\left(\frac{2}{x}\right)^4$$

$$=\left(1+\frac{x}{2}\right)^4-4\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right)+6\left(1+x+\frac{x^2}{4}\right)\left(\frac{4}{x^2}\right)-4\left(1+\frac{x}{2}\right)\left(\frac{8}{x^3}\right)+\frac{16}{x^4}$$

$$=\left(1+\frac{x}{2}\right)^4-\frac{8}{x}\left(1+\frac{x}{2}\right)^3+\frac{24}{x^2}+\frac{24}{x}+6-\frac{32}{x^3}-\frac{16}{x^2}+\frac{16}{x^4}$$

$$=\left(1+\frac{x}{2}\right)^4-\frac{8}{x}\left(1+\frac{x}{2}\right)^3+\frac{8}{x^2}+\frac{24}{x}+6-\frac{32}{x^3}+\frac{16}{x^4}\ \ldots(1)$$

Again by using Binomial Theorem, we obtain

$$\left(1+\frac{x}{2}\right)^4={}^4C_0(1)^4+{}^4C_1(1)^3\left(\frac{x}{2}\right)+{}^4C_2(1)^2\left(\frac{x}{2}\right)^2+{}^4C_3(1)^3\left(\frac{x}{2}\right)^3+{}^4C_4\left(\frac{x}{2}\right)^4$$

$$=1+4\times\frac{x}{2}+6\times\frac{x^4}{4}+4\times\frac{x^3}{8}+\frac{x^4}{16}$$

$$=1+2x+\frac{3x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}......(2)$$

$$\left(1+\frac{x}{2}\right)^3={}^3C_0(1)^3+{}^3C_1(1)^2\left(\frac{x}{2}\right)+{}^3C_2(1)\left(\frac{x}{2}\right)+{}^3C_3\left(\frac{x}{2}\right)^3$$

$$=1+\frac{3x}{2}+\frac{3x^2}{4}+\frac{x^3}{8}\ ....(3)$$

From (1), (2) and (3), we obtain

$$\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^4$$

$$=1+2x+\frac{3x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}-\frac{8}{x}\left(1+\frac{3x}{2}+\frac{3x^2}{4}+\frac{x^3}{8}\right)+\frac{8}{x^2}+\frac{24}{x}+6-\frac{32}{x^3}+\frac{16}{x^4}$$

$$=1+2x+\frac{3}{2}x^2+\frac{x^3}{2}+\frac{x^4}{16}-\frac{8}{x}-12-6x-x^2+\frac{8}{x^2}+\frac{24}{x}+6-\frac{32}{x^3}+\frac{16}{x^4}$$

$$=\frac{16}{x}+\frac{8}{x^2}-\frac{32}{x^3}+\frac{16}{x^4}-4x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}-5$$

Question 10:

Find the expansion of $\left(3x^2-2ax+3a^2\right)^3$ using binomial theorem.

Solution 10:

Using Binomial Theorem, the given expression $\left(3x^2-2ax+3a^2\right)^3$ can be expanded as

$$\left[\left(3x^2-2ax\right)+3a^2\right]^3$$

$$={}^3C_0\left(3x^2-2ax\right)^3+{}^3C_1\left(3x^2-2ax\right)^2\left(3a^2\right)+{}^3C_2\left(3x^2-2ax\right)\left(3a^2\right)^2+{}^3C_3\left(3a^2\right)^3$$

$$=\left(3x^2-2ax\right)^3+3\left(9x^4-12ax^3+4a^2x^2\right)\left(3a^2\right)+3\left(3x^2-2ax\right)\left(9a^4\right)+27a^6$$

$$=\left(3x^2-2ax\right)^3+81a^2x^4-108a^3x^3+36a^4x^2+81a^4x^2-54a^5x+27a^6$$

$$=\left(3x^2-2ax\right)^3+81a^2x^4-108a^3x^3+117a^4x^2-54a^5x+27a^6 \qquad .... (1)$$

Again by using Binomial Theorem, we obtain

$$\left(3x^2-2ax\right)^3$$

$$={}^3C_0\left(3x^2\right)^3-{}^3C_1\left(3x^2\right)^2(2ax)+{}^3C_2\left(3x^2\right)(2ax)^2-{}^3C_3(2ax)^3$$

$$=27x^6-3\left(9x^4\right)(2ax)+3\left(3x^2\right)\left(4a^2x^2\right)-8a^3x^3$$

$$=27x^6-54ax^5+36a^2x^4-8a^3x^3 \qquad .... (2)$$

From (1) and (2), we obtain

$$\left(3x^2 - 2ax + 3a^2\right)^3$$
$$= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$$
$$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$$