Exercise 6.1

1. Find the rate of change of the area of a circle with respect to its radius r when

(a) $r = 3$ cm **(b) $r = 4$ cm.**

Sol. Let z denote the area of a circle of variable radius r.

We know that z (area of circle) $= \pi r^2$

$\therefore$ By Note 1 above, rate of change of area z w.r.t. radius r

$$= \frac{dz}{dr} = \pi(2r) = 2\pi r \qquad ...(i)$$

(a) **When $r = 3$ cm** (given), $\therefore$ From (i), $\frac{dz}{dr} = 2\pi(3) = 6\pi$ sq. cm

(b) **When $r = 4$ cm** (given), $\therefore$ From (i), $\frac{dz}{dr} = 2\pi(4) = 8\pi$ sq. cm.

2. The volume of a cube is increasing at the rate of 8 cm^3/sec. How fast is the surface area increasing when the length of an edge is 12 cm?

Sol. Let x cm be the edge of a cube (for example, a room whose length, breadth and height are equal) at time t.

Given: Rate of **Increase** of volume of cube = 8 cm^3/sec.

$$\Rightarrow \quad \frac{d}{dt}(x.x.x) \quad i.e., \quad \frac{d}{dt}\, \boldsymbol{x^3} \text{ is positive and } = 8$$

(8 cm^3/sec $\Rightarrow$ rate of Increase w.r.t. time)

$$\Rightarrow \quad 3x^2 \frac{\boldsymbol{d}}{\boldsymbol{dt}} \boldsymbol{x} = 8 \quad \Rightarrow \quad \frac{dx}{dt} = \frac{8}{3x^2} \qquad ...(i)$$

Let z denote the surface area of the cube. $\therefore$ $z = \boldsymbol{6x^2}$

(Area of four walls + Area of floor + Area of ceiling)

$\therefore$ Rate of change of surface area of cube

$$= \frac{dz}{dt} = 6\frac{d}{dt}x^2 = 6\left(2x\frac{dx}{dt}\right) = 12x\left(\frac{8}{3x^2}\right) \quad \text{[By } (i)]$$

$$= 4\left(\frac{8}{x}\right) = \frac{32}{x} \text{ cm}^2/\text{sec.}$$

Putting x = 12 cm (given), $\frac{dz}{dt} = \frac{32}{12} = \frac{8}{3}$ cm^2/sec

Since $\frac{dz}{dt}$ is positive, therefore, surface area is increasing at the rate of $\frac{8}{3}$ cm^2/sec.

3. The radius of a circle is increasing uniformely at the rate of 3 cm per second. Find the rate at which the area of the circle is increasing when the radius is 10 cm.

Sol. Let x cm denote the radius of a circle at time t.

Given: Rate of increase of radius of circle = 3 cm/sec.

$$\Rightarrow \quad \frac{dx}{dt} \text{ is positive and } = 3 \text{ cm/sec} \qquad ...(i)$$

Let z denote the area of the circle.

$\therefore$ $z = \boldsymbol{\pi x^2}$.

$\therefore$ Rate of change of area of circle $= \frac{dz}{dt} = \pi\frac{d}{dt}x^2$

$$= \pi.2x\frac{\boldsymbol{dx}}{\boldsymbol{dt}} = 2\pi x(3) \qquad \text{[By } (i)]$$

$$= 6\pi x.$$

Putting x = 10 cm (given), $\frac{dz}{dt} = 6\pi(10) = 60\pi$ cm^2/sec

Since $\frac{dz}{dt}$ is positive, therefore area of circle is **increasing** at the rate of 60π cm^2/sec.

4. An edge of a variable cube is increasing at the rate of 3 cm per second. How fast is the volume of the cube increasing when the edge is 10 cm long?

Sol. Let x cm be the edge of variable cube at time t.

Given: Rate of increase of edge x is 3 cm/sec.

$\therefore \quad \frac{dx}{dt}$ is positive and = 3 cm/sec ...(*i*)

Let z denote the volume of the cube.

$\therefore \quad z = x^3$

$\therefore$ Rate of change of volume of cube

$$= \frac{dz}{dt} = \frac{d}{dt}x^3 = 3x^2 \frac{dx}{dt} = 3x^2(3) \qquad \text{[By } (i)]$$

or $\quad \frac{dz}{dt} = 9x^2$ cm^3/sec.

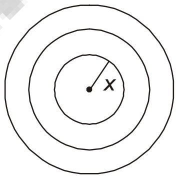
Putting x = 10 cm (given), $\frac{dz}{dt} = 9(10)^2 = 9(100) = 900$ cm^3/sec.

Since $\frac{dz}{dt}$ is positive, therefore volume of the cube is **increasing** at the rate of 900 cm^3/sec.

5. A stone is dropped into a quite lake and waves move in circles at the rate of 5 cm/sec. At the instant when radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

Sol. Let x cm be radius of circular wave at time t.

Given: Waves move in circles at the rate of 5 cm/sec.



$\Rightarrow$ Radius x of circular wave increases at the rate of 5 cm/sec.

$\Rightarrow \quad \frac{dx}{dt}$ is positive and = 5 cm/sec. ...(*i*)

Let z denote the enclosed area of the circular wave at time t.

$\therefore \quad z = \pi x^2$.

$\therefore$ Rate of change of area $= \frac{dz}{dt} = \pi \frac{d}{dt}x^2 = \pi.2x \frac{dx}{dt}$

$= 2\pi x(5)$ [By (*i*)] $= 10\pi x$

Putting x = 8 cm (given), $\frac{dz}{dt} = 10\pi(8) = 80\pi$ cm^2/sec.

Since $\frac{dz}{dt}$ is positive, therefore area of circular wave is increasing at the rate of 80π cm^2/sec.

6. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

Sol. Let x be the radius of the circle at time t.

Given: Rate of increase of radius of circle = 0.7 cm/sec.

$\Rightarrow \quad \frac{dx}{dt}$ is positive and = 0.7 cm/sec. ...(*i*)

Let z denote the circumference of the circle at time t.

$\therefore$ **$z = 2\pi x$ (Formula)**

$\therefore$ Rate of change of circumference of circle

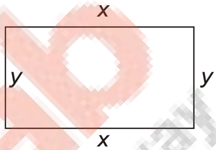
$$= \frac{dz}{dt} = \frac{d}{dt}(2\pi x) = 2\pi \frac{dx}{dt} = 2\pi(0.7) \qquad \text{(By } (i))$$

$$= 1.4\pi \text{ cm/sec.}$$

7. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When x = 8 cm and y = 6 cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

Sol. Given: Rate of decrease of length x of rectangle is 5 cm/minute.

$\Rightarrow \frac{dx}{dt}$ is negative and $= -5$ cm/minute ...(i)



Given: Rate of increase of width y of rectangle is 4 cm/minute.

$\Rightarrow \frac{dy}{dt}$ is positive and $= 4$ cm/minute ...(ii)

(a) Let z denote the perimeter of rectangle.

$\therefore \quad z = x + y + x + y = 2x + 2y$

$\therefore \quad \frac{dz}{dt} = 2\frac{dx}{dt} + 2\frac{dy}{dt}$

Putting values from (i) and (ii),

$$\frac{dz}{dt} = 2(-5) + 2(4) = -10 + 8 = -2 \text{ is negative.}$$

$\therefore$ Perimeter z of the rectangle is **decreasing** at the rate of 2 cm/sec.

(Even when x = 8 cm and y = 6 cm).

(b) Let z denote the area of rectangle.

$\therefore$ **$z = xy$**

$\therefore \quad \frac{dz}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$ | By Product Rule

Putting x = 8 cm and y = 6 cm (given) and putting values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ from (i) and (ii),

$$\frac{dz}{dt} = 8(4) + 6(-5) = 32 - 30 = 2 \text{ is positive.}$$

$\therefore$ Area z of the rectangle is **increasing** at the rate of 2 sq cm/minute even when x = 8 cm and y = 6 cm.

8. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

Sol. Let x cm be the radius of the spherical balloon at time t.

Given: Rate at which the balloon is being inflated *i.e.,* rate at which the volume of the balloon is increasing = 900 cu. cm sec.

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{4\pi}{3}x^3\right) = 900$$

$$\Rightarrow \quad \frac{4\pi}{3}\frac{d}{dt}x^3 = 900 \quad \Rightarrow \quad \frac{4\pi}{3} \cdot 3x^2 \frac{dx}{dt} = 900$$

$$\Rightarrow \quad 4\pi x^2 \frac{dx}{dt} = 900 \quad \Rightarrow \quad \frac{dx}{dt} = \frac{900}{4\pi x^2}$$

Putting $x = 15$ cm (given), $\frac{dx}{dt} = \frac{900}{4\pi(15)^2} = \frac{900}{4\pi(225)}$

$$= \frac{900}{900\pi} = \frac{1}{\pi} \text{ is positive.}$$

$\therefore$ Radius of balloon is **increasing** at the rate of $\frac{1}{\pi}$ cm sec.

9. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.

Sol. We know that the volume V of a balloon with radius x is

$$V = \frac{4}{3}\pi x^3$$

$\therefore$ Rate of change of volume with respect to radius x is given by

$$\frac{dV}{dx} = \frac{d}{dx}\left(\frac{4}{3}\pi x^3\right) = \frac{4}{3}\pi \cdot 3x^2 = 4\pi x^2$$

$\therefore$ When $x = 10$ cm, $\frac{dV}{dx} = 4\pi(10)^2 = 400\pi$

i.e., the volume is increasing at the rate of $4\pi(10)^2 = 400\pi$ cm^3/cm.

10. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall? (*Important*)

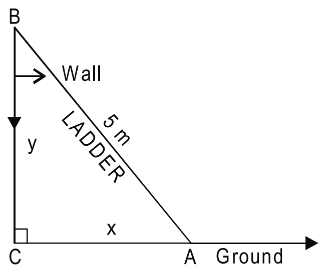
Sol. Let AB be the ladder and C, the junction of wall and ground, AB = 5 m

Let CA = x metres, CB = y metres.

We know that as the end A moves away from C, the end B moves towards C.

[$\because$ Length of ladder can't change]

i.e., as x increases, y decreases.



$$\text{Now } \frac{dx}{dt} = 2 \text{ cm/s}$$

...(*i*) (given)

In ΔABC, by Pythagoras Theorem $AC^2 + BC^2 = AB^2$

or $\quad x^2 + y^2 = 5^2 = 25 \quad$ ...(*ii*)

Differentiating both sides w.r.t. t, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

or $\quad 2x(2) + 2y = 0 \quad$ or $\quad 2y\frac{dy}{dt} = -4x$

$$\therefore \quad \frac{dy}{dt} = -\frac{2x}{y} \quad ...(iii)$$

When $x = 4$ (given), from (*ii*), $16 + y^2 = 25$ or $y^2 = 9$ or $y = 3$

$\therefore$ From (*iii*), $\dfrac{dy}{dt} = -\dfrac{2\times 4}{3} = -\dfrac{8}{3}$ cm/s.

Note. The negative sign indicates that y decreases as t increases.

11. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the *y*-coordinate is changing 8 times as fast as the *x*-coordinate.

Sol. Given: Equation of the curve is $6y = x^3 + 2 \quad$ ...(*i*)

Let (x, y) be the required point on curve (*i*).

Given: y-coordinate is changing 8 times as fast as the x coordinate.

$\Rightarrow$ Rate of change of y w.r.t. x is 8

$$\Rightarrow \quad \frac{dy}{dx} = 8 \quad ...(ii)$$

Differentiating both sides of (*i*) w.r.t. x, we have $6\dfrac{dy}{dx} = 3x^2$

Putting $\dfrac{dy}{dx} = 8$ from (*ii*), $48 = 3x^2 \Rightarrow x^2 = \dfrac{48}{3} = 16 \therefore x = \pm 4$

When $x = 4$, from (*i*), $6y = 64 + 2 = 66 \quad \therefore \quad y = \dfrac{66}{6} = 11$

$\therefore$ One required point is (4, 11).

When $x = -4$, from (*i*) $6y = -64 + 2 = -62$,

$$\therefore \quad y = \frac{-62}{6} = \frac{-31}{3}$$

$\therefore$ Second required point is $\left(-4, \dfrac{-31}{3}\right)$.

$\therefore$ Required points on curve (*i*) are (4, 11) and $\left(-4, \dfrac{-31}{3}\right)$.

12. The radius of an air bubble is increasing at the rate of $\dfrac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?

Sol. Let x cm be the radius of the air bubble at time t.

Given: Rate of increase of radius of air bubble (spherical as we all know) $= \dfrac{1}{2}$ cm/sec.

$\Rightarrow \quad \frac{dx}{dt}$ is positive and $= \frac{1}{2}$ cm/sec. ...(*i*)

Let z denote the volume of the air bubble.

$\therefore \qquad \boldsymbol{z = \frac{4\pi}{3} x^3}$

$\therefore \quad \frac{dz}{dt}$ = Rate of change of volume of air bubble

$$= \frac{4\pi}{3} \frac{d}{dt} x^3 = \frac{4\pi}{3} \cdot 3x^2 \frac{dx}{dt} = 4\pi x^2 \left(\frac{1}{2}\right) \text{ [By } (i)] = 2\pi x^2$$

Putting $x = 1$ cm (given), $\frac{dz}{dt} = 2\pi(1)^2 = 2\pi$ which is positive.

$\therefore$ Required rate of increase of volume of air bubble is 2π cm^3/sec.

13. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x + 1)$. Find the rate of change of its volume with respect to x.

Sol. Diameter of the balloon $= \frac{3}{2}(2x + 1)$ (given)

$\therefore$ Radius of balloon $= \frac{1}{2}$(diameter) $= \frac{1}{2} \cdot \frac{3}{2}(2x + 1) = \frac{3}{4}(2x + 1)$

$\therefore$ Volume of balloon (V) $= \boldsymbol{\frac{4}{3} \pi \text{ (radius)}^3}$

$$= \frac{4\pi}{3} \left(\frac{3}{4}(2x+1)\right)^3 = \frac{4}{3}\pi \cdot \frac{27}{64}(2x + 1)^3$$

$$= \frac{9\pi}{16}(2x + 1)^3 \text{ cu. units}$$

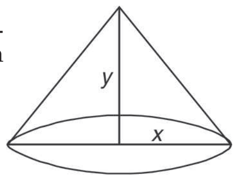
$\therefore$ Rate of change of volume w.r.t. x,

$$= \frac{dV}{dx} = \frac{9\pi}{16} \cdot 3(2x + 1)^2 \cdot \frac{d}{dx}(2x + 1)$$

$$= \frac{27\pi}{16}(2x + 1)^2 \cdot 2 = \frac{27\pi}{8}(2x + 1)^2.$$

14. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

Sol. Let the height and base radius of the sand-cone formed at time t sec be y cm and x cm respectively. Then $y = \frac{1}{6}x$ (given) or $x = 6y$.



Volume of cone (V) $= \frac{1}{3}\pi x^2 y$

Putting $x = 6y$, $\qquad$ V $= \frac{1}{3}\pi(6y)^2 y = 12\pi y^3$

$\therefore \quad \frac{dV}{dy} = 36\pi y^2$ ...(*i*)

It is given that sand is pouring from a pipe to form a sand-cone at the rate of 12 cm^3/sec.

$\therefore \quad \dfrac{dV}{dt} = 12 \quad \Rightarrow \quad \dfrac{dV}{dy} \times \dfrac{dy}{dt} = 12$

$\Rightarrow \quad 36\pi y^2 \times \dfrac{dy}{dt} = 12 \quad$ (By (i)) $\quad \Rightarrow \dfrac{dy}{dt} = \dfrac{1}{3\pi y^2}$

When $y = 4$ cm, (given); $\dfrac{dy}{dt} = \dfrac{1}{3\pi \times 4^2} = \dfrac{1}{48\pi}$ cm/sec.

15. The total cost C(x) in rupees associated with the production of x units of an item is given by

$$\mathbf{C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000.}$$

Find the marginal cost when 17 units are produced.

Sol. Marginal cost *is defined as the rate of change of total cost with respect to the number of units produced.*

$\therefore$ Marginal cost (MC) $= \dfrac{dC}{dx}$

$$= \frac{d}{dx}(0.007x^3 - 0.003x^2 + 15x + 4000)$$

$$= 0.021x^2 - 0.006x + 15$$

$\therefore$ When $x = 17$, MC $= 0.021 \times (17)^2 - 0.006 \times (17) + 15$

$$= 0.021(289) - 0.102 + 15$$

$$= 6.069 - 0.102 + 15 = 20.967$$

Hence, the required marginal cost = ₹ 20.97.

16. The total revenue in rupees received from the sale of x units of a product is given by

$$\mathbf{R(x) = 13x^2 + 26x + 15.}$$

Find the marginal revenue when x = 7.

Sol. Marginal Revenue *is defined as the rate of change of total revenue with respect to the number of units sold.*

$\therefore$ Marginal revenue (MR) $= \dfrac{dR}{dx}$

$$= \frac{d}{dx}(13x^2 + 26x + 15) = 26x + 26$$

When $x = 7$, MR $= 26 \times 7 + 26 = 208$

Hence, the required marginal revenue = ₹ 208.

Choose the correct answer in Exercises 17 and 18.

17. The rate of change of the area of a circle with respect to its radius r at r = 6 cm is

(A) 10π (B) 12π (C) 8π (D) 11π.

Sol. Let z denote the area of a circle of radius r.

$\therefore \quad z = \pi r^2$

$\therefore \quad$ Rate of change of area z w.r.t. radius $r = \dfrac{dz}{dr} = 2\pi r$

Putting r = 6 cm (given), $\frac{dz}{dr} = 2\pi(6) = 12\pi$

$\therefore$ Option (B) is the correct answer.

18. The total revenue in Rupees received from the sale of x units of a product is given by R(x) = $3x^2 + 36x + 5$. The marginal revenue, when x = 15 is

(A) 116 **(B) 96** **(C) 90** **(D) 126**

Sol. Given: Total revenue $R(x) = 3x^2 + 36x + 5$

$\therefore$ Marginal revenue = $\frac{d}{dx} R(x) = 6x + 36$

Putting x = 15 (given), $\frac{d(R(x))}{dx} = 6(15) + 36$

$\therefore$ $90 + 36 = 126$

$\therefore$ Option (D) is the correct answer.

Exercise 6.2

1. Show that the function given by $f(x) = 3x + 17$ is strictly increasing on R.

Sol. Given: $f(x) = 3x + 17$

$\therefore \quad f'(x) = 3(1) + 0 = 3 > 0$ *i.e.,* + ve for all $x \in$ R.

$\therefore$ $f(x)$ is strictly increasing on R.

2. Show that the function given by $f(x) = e^{2x}$ is strictly increasing on R.

Sol. Given: $f(x) = e^{2x}$

$\therefore \quad f'(x) = e^{2x} \frac{d}{dx} 2x = e^{2x}(2) = 2e^{2x} > 0$ *i.e.,* + ve for all $x \in$ R.

[$\because$ We know that e is approximately equal to 2.718 and is always positive]

$\therefore$ $f(x)$ is strictly increasing on R.

Remark. $e^{-2} = \frac{1}{(e^2)} > 0$ and $e^0 = 1 > 0$.

3. Show that the function given by $f(x) = \sin x$ is (*a*) strictly increasing in $\left(0, \frac{\pi}{2}\right)$ (*b*) strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$ (*c*) neither increasing nor decreasing in $(0, \pi)$.

Sol. Given: $f(x) = \sin x$

$\therefore \quad f'(x) = \cos x$

(*a*) We know that $f'(x) = \cos x > 0$ *i.e.,* + ve in first quadrant *i.e.,* in $\left(0, \frac{\pi}{2}\right)$.

$\therefore$ $f(x)$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$.

(b) We know that $f'(x) = \cos x < 0$ *i.e.,* – ve in second quadrant *i.e.,* in $\left(\frac{\pi}{2}, \pi\right)$.

$\therefore$ $f(x)$ is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

(c) Because $f'(x) = \cos x > 0$ *i.e.,* + ve in $\left(0, \frac{\pi}{2}\right)$ and $f'(x) = \cos x < 0$ *i.e.,* – ve in $\left(\frac{\pi}{2}, \pi\right)$ and $f'\left(\frac{\pi}{2}\right) = \cos \frac{\pi}{2} = 0$

$\therefore$ $f'(x)$ does not keep the same sign in the interval $(0, \pi)$.

Hence $f(x)$ is neither increasing nor decreasing in $(0, \pi)$.

4. Find the intervals in which the function f given by

$$\boldsymbol{f(x) = 2x^2 - 3x} \textbf{ is}$$

(a) strictly increasing (b) strictly decreasing.

Sol. Given: $\qquad f(x) = 2x^2 - 3x$

$\therefore \qquad f'(x) = 4x - 3 \qquad ...(i)$

Step I. Let us put $f'(x) = 0$ to find turning points *i.e.,* points on the given curve where tangent is parallel to x-axis.

$\therefore$ From (i), $4x - 3 = 0$ *i.e.,* $4x = 3$

or $\qquad x = \frac{3}{4}$ (= 0.75).

This turning point divides the real line in two disjoint sub-intervals $\left(-\infty, \frac{3}{4}\right)$ and $\left(\frac{3}{4}, \infty\right)$.

Step II.

Interval	**sign of $f'(x) = 4x - 3$ ...(i)**	**Nature of function f**
$\left(-\infty, \frac{3}{4}\right)$	Take $x = 0.5$ (say) then from (i) $f'(x) < 0$	$\therefore$ f is strictly decreasing $\downarrow$
$\left(\frac{3}{4}, \infty\right)$	Take $x = 1$ (say) then from (i), $f'(x) > 0$	$\therefore$ f is strictly increasing $\uparrow$

Thus, (a) f is strictly increasing in $\left(\frac{3}{4}, \infty\right)$.

(b) f is strictly decreasing in $\left(-\infty, \frac{3}{4}\right)$.

5. Find the intervals in which the function f given by

$$\boldsymbol{f(x) = 2x^3 - 3x^2 - 36x + 7} \textbf{ is}$$

(a) strictly increasing (b) strictly decreasing.

Sol. Given: $f(x) = 2x^3 - 3x^2 - 36x + 7$

$\therefore \quad f'(x) = 6x^2 - 6x - 36$

Step I. Form factors of $f'(x)$

$$f'(x) = 6(x^2 - x - 6)$$

(**Caution:** Don't omit 6. It can't be cancelled only from R.H.S.)

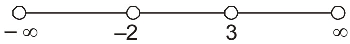
or $\quad f'(x) = 6(x^2 - 3x + 2x - 6) = 6[x(x - 3) + 2(x - 3)]$

$\quad = 6(x + 2)(x - 3) \qquad ...(i)$

Step II. Put $f'(x) = 0 \Rightarrow 6(x + 2)(x - 3) = 0$

But $6 \neq 0 \quad \therefore$ Either $x + 2 = 0$ or $x - 3 = 0$

i.e., $x = -2, x = 3$.



These turning points $x = -2$ and $x = 3$ divide the real line into three disjoint sub-intervals $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$.

Step III.

Interval	**sign of $f'(x)$ $= 6(x + 2)(x - 3)$ ...(i)**	**Nature of function f**
$(-\infty, -2)$	Take $x = -3$ (say). Then from (*i*), $f'(x) = (+)(-)(-)$ $= (+)$ *i.e.,* > 0	$\therefore$ f is strictly increasing $\uparrow$ in $(-\infty, -2)$
$(-2, 3)$	Take $x = 2$ (say). Then from (*i*), $f'(x) = (+)(+)(-)$ $= (-)$ *i.e.,* < 0	$\therefore$ f is strictly decreasing $\downarrow$ in $(-2, 3)$
$(3, \infty)$	Take $x = 4$ (say). Then from (*i*), $f'(x) = (+)(+)(+)$ $= (+)$ *i.e.,* > 0	$\therefore$ f is strictly increasing $\uparrow$ in $(3, \infty)$

Thus, (*a*) f is strictly increasing in $(-\infty, -2)$ and $(3, \infty)$.

(*b*) f is strictly decreasing in $(-2, 3)$.

6. Find the intervals in which the following functions are strictly increasing or decreasing.

(*a*) $x^2 + 2x - 5$ **(*b*) $10 - 6x - 2x^2$**

(*c*) $-2x^3 - 9x^2 - 12x + 1$ **(*d*) $6 - 9x - x^2$**

(*e*) $(x + 1)^3(x - 3)^3$.

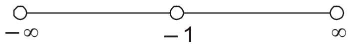
Sol. (*a*) **Given:** $f(x) = x^2 + 2x - 5$

$\therefore \quad f'(x) = 2x + 2 = 2(x + 1) \qquad ...(i)$

Step I. Put $f'(x) = 0 \Rightarrow 2(x + 1) = 0$

But $2 \neq 0$. Therefore, $x + 1 = 0$ *i.e.,* $x = -1$.

This turning point $x = -1$ divides the real line into two disjoint sub-intervals $(-\infty, -1)$ and $(-1, \infty)$.



Step II.

Interval	sign of $f'(x)$ $= 2(x + 1)$ ...(*i*)	Nature of function f
$(-\infty, -1)$	Take $x = -2$ (say). Then from (*i*), $f'(x) = (-)$ *i.e.,* < 0	$\therefore$ f is strictly decreasing $\downarrow$
$(-1, \infty)$	Take $x = 0$ (say). Then from (*i*), $f'(x) = (+)$ *i.e.,* > 0	$\therefore$ f is strictly increasing $\uparrow$

Thus, f is strictly increasing in $(-1, \infty)$ (*i.e.,* $x > -1$) and strictly decreasing in $(-\infty, -1)$ (*i.e.,* $x < -1$).

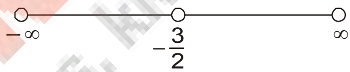
(*b*) **Given:** $f(x) = 10 - 6x - 2x^2$

$\therefore \quad f'(x) = -6 - 4x = -2(3 + 2x) \quad$ ...(*i*)

Step I. Put $f'(x) = 0 \Rightarrow -2(3 + 2x) = 0$

But $-2 \neq 0$. Therefore, $3 + 2x = 0$ *i.e.,* $2x = -3$

i.e., $x = -\dfrac{3}{2}$.



This turning point $x = -\dfrac{3}{2}$ divides the real line into two disjoint sub-intervals $\left(-\infty, -\dfrac{3}{2}\right)$ and $\left(-\dfrac{3}{2}, \infty\right)$.

Step III.

Interval	sign of $f'(x)$ $= -2(3 + 2x)$ ...(*i*)	Nature of function f
$\left(-\infty, -\frac{3}{2}\right)$	Take $x = -2$ (say). Then from (*i*), $f'(x) = (-)(-)$ $= (+)$ *i.e.,* > 0	$\therefore$ f is strictly increasing $\uparrow$
$\left(-\frac{3}{2}, \infty\right)$	Take $x = -1$ (say). Then from (*i*), $f'(x) = (-)(+)$ $= (-)$ *i.e.,* < 0	$\therefore$ f is strictly decreasing $\downarrow$

Thus, f is strictly increasing in $\left(-\infty, -\dfrac{3}{2}\right)$ (*i.e.,* for $x < -\dfrac{3}{2}$) and strictly decreasing in $\left(-\frac{3}{2}, \infty\right)$ (*i.e.,* for $x > -\frac{3}{2}$).

(c) Let $f(x) = -2x^3 - 9x^2 - 12x + 1$

$\therefore \quad f'(x) = -6x^2 - 18x - 12 = -6(x^2 + 3x + 2)$

Step I. Forming factors of $f'(x)$

$= -6(x^2 + x + 2x + 2) = -6[x(x+1) + 2(x+1)]$

or $\quad f'(x) = -6(x+1)(x+2)$ ...(i)

Step II. $f'(x) = 0$ gives $x = -1$ or $x = -2$

The points $x = -2$ and $x = -1$ (arranged in ascending order) divide the real line into 3 disjoint intervals, namely, $(-\infty, -2)$, $(-2, -1)$ and $(-1, \infty)$.

Step III. Nature of $f(x)$

Interval	**sign of $f'(x)$ $= -6(x+1)(x+2)$ (i)**	**Nature of function f**
$(-\infty, -2)$	Take $x = -3$ (say), Then from (i), $f'(x) = (-)(-)(-)$ $= (-)$ *i.e.,* < 0	$\therefore$ f is strictly decreasing in $(-\infty, -2)$ $\downarrow$
$(-2, -1)$	Take $x = -1.5$ (say), Then from (i), $f'(x) = (-)(-)(+) = +$ *i.e.,* > 0	$\therefore$ f is strictly increasing in $(-2, -1)$ $\uparrow$
$(-1, \infty)$	Take $x = 0$ (say), then from (i) $f'(x) = (-)(+)(+) = (-)$ *i.e.,* < 0	$\therefore$ f is strictly decreasing in $(-1, \infty)$ $\downarrow$

$\therefore$ f is strictly increasing in $(-2, -1)$ and strictly decreasing in $(-\infty, -2)$ and $(-1, \infty)$

(d) Let $f(x) = 6 - 9x - x^2 \quad \therefore \quad f'(x) = -9 - 2x.$

$f(x)$ is strictly **increasing** if $f'(x) > 0$, *i.e.,* if $-9 - 2x > 0$

or $\quad -2x > 9 \quad$ or $\quad x < -\frac{9}{2}$

$\therefore$ f is strictly increasing $\uparrow$, in the interval $\left(-\infty, -\frac{9}{2}\right)$.

$f(x)$ is strictly **decreasing** if $f'(x) < 0$, *i.e.,* if $-9 - 2x < 0$

or $\quad -2x < 9 \quad$ or $\quad x > -\frac{9}{2}$

$\therefore$ f is strictly decreasing $\downarrow$ in the interval $\left(-\frac{9}{2}, \infty\right)$.

(*e*) Let $f(x) = (x + 1)^3 (x - 3)^3$

then $f'(x) = (x + 1)^3 \cdot 3(x - 3)^2 + (x - 3)^3 \cdot 3(x + 1)^2$

$= 3(x + 1)^2 (x - 3)^2 (x + 1 + x - 3)$

$= 3(x + 1)^2 (x - 3)^2 (2x - 2)$

$= 6(x + 1)^2 (x - 3)^2 (x - 1)$

The factors $(x + 1)^2$ and $(x - 3)^2$ are non-negative for all x.

$\therefore$ $f(x)$ is strictly **increasing** if

$f'(x) > 0$, *i.e.,* if $x - 1 > 0$ or $x > 1$

$f(x)$ is strictly **decreasing** if $f'(x) < 0$, *i.e.,* if $x - 1 < 0$

or $x < 1$.

Thus, f is strictly increasing $\uparrow$ in $(1, \infty)$ and strictly decreasing $\downarrow$ in $(-\infty, 1)$.

7. Show that $y = \log (1 + x) - \dfrac{2x}{2 + x}$, $x > -1$ is an increasing function of x throughout its domain.

Sol. Given: $y = \log (1 + x) - \dfrac{2x}{2 + x}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+x} \frac{d}{dx}(1 + x) - \left[\frac{(2+x)\frac{d}{dx}(2x) - 2x\frac{d}{dx}(2+x)}{(2+x)^2} \right]$$

$$= \frac{1}{1+x} - \left[\frac{(2+x)2 - 2x}{(2+x)^2} \right] = \frac{1}{1+x} - \frac{(4 + 2x - 2x)}{(2+x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{1+x} - \frac{4}{(2+x)^2} = \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2}$$

$$= \frac{4 + x^2 + 4x - 4 - 4x}{(1+x)(2+x)^2} = \frac{x^2}{(1+x)(2+x)^2} \quad ...(i)$$

Domain of the given function is given to be $x > -1$

$\Rightarrow$ $x + 1 > 0$. Also $(2 + x)^2 > 0$ and $x^2 \geq 0$

$\therefore$ From (*i*), $\dfrac{dy}{dx} \geq 0$ for all x in the domain $(x > -1)$.

$\therefore$ The given function is an increasing function of x (in its domain namely $x > -1$).

Note 1. For an increasing function $\dfrac{dy}{dx} = f'(x) \geq 0$ and for a strictly increasing function $\dfrac{dy}{dx} = f'(x) > 0$.

Note 2. For a decreasing function $\dfrac{dy}{dx} = f'(x) \leq 0$ and for a strictly decreasing function $\dfrac{dy}{dx} = f'(x) < 0$.

8. Find the value of x for which $y = (x(x - 2))^2$ is an increasing function.

Sol. Given: $y\ (= f(x)) = (x(x-2))^2$.

Step I. Find $\frac{dy}{dx}$ and form factors of R.H.S. of value of $\frac{dy}{dx}$.

$$\therefore \quad \frac{dy}{dx} = 2x(x-2)\frac{d}{dx}[x(x-2)]$$

$$\left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = 2x(x-2)\left[x\frac{d}{dx}(x-2) + (x-2)\frac{d}{dx}x\right] \quad \text{(Product Rule)}$$

$$= 2x(x-2)[x + x - 2] = 2x(x-2)(2x-2)$$

$$\text{or} \quad \frac{dy}{dx} = 4x(x-2)(x-1) \qquad ...(i)$$

Step II. Put $\frac{dy}{dx} = 0$.

$\therefore$ From (i) $4x(x-2)(x-1) = 0$

But $4 \neq 0$ $\therefore$ Either $x = 0$ or $x - 2 = 0$ or $x - 1 = 0$

$\Rightarrow$ $x = 0$, $x = 2$, $x = 1$

These three turning points $x = 0$, $x = 1$, $x = 2$ (arranged in their ascending order divide the real line into three sub-intervals $(-\infty, 0]$, $[0, 1]$, $[1, 2]$, $[2, \infty)$.

Step III

Interval	**sign of $\frac{dy}{dx}$ $= 4x(x-2)(x-1)$ ...(i)**	**Nature of $y = f(x)$**
$(-\infty, 0]$	Take $x = -1$ (say). Then from (i), $\frac{dy}{dx} = (-)(-)(-)$ $= (-)$ (or $= 0$ at $x = 0$) *i.e.*, ≤ 0	$\therefore$ $f(x)$ is decreasing $\downarrow$
$[0, 1]$	Take $x = \frac{1}{2}$ (say). Then from (i), $\frac{dy}{dx} = (+)(-)(-)$ $= (+)$ (or $= 0$ at $x = 0$, $x = 1$) *i.e.*, ≥ 0	$\therefore$ $f(x)$ is increasing $\uparrow$
$[1, 2]$	Take $x = 1.5$ (say). Then from (i), $\frac{dy}{dx} = (+)(-)(+)$	$\therefore$ $f(x)$ is decreasing $\downarrow$

	$= (-)$ (or $= 0$ at $x = 1, x = 2$) *i.e.,* ≤ 0	
$[2, \infty)$	Take $x = 3$ (say). Then from (*i*), $\frac{dy}{dx} = (+)\,(+)\,(+)$ $= (+)$ (or $= 0$ at $x = 2$) *i.e.,* ≥ 0	$\therefore$ $f(x)$ is increasing $\uparrow$

Therefore, $f(x)$ is an increasing function in the intervals $[0, 1]$ (*i.e.,* $0 \le x \le 1$) and $[2, \infty)$ (*i.e.,* $x \ge 2$).

Remark. (We have included the turning points in the sub-intervals because we are to discuss for increasing function and not for strictly increasing function. See Notes 1 and 2 at the end of solution of Q. No. 7).

9. Prove that $y = \dfrac{4 \sin \theta}{(2 + \cos \theta)} - \theta$ is an increasing function of θ in $\left[0, \dfrac{\pi}{2}\right]$.

Sol. Here $y = \dfrac{4 \sin \theta}{(2 + \cos \theta)} - \theta$

$$\Rightarrow \quad \frac{dy}{d\theta} = \frac{(2 + \cos \theta) \,.\, 4 \cos \theta - 4 \sin \theta\, (- \sin \theta)}{(2 + \cos \theta)^2} - 1$$

$$= \frac{8 \cos \theta + 4 \cos^2 \theta + 4 \sin^2 \theta}{(2 + \cos \theta)^2} - 1$$

$$= \frac{8 \cos \theta + 4 (\cos^2 \theta + \sin^2 \theta) - (2 + \cos \theta)^2}{(2 + \cos \theta)^2} \qquad \text{(Taking L.C.M.)}$$

$$= \frac{8 \cos \theta + 4 - (2 + \cos \theta)^2}{(2 + \cos \theta)^2} = \frac{(8 \cos \theta + 4) - (4 + 4 \cos \theta + \cos^2 \theta)}{(2 + \cos \theta)^2}$$

or $$\frac{dy}{d\theta} = \frac{4 \cos \theta - \cos^2 \theta}{(2 + \cos \theta)^2} = \frac{\cos \theta\, (4 - \cos \theta)}{(2 + \cos \theta)^2} \qquad ...(i)$$

Since $0 \le \theta \le \dfrac{\pi}{2}$,

we have $0 \le \cos \theta \le 1$ and, therefore, $4 - \cos \theta > 0$. Also $(2 + \cos \theta)^2 > 0$

$\therefore$ From (*i*), $\dfrac{dy}{d\theta} \ge 0$ for $0 \le \theta \le \dfrac{\pi}{2}$.

Hence, y is an increasing function of θ in $\left[0, \dfrac{\pi}{2}\right]$.

10. Prove that the logarithmic function is strictly increasing on $(0, \infty)$.

Sol. Given: $f(x) = \log x$

$\therefore \quad f'(x) = \frac{1}{x} > 0$ for all x in $(0, \infty)$ $\quad [\because \quad x \in (0, \infty) \quad \Rightarrow \quad x > 0]$

$\therefore \quad f(x)$ is strictly increasing on $(0, \infty)$.

11. Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor strictly decreasing on $(-1, 1)$.

Sol. Given: $\quad f(x) = x^2 - x + 1$

$\therefore \quad f'(x) = 2x - 1$

$f(x)$ is strictly increasing if $f'(x) > 0$ *i.e.,* if $2x - 1 > 0$

i.e., if $2x > 1$ or $x > \frac{1}{2}$

$f(x)$ is strictly decreasing if

$$f'(x) < 0 \text{ i.e., if } 2x - 1 < 0 \text{ i.e., } x < \frac{1}{2}$$

$\therefore f(x)$ is strictly increasing for $x > \frac{1}{2}$ *i.e.,* on the interval $\left(\frac{1}{2}, 1\right)$

$[\because$ The given interval is $(-1, 1)]$

and $f(x)$ is strictly decreasing for $x < \frac{1}{2}$ *i.e.,* on the interval

$\left(-1, \frac{1}{2}\right)$. $\quad [\because$ The given interval is $(-1, 1)]$

$\therefore f(x)$ is neither strictly increasing nor strictly decreasing on the interval $(-1, 1)$.

12. Which of the following functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$?

(A) $\cos x$ (B) $\cos 2x$ (C) $\cos 3x$ (D) $\tan x$.

Sol. (A) Let $f(x) = \cos x$ then $f'(x) = -\sin x$

$\because \quad 0 < x < \frac{\pi}{2}$ in $\left(0, \frac{\pi}{2}\right)$, therefore $\sin x > 0$

[Because $\sin x$ is positive in both first and second quadrants]

$\Rightarrow \quad -\sin x < 0 \quad \therefore \quad f'(x) = -\sin x < 0$ on $\left(0, \frac{\pi}{2}\right)$

$\Rightarrow \quad f(x)$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

(B) Let $f(x) = \cos 2x$ then $f'(x) = -2 \sin 2x$

$\because \quad 0 < x < \frac{\pi}{2}, \quad \therefore \quad 0 < 2x < \pi$

$\Rightarrow \quad \sin 2x > 0 \quad \Rightarrow \quad -2 \sin 2x < 0$

$\therefore\ f'(x) = -\,2\sin 2x < 0$ on $\left(0, \frac{\pi}{2}\right)$

$\Rightarrow f(x)$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

(C) Let $f(x) = \cos 3x$ then $f'(x) = -\,3\sin 3x$

$\because\quad 0 < x < \frac{\pi}{2}, \quad \therefore \quad 0 < 3x < \frac{3\pi}{2} = 270°$

Now for $0 < 3x < \pi$, $\left(i.e., 0 < x < \frac{\pi}{3}\right)$ $\sin 3x > 0$

($\because$ $\sin\theta$ is positive in first two quadrants)

$\Rightarrow\ f'(x) = -\,3\sin 3x < 0 \Rightarrow f'(x) < 0$

$\Rightarrow f(x)$ is strictly decreasing on $\left(0, \frac{\pi}{3}\right)$

and for $\pi < 3x < \frac{3\pi}{2}$, $\sin 3x < 0$

[Because $\sin\theta$ is negative in third quadrant]

$\therefore\ f'(x) = -\,3\sin 3x > 0 \Rightarrow f'(x) > 0$

$\Rightarrow f(x)$ is strictly increasing on $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

$\therefore$ $f(x)$ is neither strictly increasing nor strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

(D) Let $f(x) = \tan x$ then $f'(x) = \sec^2 x > 0$

$\Rightarrow f(x)$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

Hence, only the functions in options **(A)** and **(B)** are strictly decreasing.

13. On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ is strictly decreasing?

(A) (0, 1) **(B) $\left(\frac{\pi}{2}, \pi\right)$** **(C) $\left(0, \frac{\pi}{2}\right)$** **(D) None of these.**

Sol. Given: $f(x) = x^{100} + \sin x - 1$

$\therefore\quad f'(x) = 100\, x^{99} + \cos x \quad \ldots(i)$

Let us test option (A) (0, 1)

On (0, 1); $x > 0$ and hence $100\, x^{99} > 0$

For $\cos x$; interval (0, 1) $\Rightarrow$ (0, 1 radian)

$\Rightarrow$ (0, 57° nearly) ($\because$ π radians = 180°

$\Rightarrow\quad 1 \text{ radian} = \frac{180°}{\pi}$

$= \frac{180°}{\left(\frac{22}{7}\right)} = 180° \times \frac{7}{22} = \frac{90° \times 7}{11} = \frac{630°}{11} = 57°$ nearly)

$\Rightarrow$ x is in first quadrant and hence cos x is positive.

$\therefore$ From (i), $f'(x) = 100x^{99} + \cos x > 0$ and hence $f(x)$ is strictly increasing on (0, 1).

$\therefore$ Option (A) is not the correct option.

Let us test option (B) $\left(\frac{\pi}{2}, \pi\right)$

For $100x^{99}$, $x \in \left(\frac{\pi}{2}, \pi\right)$

$$\Rightarrow \quad x \in \left(\frac{\left(\frac{22}{7}\right)}{2}, \frac{22}{7}\right) = \left(\frac{11}{7}, \frac{22}{7}\right) = (1.5, 3.1)$$

$\Rightarrow$ $x > 1$ $\Rightarrow$ $x^{99} > 1$ and hence $100x^{99} > 100$.

For cos x, $\left(\frac{\pi}{2}, \pi\right)$ $\Rightarrow$ Second quadrant and hence cos x is negative and has value between -1 and 0.

($\because$ $-1 \le \cos\theta \le 1$)

$\therefore$ From (i), $f'(x) = 100x^{99} + \cos x > 100 - 1 = 99 > 0$

$\therefore$ $f(x)$ is strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$.

$\therefore$ Option (B) is not the correct option.

Let us test option (C) $\left(0, \frac{\pi}{2}\right)$

On $\left(0, \frac{\pi}{2}\right)$ *i.e.,* (0, 1.5) both terms $100x^{99}$ and cos x are positive and hence from (i), $f'(x) = 100x^{99} + \cos x$ is positive.

$\therefore$ $f(x)$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ also.

$\therefore$ Option (C) is also not the correct option.

$\therefore$ Option (D) is the correct answer.

14. Find the least value of a such that the function f given by $f(x) = x^2 + ax + 1$ strictly increasing on (1, 2).

Sol. Here $f(x) = x^2 + ax + 1$ ...(i)

Differentiating (i) w.r.t. x, $f'(x) = 2x + a$ ...(ii)

Because $f(x)$ is strictly increasing on (1, 2) (given),

$\therefore$ $f'(x) = 2x + a > 0$ for all x in (1, 2) ...(iii)

Now on (1, 2), $1 < x < 2$

Multiplying by 2, $2 < 2x < 4$ for all x in (1, 2).

Adding a to all sides

$2 + a < 2x + a < 4 + a$ for all x in (1, 2)

or $2 + a < f'(x) < 4 + a$ for all x in (1, 2) [By (ii)]

$\therefore$ Minimum value of $f'(x)$ is $2 + a$ and maximum value of

$f'(x)$ is $4 + a$. ...(*iv*)

But from (*iii*), $f'(x) > 0$ for all x in (1, 2)

$\therefore\ 2 + a > 0$ and $4 + a > 0$ [By (*iv*)]

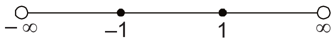
$\therefore\ a > -2$ and $a > -4$

$\therefore\ a > -2$ [$\because\ a > -2 \Rightarrow a > -4$ automatically]

$\therefore$ Least value of a is -2.

15. Let I be any interval disjoint from [– 1, 1]. Prove that the function *f* given by $f(x) = x + \dfrac{1}{x}$ is strictly increasing on I.

Sol. Given: $f(x) = x + \dfrac{1}{x} = x + x^{-1}$



$$\therefore\quad f'(x) = 1 + (-1)x^{-2} = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

Forming factors, $f'(x) = \dfrac{(x - 1(x + 1)}{x^2}$ ...(*i*)

Given: I is an interval disjoint from [– 1, 1].

i.e., I = (– ∞, ∞) – [– 1, 1] = (– ∞, – 1) ∪ (1, ∞)

$\therefore$ For every $x \in$ I, either $x < -1$ or $x > 1$

For $x < -1$ (For example, $x = -2$ (say)),

from (*i*), $f'(x) = \dfrac{(-)\,(-)}{(+)} = (+)$ *i.e.,* > 0

For $x > 1$ (For example, $x = 2$ (say)),

from (*i*), $f'(x) = \dfrac{(+)\,(+)}{(+)} = (+)$ *i.e.,* > 0

$\therefore\ f'(x) > 0$ for all $x \in$ I $\quad\therefore\ f(x)$ is strictly increasing on I.

16. Prove that the function *f* given by $f(x) = \log \sin x$ is strictly increasing on $\left(0, \dfrac{\pi}{2}\right)$ and strictly decreasing on $\left(\dfrac{\pi}{2}, \pi\right)$.

Sol. Given: $f(x) = \log \sin x$

$$\therefore\quad f'(x) = \frac{1}{\sin x}\frac{d}{dx}\sin x = \frac{1}{\sin x}(\cos x) = \cot x \qquad ...(i)$$

On the interval $\left(0, \dfrac{\pi}{2}\right)$ *i.e.,* in first quadrant, from (*i*), $f'(x) = \cot x > 0$

$\therefore\ f(x)$ is strictly increasing on $\left(0, \dfrac{\pi}{2}\right)$.

On the interval $\left(\dfrac{\pi}{2}, \pi\right)$ *i.e.,* in second quadrant, from (*i*), $f'(x) = \cot x < 0$.

$\therefore\ f(x)$ is strictly decreasing on $\left(\dfrac{\pi}{2}, \pi\right)$.

17. Prove that the function f given by $f(x)$ = log cos x is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ and strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$.

Sol. Given: $f(x) = \log \cos x$

$\therefore\ f'(x) = \frac{1}{\cos x} \frac{d}{dx} (\cos x) = \frac{1}{\cos x} (-\sin x) = -\tan x \quad ...(i)$

We know that on the interval $\left(0, \frac{\pi}{2}\right)$ *i.e.,* in first quadrant, tan x is positive and hence from (i), $f'(x) = -\tan x$ is negative *i.e.,* < 0.

$\therefore\ f(x)$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

We know that on the interval $\left(\frac{\pi}{2}, \pi\right)$ *i.e.,* in second quadrant; tan x is negative and hence from (i),

$f'(x) = -\tan x$ is positive *i.e.,* > 0.

$\therefore\ f(x)$ is strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$.

18. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in R.

Sol. Given: $f(x) = x^3 - 3x^2 + 3x - 100$.

Then $f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1)$

$= 3(x-1)^2 \geq 0$ for all x in R

$\therefore\ f(x)$ is increasing on R.

19. The interval in which $y = x^2 e^{-x}$ is increasing is

(A) $(-\infty, \infty)$ (B) $(-2, 0)$ (C) $(2, \infty)$ (D) $(0, 2)$.

Sol. Given: $y\ (= f(x)) = x^2 e^{-x}$

$$\therefore\ \frac{dy}{dx} = x^2 \frac{d}{dx} e^{-x} + e^{-x} \frac{d}{dx} x^2 = x^2 e^{-x} (-1) + e^{-x} (2x)$$

$$= -x^2 e^{-x} + 2x e^{-x} = x e^{-x} (-x + 2)$$

or $\frac{dy}{dx} = \frac{x(2-x)}{e^x}$

Out of the intervals mentioned in the options (A), (B), (C) and (D), $\frac{dy}{dx} > 0$ for all x in interval $(0, 2)$ of option (D).

$\therefore\ y\ (= f(x))$ is strictly increasing and hence increasing in interval $(0, 2)$ of option D.

Note. For a subjective solution of this question, proceed as in solution of Q. No. 6 (a), (b), (c).

Remark. Increasing (decreasing) function or monotonically increasing (or monotonically decreasing) function have the same meaning.

Exercise 6.3

1. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.

Sol. Given: Equation of the curve is $y = 3x^4 - 4x$ ...(*i*)

$\therefore$ Slope of the tangent to the curve $y = f(x)$ at the point (x, y)

$= \text{Value of } \dfrac{dy}{dx} \text{ at the point } (x, y)$

$= 3(4x^3) - 4 = 12x^3 - 4$

$\therefore$ Slope of the tangent at (point) $x = 4$ to curve (*i*) is

$12(4)^3 - 4 = 12 \times 64 - 4 = 768 - 4 = 764.$

2. Find the slope of the tangent to the curve $y = \dfrac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.

Sol. Given: Equation of the curve is $y = \dfrac{x-1}{x-2}$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = \frac{(x-2)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x-2)}{(x-2)^2}$$

$$\text{or} \quad \frac{dy}{dx} = \frac{(x-2)-(x-1)}{(x-2)^2} = \frac{x-2-x+1}{(x-2)^2} = \frac{-1}{(x-2)^2} \quad ...(ii)$$

Putting $x = 10$ (given) in (*ii*), slope of the tangent to the given curve (*i*), at $x = 10$ (= value of $\dfrac{dy}{dx}$ at $x = 10$)

$$= \frac{-1}{(10-2)^2} = \frac{-1}{(8)^2} = \frac{-1}{64}.$$

3. Find the slope of the tangent to the curve $y = x^3 - x + 1$ at the point whose x-coordinate is 2.

Sol. Given: Equation of the curve is $y = x^3 - x + 1$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 1$$

Slope of the tangent to curve (*i*) at $x = 2$ (given)

$= \text{Value of } \dfrac{dy}{dx} \text{ (at } x = 2) = 3.2^2 - 1 = 3(4) - 1$

$= 12 - 1 = 11.$

4. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.

Sol. Given: Equation of the curve is $y = x^3 - 3x + 2$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 3$$

Slope of the tangent of curve (*i*) at $x = 3$ (given)

$= \text{Value of } \dfrac{dy}{dx} \text{ (at } x = 3) = 3 \,.\, 3^2 - 3 = 3 \,.\, 9 - 3$

$= 27 - 3 = 24.$

5. Find the slope of the normal to the curve

$$\boldsymbol{x = a \cos^3 \theta,\ y = a \sin^3 \theta \quad \text{at} \quad \theta = \frac{\pi}{4}}.$$

Sol. Given: Equations of the curve are

$x = a \cos^3 \theta,\ y = a \sin^3 \theta$

$$\therefore \quad \frac{dx}{d\theta} = a \frac{d}{d\theta} (\cos \theta)^3 \quad \text{and} \quad \frac{dy}{d\theta} = a \frac{d}{d\theta} (\sin \theta)^3$$

$$= a.3 (\cos \theta)^2 \frac{d}{d\theta} (\cos \theta) \quad \text{and}$$

$$= a.3 (\sin \theta)^2 \frac{d}{d\theta} \sin \theta$$

$$\text{or} \quad \frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta \quad \text{and} \quad \frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta} = \frac{-\sin \theta}{\cos \theta} = -\tan \theta$$

Slope of the tangent at $\theta = \dfrac{\pi}{4}$ (given) = value of $\dfrac{dy}{dx}$ at $\left(\theta = \dfrac{\pi}{4}\right)$

$$= -\tan \frac{\pi}{4} = -1.$$

$\therefore$ Slope of the normal $\left(\text{at } \theta = \dfrac{\pi}{4}\right)$ = negative reciprocal of slope of tangent = 1. $\left(\because \dfrac{-1}{m} = \dfrac{-1}{-1} = 1\right)$.

6. Find the slope of the normal to the curve

$$\boldsymbol{x = 1 - a \sin \theta,\ y = b \cos^2 \theta \text{ at } \theta = \frac{\pi}{2}}.$$

Sol. Given: Equations of the curve are

$x = 1 - a \sin \theta,\ y = b \cos^2 \theta$

$$\therefore \quad \frac{dx}{d\theta} = 0 - a \cos \theta \quad \text{and} \quad \frac{dy}{d\theta} = b \frac{d}{d\theta} (\cos \theta)^2$$

$$\Rightarrow \quad \frac{dx}{d\theta} = -a \cos \theta \quad \text{and} \quad \frac{dy}{d\theta} = b.2 (\cos \theta) \frac{d}{d\theta} \cos \theta$$

$$= -2b \cos \theta \sin \theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2b \cos \theta \sin \theta}{-a \cos \theta} = \frac{2b}{a} \sin \theta$$

Slope of the tangent (at $\theta = \dfrac{\pi}{2}$ (given))

$$= \text{value of } \frac{dy}{dx} \left(\text{at } \theta = \frac{\pi}{2}\right)$$

$$= \frac{2b}{a} \sin \frac{\pi}{2} = \frac{2b}{a} (1)$$

$$= \frac{2b}{a} \quad (= m \text{ say})$$

$\therefore$ Slope of the normal $\left(\text{at } \theta = \frac{\pi}{2}\right) = \frac{-1}{m} = -\frac{a}{2b}$.

7. Find the points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis.

Sol. Equation of curve is $y = x^3 - 3x^2 - 9x + 7$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = 3x^2 - 6x - 9$ = Slope of tangent at (x, y)

Since the tangent is parallel to the x-axis, $\frac{dy}{dx} = 0$

$\Rightarrow \quad 3x^2 - 6x - 9 = 0$ or $x^2 - 2x - 3 = 0$

or $(x - 3)(x + 1) = 0 \qquad \therefore \quad x = 3, -1$

When $x = 3$, from (*i*), $y = 27 - 27 - 27 + 7 = -20$

When $x = -1$, from (*i*), $y = -1 - 3 + 9 + 7 = 12$.

$\therefore$ The required points are $(3, -20)$ and $(-1, 12)$.

8. Find a point on the curve $y = (x - 2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

Sol. Let A(2, 0) and B(4, 4) be the given points.

Slope of chord AB $= \frac{4-0}{4-2} = 2 \qquad \left[\because m = \frac{y_2 - y_1}{x_2 - x_1}\right]$

Equation of curve is $y = (x - 2)^2$

$\therefore$ Slope of tangent at $(x, y) = \frac{dy}{dx} = 2(x - 2)$.

If the tangent is parallel to the chord AB, then

slope of tangent = slope of chord

$\Rightarrow \quad 2(x - 2) = 2 \Rightarrow 2x - 4 = 2 \Rightarrow 2x = 6 \Rightarrow x = 3$

$\therefore \qquad y = (3 - 2)^2 = 1$

Hence, the required point is (3, 1).

9. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

Sol. Equation of curve is $\quad y = x^3 - 11x + 5$ ...(*i*)

Equation of tangent is $\quad y = x - 11$

or $\qquad x - y - 11 = 0$ ...(*ii*)

From (*i*), $\qquad \frac{dy}{dx} = 3x^2 - 11$

$\Rightarrow$ Slope of tangent at (x, y) is $3x^2 - 11$.

But slope of tangent from (*ii*) is $\frac{-a}{b} = \frac{-1}{-1} = 1$.

$\therefore \quad 3x^2 - 11 = 1$ or $3x^2 = 12$ or $x^2 = 4 \quad \therefore \quad x = \pm 2$

From (*i*), when $x = 2, \quad y = 8 - 22 + 5 = -9$

when $\qquad x = -2 \quad y = -8 + 22 + 5 = 19$

$\therefore$ We get two points $(2, -9)$ and $(-2, 19)$. Of these, $(-2, 19)$

does not satisfy eqn. (*ii*) while (2, – 9) does. Hence, the required point is (2, – 9).

10. Find the equation of all lines having slope – 1 that are tangents to the curve $y = \dfrac{1}{x-1}$, $x \neq 1$.

Sol. Given: Equation of the curve is $y = \dfrac{1}{x-1} = (x-1)^{-1}$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = (-1)(x-1)^{-2}\frac{d}{dx}(x-1) = \frac{-1}{(x-1)^2}$$

= Slope of the tangent to the given curve at any point (x, y).

But the slope is given to be – 1

$$\therefore \quad \frac{-1}{(x-1)^2} = -1 \Rightarrow -(x-1)^2 = -1$$

$$\Rightarrow \quad (x-1)^2 = 1 \quad \Rightarrow \quad x - 1 = \pm 1 \quad \Rightarrow \quad x = 1 \pm 1$$

$$\Rightarrow \quad x = 1 + 1 = 2 \quad \text{or} \quad x = 1 - 1 = 0$$

Putting $x = 2$ in (*i*), $y = \dfrac{1}{2-1} = \dfrac{1}{1} = 1$

$\therefore$ One point of contact is (2, 1).

$\therefore$ Equation of one required tangent is $y - 1 = -1(x - 2)$

$[\because \ y - y_1 = m(x - x_1)]$

i.e., $y - 1 = -x + 2$ or $x + y - 3 = 0$

Putting $x = 0$ in (*i*), $y = \dfrac{1}{0-1} = \dfrac{1}{-1} = -1$

$\therefore$ The other point of contact is (0, – 1).

$\therefore$ Equation of the other tangent is

$$y - (-1) = -1(x - 0) \quad \text{or} \quad y + 1 = -x$$

or $\quad x + y + 1 = 0$

$\therefore$ Equations of required tangents are

$$x + y - 3 = 0 \quad \text{and} \quad x + y + 1 = 0.$$

11. Find the equations of all lines having slope 2 which are tangents to the curve $y = \dfrac{1}{x-3}$, $x \neq 3$.

Sol. Equation of curve is $y = \dfrac{1}{x-3} = (x-3)^{-1}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = (-1)(x-3)^{-2} = \frac{-1}{(x-3)^2}$$

= Slope of tangent to the given curve at any point (x, y)

But the slope is given to be 2.

$$\therefore \quad \frac{-1}{(x-3)^2} = 2 \quad \text{or} \quad 2(x-3)^2 = -1 \quad \text{or} \quad (x-3)^2 = -\frac{1}{2} < 0$$

which is not possible since $(x - 3)^2 > 0$.

Hence, there is no tangent to the given curve having slope 2.

12. Find the equations of all lines having slope 0 which are tangents to the curve

$$y = \frac{1}{x^2 - 2x + 3}.$$

Sol. Equation of curve is $y = \dfrac{1}{x^2 - 2x + 3}$ ...(*i*)

Differentiating w.r.t. x, we have

$= \dfrac{dy}{dx} = \dfrac{d}{dx}[(x^2 - 2x + 3)^{-1}] = -(x^2 - 2x + 3)^{-2} . (2x - 2)$

$= \dfrac{-2(x-1)}{(x^2 - 2x + 3)^2}$

= Slope of tangent to the given curve at any point (x, y)

But the slope (of tangent) is given to be 0

$\therefore \quad \dfrac{-2(x-1)}{(x^2 - 2x + 3)^2} = 0 \quad \Rightarrow -2(x-1) = 0$

$\Rightarrow \quad x - 1 = 0 \quad \Rightarrow \quad x = 1$

Putting $x = 1$ in (*i*), we have $y = \dfrac{1}{1 - 2 + 3} = \dfrac{1}{2}$

Thus the point on the curve at which tangent has slope 0 is $\left(1, \dfrac{1}{2}\right)$.

$\therefore$ Equation of tangent is $y - \dfrac{1}{2} = 0\ (x - 1)$

or $y - \dfrac{1}{2} = 0$ or $y = \dfrac{1}{2}$.

13. Find the points on the curvem $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ at which the tangents are

(*i*) parallel to *x*-axis **(*ii*) parallel to *y*-axis.**

Sol. Given: Equation of the curve is $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ ...(*i*)

Differentiating both sides of eqn. (*i*) w.r.t. x, we have

$\dfrac{2x}{9} + \dfrac{2y}{16}\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad \dfrac{2y}{16}\dfrac{dy}{dx} = -\dfrac{2x}{9}$

$\Rightarrow \quad 18y\dfrac{dy}{dx} = -32x \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{-32x}{18y} = \dfrac{-16x}{9y}$ ...(*ii*)

(*i*) If tangent is **parallel to *x*-axis,** $\Rightarrow$ Slope of tangent = 0

$\Rightarrow \quad \boldsymbol{\dfrac{dy}{dx} = 0} \quad \Rightarrow \quad$ From (*ii*), $\dfrac{-16x}{9y} = 0 \Rightarrow -16x = 0$

$\Rightarrow \quad x = \dfrac{0}{-16} = 0$

Putting $x = 0$ in (i), $\frac{y^2}{16} = 1$ or $y^2 = 16$. Therefore, $y = \pm 4$.

$\therefore$ The points on curve (i) where tangents are parallel to x-axis are $(0, \pm 4)$.

(ii) If the tangent is **parallel to y-axis**

$\Rightarrow$ Slope of the tangent $= \pm \infty$ $\Rightarrow$ $\frac{dy}{dx} = \pm \infty$

$$\Rightarrow \frac{dx}{dy} = 0$$

$\therefore$ From (ii), $\frac{9y}{-16x} = 0 \Rightarrow 9y = 0 \Rightarrow y = \frac{0}{9} = 0$

Putting $y = 0$ in (i), $\frac{x^2}{9} = 1$ or $x^2 = 9$

$\therefore$ $x = \pm 3$.

Hence the points on the curve at which the tangent are parallel to y-axis are $(\pm 3, 0)$.

14. Find the equations of the tangent and normal to the given curves at the indicated points:

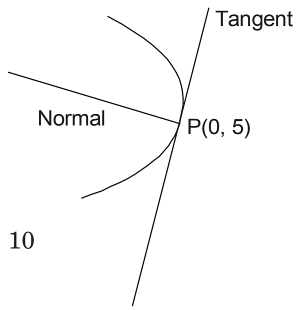
(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5).
(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3).
(iii) $y = x^3$ at (1, 1).
(iv) $y = x^2$ at (0, 0).
(v) $x = \cos t$, $y = \sin t$ at $t = \frac{\pi}{4}$.

Sol. (i) **Given:** Equation of the curve is

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5 \quad ...(i)$$

$\therefore$ $\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$

= Slope of the tangent at point (x, y)



$\therefore$ Slope of tangent at (0, 5)

$= $ Value of $\frac{dy}{dx}$ at (0, 5)

(Putting $x = 0$)

$= 4(0)^3 - 18(0)^2 + 26(0) - 10$

$= -10$ ($= m$ say)

$\therefore$ Slope of the normal at (0, 5)

$$= \frac{-1}{m} = \frac{-1}{-10} = \frac{1}{10}$$

$\therefore$ Equation of the tangent at (0, 5) is

$y - 5 = -10(x - 0)$ $\quad | \ y - y_1 = m(x - x_1)$

i.e., $y - 5 = -10x$ or $10x + y = 5$

and equation of the normal at (0, 5) is

$$y - 5 = \frac{1}{10}(x - 0)$$

$\Rightarrow \quad 10y - 50 = x \quad$ *i.e.,* $\quad - x + 10y - 50 = 0$

or $\quad x - 10y + 50 = 0.$

(*ii*) **Given:** Equation of the curve is

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5 \qquad ...(i)$$

$$\therefore \quad \frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

= Slope of the tangent at the point (x, y)

$\therefore$ Slope of the tangent at (1, 3) = Value of $\frac{dy}{dx}$ at (1, 3).

(Putting $x = 1$) $= 4(1)^3 - 18(1)^2 + 26(1) - 10$

$= 4 - 18 + 26 - 10 = 30 - 28 = 2$ (= m say)

$\therefore$ Slope of the normal at (1, 3) $= \frac{-1}{m} = \frac{-1}{2}$

$\therefore$ Equation of the tangent at (1, 3) is $y - 3 = 2(x - 1)$

$\Rightarrow \quad y - 3 = 2x - 2 \quad \Rightarrow \quad y = 2x + 1$

and equation of the normal at (1, 3) is $y - 3 = \frac{-1}{2}(x - 1)$

$\Rightarrow \quad 2(y - 3) = -(x - 1) \quad \Rightarrow \quad 2y - 6 = -x + 1$

$\Rightarrow \quad x + 2y - 7 = 0.$

(*iii*) **Given:** Equation of the curve is

$$y = x^3 \qquad ...(i)$$

$\therefore \quad \frac{dy}{dx} = 3x^2$ = Slope of the tangent at the point (x, y).

$\therefore$ Slope of the tangent at (1, 1) = Value of $\frac{dy}{dx}$ at (1, 1).

(Putting $x = 1$) $= 3.1^2 = 3 = m$(say)

$\therefore$ Slope of the normal at (1, 1) $= \frac{-1}{m} = \frac{-1}{3}$

$\therefore$ Equation of the tangent at (1, 1) is $y - 1 = 3(x - 1)$

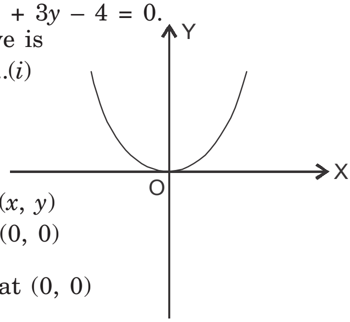
$\Rightarrow \quad y - 1 = 3x - 3 \quad \Rightarrow \quad y = 3x - 2$

and equation of the normal at (1, 1) is $y - 1 = \frac{-1}{3}(x - 1)$

$\Rightarrow \quad 3y - 3 = -x + 1 \quad \Rightarrow \quad x + 3y - 4 = 0.$

(*iv*) **Given:** Equation of the curve is

$$y = x^2 \qquad ...(i)$$



$\therefore \quad \frac{dy}{dx} = 2x$

= Slope of the tangent at (x, y)

$\therefore$ Slope of the tangent at (0, 0)

= Value of $\frac{dy}{dx}$ at (0, 0)

(Putting $x = 0$) $= 2 \times 0 = 0$ (= m say)

$\therefore$ Tangent at (0, 0) to curve (*i*) is $(y - 0) = 0\ (x - 0)$ or $y = 0$ *i.e.* x-axis and hence normal at (0, 0) to curve (*i*) is y-axis.

(*v*) **Given:** Equations of the curve are

$$x = \cos t, \quad y = \sin t$$

$$\therefore \quad \frac{dx}{dt} = -\sin t \text{ and } \frac{dy}{dt} = \cos t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t$$

$$= \text{Slope of the tangent at } (x, y)$$

$\therefore$ Slope of the tangent at $t = \frac{\pi}{4}$ is value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$

$$= -\cot \frac{\pi}{4} = -1 \ (= m \text{ say})$$

$\therefore$ Slope of the normal at $t = \frac{\pi}{4}$ is $\frac{-1}{m} = \frac{-1}{-1} = 1$

Point $t = \frac{\pi}{4} \Rightarrow$ Point $(x, y) = (\cos t, \sin t)$

$$= \left(\cos \frac{\pi}{4}, \sin \frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$\therefore$ Equation of the tangent is $y - \frac{1}{\sqrt{2}} = -1\left(x - \frac{1}{\sqrt{2}}\right)$

$$\Rightarrow y - \frac{1}{\sqrt{2}} = -x + \frac{1}{\sqrt{2}} \Rightarrow x + y = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

or $\quad x + y = \sqrt{2}$ $\quad \left[\because \frac{2}{\sqrt{2}} = \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}} = \sqrt{2}\right]$

and equation of the normal at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is

$$y - \frac{1}{\sqrt{2}} = 1\left(x - \frac{1}{\sqrt{2}}\right) \text{ or } y - \frac{1}{\sqrt{2}} = x - \frac{1}{\sqrt{2}}$$

or $\quad y = x.$

15. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is

(*a*) parallel to the line $2x - y + 9 = 0$

(*b*) perpendicular to the line $5y - 15x = 13$.

Sol. Given: Equation of the curve is $y = x^2 - 2x + 7$ ...(*i*)

$\therefore$ Slope of the tangent $= \frac{dy}{dx} = 2x - 2$ ...(*ii*)

(*a*) Slope of the given line $2x - y + 9 = 0$ is

$$\frac{-\text{coeff. of } x}{\text{coeff. of } y} \left(\frac{-a}{b}\right) = \frac{-2}{-1} = 2$$

$\therefore$ Slope of tangent parallel to this line is also = 2

($\because$ Parallel lines have same slope)

$\Rightarrow$ (By (*ii*)), $2x - 2 = 2 \Rightarrow 2x = 2 + 2 = 4$

$\Rightarrow \qquad x = \frac{4}{2} = 2$

Putting $x = 2$ in (*i*), $y = 4 - 4 + 7 = 7$

$\therefore$ Point of contact is (2, 7)

$\therefore$ Equation of the tangent at (2, 7) is

$y - 7 = 2(x - 2)$ or $y - 7 = 2x - 4$

or $\qquad y - 2x - 3 = 0.$

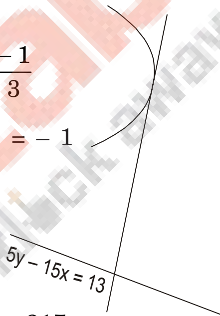
(*b*) Slope of the given line

$5y - 15x = 13$ *i.e.*, $-15x + 5y = 13$

is $\frac{-a}{b} = -\left(\frac{-15}{5}\right) = 3 = (m \text{ say})$

$\therefore$ Slope of the required tangent

perpendicular to this line $= \frac{-1}{m} = \frac{-1}{3}$



$\Rightarrow$ (By (*ii*)) $2x - 2 = \frac{-1}{3} \Rightarrow 6x - 6 = -1$

$\Rightarrow \qquad 6x = 6 - 1 = 5 \Rightarrow x = \frac{5}{6}$

Putting $x = \frac{5}{6}$ in (*i*), $y = \frac{25}{36} - \frac{5}{3} + 7$

$$= \frac{25 - 60 + 252}{36} = \frac{277 - 60}{36} = \frac{217}{36}$$

$\therefore$ Point of contact is $\left(\frac{5}{6}, \frac{217}{36}\right)$

$\therefore$ Equation of the required tangent at $\left(\frac{5}{6}, \frac{217}{36}\right)$ is

$$y - \frac{217}{36} = \frac{-1}{3}\left(x - \frac{5}{6}\right)$$

$\Rightarrow 3y - \frac{217}{12} = -x + \frac{5}{6} \qquad \Rightarrow x + 3y = \frac{217}{12} + \frac{5}{6}$

$\Rightarrow \qquad x + 3y = \frac{217 + 10}{12} = \frac{227}{12}$

Cross-multiplying, $12x + 36y = 227.$

16. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel.

Sol. Given: Equation of the given curve is $y = 7x^3 + 11$

$\therefore \quad \frac{dy}{dx} = 21x^2$ = Slope of the tangent to the curve at (x, y)

Putting $x = 2$, slope of the tangent $= 21(2)^2 = 21 \times 4 = 84$

Putting $x = -2$, slope of the tangent $= 21(-2)^2 = 21 \times 4 = 84$
Since the slopes of the two tangents are equal (each = 84), therefore, tangents at $x = 2$ and $x = -2$ are parallel.

17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

Sol. Given: Equation of the curve is $y = x^3$ ...(*i*)

$\therefore \dfrac{dy}{dx} = 3x^2 =$ Slope of the tangent at the point (x, y) ...(*ii*)

Given: Slope of the tangent = y-coordinate of the point.
Putting values from (*ii*) and (*i*),

$$3x^2 = x^3 \Rightarrow 3x^2 - x^3 = 0 \Rightarrow x^2(3 - x) = 0$$

$\therefore$ Either $x^2 = 0$ *i.e.,* $x = 0$ or $3 - x = 0$ *i.e.,* $x = 3$
Putting $x = 0$ in (*i*), $y = 0$ $\therefore$ Point is $(0, 0)$
Putting $x = 3$ in (*i*), $y = 3^3 = 27$ $\therefore$ Point is $(3, 27)$
$\therefore$ The required points are $(0, 0)$ and $(3, 27)$.

18. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.

Sol. Equation of curve is

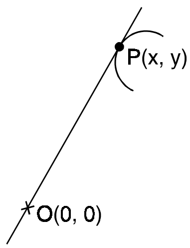
$$y = 4x^3 - 2x^5 \quad ...(i)$$

Let the required point be $P(x, y)$, the tangent at which passes through the origin $O(0, 0)$.

Differentiating both sides of eqn. (*i*) w.r.t. x, $\dfrac{dy}{dx} = 12x^2 - 10x^4$

$\therefore$ Slope of the tangent OP at $P(x, y) = \dfrac{dy}{dx} = 12x^2 - 10x^4 = \dfrac{y - 0}{x - 0}$

or $\dfrac{y}{x} = 12x^2 - 10x^4$ or $y = 12x^3 - 10x^5$



Putting this value of y in eqn. (*i*), we have

$$12x^3 - 10x^5 = 4x^3 - 2x^5 \text{ or } 8x^3 - 8x^5 = 0$$

or $8x^3 (1 - x^2) = 0$
$\therefore$ Either $x = 0$ or $1 - x^2 = 0$
i.e., $x^2 = 1$ $\therefore$ $x = \pm 1$
Putting $x = 0$ in (*i*), $y = 0$
Putting $x = 1$ in (*i*), $y = 4 - 2 = 2$
Putting $x = -1$ in (*i*), $y = -4 + 2 = -2$
Hence, the required points are $(0, 0), (1, 2)$ and $(-1, -2)$.

19. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.

Sol. Equation of curve is $x^2 + y^2 - 2x - 3 = 0$ ...(*i*)
Differentiating w.r.t. x, we get

$$2x + 2y \frac{dy}{dx} - 2 = 0 \quad \text{or } 2y \frac{dy}{dx} = 2 - 2x$$

Dividing by 2, $y \dfrac{dy}{dx} = 1 - x$

$\therefore \qquad \dfrac{dy}{dx} = \dfrac{1-x}{y}$

Now the tangent is parallel to the x-axis if the slope of tangent is zero

i.e., $\dfrac{dy}{dx} = 0$ or $\dfrac{1-x}{y} = 0$ or $x = 1$

Putting $x = 1$ in (*i*), we get $1 + y^2 - 2 - 3 = 0$

or $\quad y^2 = 4 \ \therefore \ y = \pm 2$

Hence, the required points are (1, 2) and (1, – 2).

20. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

Sol. Given: Equation of the curve is $ay^2 = x^3$ ...(*i*)

Differentiating both sides of (*i*) w.r.t. x,

$$a\frac{d}{dx}y^2 = \frac{d}{dx}x^3 \Rightarrow a.2y\frac{dy}{dx} = 3x^2$$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3x^2}{2ay}$ = Slope of the tangent at the point (x, y)

$\therefore$ Slope of the tangent at the point (am^2, am^3)

(Putting $x = am^2$, $y = am^3$) $= \dfrac{3(am^2)^2}{2a.am^3} = \dfrac{3a^2m^4}{2a^2m^3} = \dfrac{3m}{2}$

$\therefore$ Slope of the normal at the point (am^2, am^3) $= -\dfrac{2}{3m}$

(Negative reciprocal)

$\therefore$ Equation of the normal at (am^2, am^3) is

$$y - am^3 = -\frac{2}{3m}(x - am^2)$$

$\Rightarrow \qquad 3m(y - am^3) = -2(x - am^2)$

$\Rightarrow \qquad 3my - 3am^4 = -2x + 2am^2$

or $\qquad 2x + 3my - 2am^2 - 3am^4 = 0$

or $\qquad 2x + 3my - am^2(2 + 3m^2) = 0.$

21. Find the equations of the normal to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

Sol. Equation of curve is $y = x^3 + 2x + 6$ ...(*i*)

Differentiating w.r.t. x, we get

Slope of tangent to the curve at $(x, y) = \dfrac{dy}{dx} = 3x^2 + 2$

$\Rightarrow$ Slope of normal to the curve at (x, y)

$$= \frac{-1}{3x^2+2} \qquad ...(ii)$$

Now the slope of given line $x + 14y + 4 = 0$ is $-\dfrac{1}{14}$. Since the normal is parallel to this line, the slope of normal is also $-\dfrac{1}{14}$ as parallel lines have equal slopes.

$\therefore$ By (*ii*), we have $\dfrac{-1}{3x^2+2} = -\dfrac{1}{14}$

or $3x^2 + 2 = 14$ or $3x^2 = 12$ or $x^2 = 4$ $\therefore$ $x = \pm 2$

Putting $x = 2$ in (*i*), $y = 8 + 4 + 6 = 18$

Putting $x = -2$ in (*i*), $y = -8 - 4 + 6 = -6$

$\therefore$ The coordinates of the feet of normals (*i.e.,* points of contact) are (2, 18) and (– 2, – 6).

$\therefore$ Equation of normal at (2, 18) is

$$y - 18 = -\frac{1}{14}(x - 2)$$

or $14y - 252 = -x + 2$ or $x + 14y - 254 = 0$

and equation of normal at (– 2, – 6) is

$$y + 6 = -\frac{1}{14}(x + 2)$$

or $14y + 84 = -x - 2$ or $x + 14y + 86 = 0.$

22. Find the equation of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Sol. Given: Equation of the parabola is $y^2 = 4ax$ ...(*i*)

Differentiating both sides of (*i*) w.r.t. x, we have

$$\frac{d}{dx}y^2 = 4a\frac{d}{dx}(x) \quad \Rightarrow \quad 2y\frac{dy}{dx} = 4a$$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{4a}{2y} = \dfrac{2a}{y}$ = Slope of the tangent at the point (x, y)

$\therefore$ Slope of the tangent at the point $(at^2, 2at)$ is

(Putting $x = at^2$, $y = 2at$) $= \dfrac{2a}{2at} = \dfrac{1}{t}$

$\therefore$ Slope of the normal $= -t$ (Negative reciprocal)

$\therefore$ Equation of the tangent at the point $(at^2, 2at)$ is

$$y - 2at = \frac{1}{t}(x - at^2) \quad \text{or} \quad ty - 2at^2 = x - at^2$$

$\Rightarrow$ $ty = x + at^2$

Again equation of the normal at the point $(at^2, 2at)$ is

$$y - 2at = -t(x - at^2) \quad \text{or} \quad y - 2at = -tx + at^3$$

or $tx + y = 2at + at^3$.

23. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

Sol. Equations of curves are $x = y^2$ ...(*i*) and $xy = k$ ...(*ii*)

To find the point(s) of intersection, we solve them simultaneously for x and y.

Putting $x = y^2$ from eqn. (*i*) in eqn. (*ii*),

we have $y^2 \,.\, y = k$ or $y^3 = k$ $\therefore$ $y = k^{1/3}$

Putting this value of y in (*i*), $x = (k^{1/3})^2 = k^{2/3}$

$\therefore$ The point of intersection is $(k^{2/3}, k^{1/3}) = (x, y)$ (say) ...(*iii*)

Differentiating (i), w.r.t. x, $\quad 1 = 2y\dfrac{dy}{dx}$

or $$\frac{dy}{dx} = \frac{1}{2y} = m_1 \qquad ...(iv)$$

Differentiating (ii) w.r.t. x, $\quad x\dfrac{dy}{dx} + y = 0$

or $$\frac{dy}{dx} = -\frac{y}{x} = m_2 \qquad ...(v)$$

Because the curves (i) and (ii) cut at right angles at their point of intersection (x, y), therefore $m_1 m_2 = -1$.
Putting values of m_1 and m_2 from (iv) and (v), we have

$$\frac{1}{2y}\left(-\frac{y}{x}\right) = -1 \quad \text{or} \quad \frac{1}{2x} = 1$$

or $\quad 2x = 1$. But from (iii), $\quad x = k^{2/3} \quad \therefore \quad 2.k^{2/3} = 1$
Cubing both sides, $\qquad 8k^2 = 1$.

24. Find the equation of the tangent and normal to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point (x_0, y_0).

Sol. Equation of hyperbola is $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad ...(i)$$

Differentiating w.r.t. x, we have $\quad \dfrac{2x}{a^2} - \dfrac{2y}{b^2} \cdot \dfrac{dy}{dx} = 0$

or $$\frac{-2y}{b^2}\frac{dy}{dx} = \frac{-2x}{a^2} \quad \text{or} \quad \frac{dy}{dx} = \frac{b^2 x}{a^2 y} \qquad ...(ii)$$

Putting $x = x_0$ and $y = y_0$ in (ii), slope of tangent at(x_0, y_0) is $\dfrac{b^2 x_0}{a^2 y_0}$

$\therefore$ Equation of tangent at (x_0, y_0) is $y - y_0 = \dfrac{b^2 x_0}{a^2 y_0}(x - x_0)$

or $\quad yy_0 - y_0^{\,2} = \dfrac{b^2}{a^2}(xx_0 - x_0^{\,2}) \quad$ or $\quad \dfrac{yy_0}{b^2} - \dfrac{y_0^{\,2}}{b^2} = \dfrac{xx_0}{a^2} - \dfrac{x_0^{\,2}}{a^2}$

or $$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = \frac{x_0^{\,2}}{a^2} - \frac{y_0^{\,2}}{b^2} \qquad ...(iii)$$

Since (x_0, y_0) lies on the hyperbola (i), $\therefore \dfrac{x_0^{\,2}}{a^2} - \dfrac{y_0^{\,2}}{b^2} = 1$

Putting this value in R.H.S. of equation (iii), equation of tangent at (x_0, y_0) becomes $\dfrac{xx_0}{a^2} - \dfrac{yy_0}{b^2} = 1$.

Now, slope of tangent at (x_0, y_0) is $\dfrac{b^2x_0}{a^2y_0}$

$\Rightarrow$ Slope of normal at (x_0, y_0) is $-\dfrac{a^2y_0}{b^2x_0}$. (Negative reciprocal)

$\therefore$ Equation of normal at (x_0, y_0) is

$$y - y_0 = -\frac{a^2y_0}{b^2x_0}(x - x_0)$$

or $$b^2x_0(y - y_0) = -a^2y_0(x - x_0)$$

Dividing every term by $a^2b^2x_0y_0$,

$$\frac{y - y_0}{a^2y_0} = -\frac{(x - x_0)}{b^2x_0} \quad \text{or} \quad \frac{(x - x_0)}{b^2x_0} + \frac{(y - y_0)}{a^2y_0} = 0$$

25. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$.

Sol. Given: Equation of the curve is $y = \sqrt{3x - 2}$ ...(*i*)

$\therefore \dfrac{dy}{dx} = \dfrac{d}{dx}(3x - 2)^{1/2} = \dfrac{1}{2}(3x - 2)^{-1/2}\dfrac{d}{dx}(3x - 2) = \dfrac{1}{2\sqrt{3x - 2}} . 3$

= Slope of the tangent at point (x, y) of curve (*i*) ...(*ii*)

Again slope of the given line $4x - 2y + 5 = 0$ is $\dfrac{-a}{b} = \dfrac{-4}{-2} = 2$ ...(*iii*)

Since required tangent is parallel to the given line, therefore

$$\frac{3}{2\sqrt{3x - 2}} = 2 \qquad \text{[Parallel lines have same slope]}$$

Cross-multiplying, $4\sqrt{3x - 2} = 3$

Squaring both sides, $16(3x - 2) = 9 \Rightarrow 48x - 32 = 9$

$\Rightarrow 48x = 32 + 9 = 41 \Rightarrow x = \dfrac{41}{48}$

Putting $x = \dfrac{41}{48}$ in (*i*), $y = \sqrt{3\left(\dfrac{41}{48}\right) - 2}$

$$= \sqrt{\frac{41}{16} - 2} = \sqrt{\frac{41 - 32}{16}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

$\therefore$ Point of contact is $(x, y) = \left(\dfrac{41}{48}, \dfrac{3}{4}\right)$

$\therefore$ Equation of the required tangent is $y - \dfrac{3}{4} = 2\left(x - \dfrac{41}{48}\right)$

$\Rightarrow y - \dfrac{3}{4} = 2x - \dfrac{41}{24} \Rightarrow y = 2x + \dfrac{3}{4} - \dfrac{41}{24} \Rightarrow y = 2x + \dfrac{18 - 41}{24}$

$\Rightarrow \quad 24y = 48x - 23 \quad$ or $\quad -48x + 24y = -23$

Dividing by -1, $48x - 24y = 23$.

Choose the correct answer in Exercise 26 and 27.

26. The slope of the normal to the curve $y = 2x^2 + 3 \sin x$ at $x = 0$ is

(A) 3 (B) $\frac{1}{3}$ (C) -3 (D) $\frac{-1}{3}$.

Sol. Given: Equation of the curve is $y = 2x^2 + 3 \sin x$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = 4x + 3 \cos x$ = Slope of the tangent at the point (x, y)

Putting $x = 0$ (given), slope of the tangent (at $x = 0$)

$= 4(0) + 3 \cos 0 = 3 = m$ (say)

$\therefore$ Slope of the normal at $x = 0$ is $\frac{-1}{m} = \frac{-1}{3}$

$\therefore$ Option (D) is the correct answer.

27. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point

(A) (1, 2) (B) (2, 1) (C) (1, – 2) (D) (– 1, 2).

Sol. Given: Equation of the curve is $y^2 = 4x$ ...(*i*)

Differentiating both sides of (*i*) w.r.t. x,

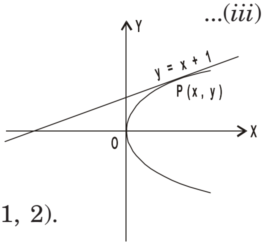
$$2y \frac{dy}{dx} = 4 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

= Slope of tangent to curve (*i*) at point(x, y) ...(*ii*)

Again slope of the given (tangent) line $y = x + 1$

i.e., $-x + y - 1 = 0$ *i.e.,* $x - y + 1 = 0$ is

$$-\frac{a}{b} = \frac{-1}{-1} = 1 \qquad ...(iii)$$



From (*ii*) and (*iii*), $\frac{2}{y} = 1$

| $\because$ Both are slopes of the same line

$\therefore \quad y = 2$

Putting $y = 2$ in (*i*), $4 = 4x$ or $x = 1$

$\therefore$ Required point of contact is $P(x, y) = (1, 2)$.

$\therefore$ Option (A) is the correct answer.

Exercise 6.4

Note. 1. Symbol for approximate value is ~.

2. Δx, a small increment (change) in the value of x, (positive or negative) is ~ dx.

3. Similarly, Δy ~ dy.

1. Using differentials, find the approximate value of each of the following up to 3 places of decimal.

(*i*) $\sqrt{25.3}$	(*ii*) $\sqrt{49.5}$	(*iii*) $\sqrt{0.6}$
(*iv*) $(0.009)^{1/3}$	(*v*) $(0.999)^{1/10}$	(*vi*) $(15)^{1/4}$
(*vii*) $(26)^{1/3}$	(*viii*) $(255)^{1/4}$	(*ix*) $(82)^{1/4}$

(x) $(401)^{1/2}$ **(xi)** $(0.0037)^{1/2}$ **(xii)** $(26.57)^{1/3}$
(xiii) $(81.5)^{1/4}$ **(xiv)** $(3.968)^{3/2}$ **(xv)** $(32.15)^{1/5}$.

Sol. (*i*) To find approximate value of $\sqrt{25.3}$.

Let $y = \sqrt{x}$ ...(*i*) by looking at square root of 25.3

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \Rightarrow dy = \frac{dx}{2\sqrt{x}} \qquad ...(ii)$$

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (*i*),

$$y + \Delta y = \sqrt{x + \Delta x} = \sqrt{25.3} = \sqrt{25 + 0.3} \qquad ...(iii)$$

(25.3 has been written as 25 + 0.3 because we know the square root of 25 as = 5)

Comparing $\sqrt{x + \Delta x}$ with $\sqrt{25 + 0.3}$, we have

$$x = 25 \text{ and } \Delta x = 0.3 \qquad ...(iv)$$

From eqn. (*iii*), $\sqrt{25.3} = y + \Delta y \sim y + dy \sim \sqrt{x} + \dfrac{dx}{2\sqrt{x}}$

(From (*i*) and (*ii*))

$$\sim \sqrt{x} + \frac{\Delta x}{2\sqrt{x}} \sim \sqrt{25} + \frac{0.3}{2\sqrt{25}} \quad \text{(From } (iv))$$

$$\sim 5 + \frac{0.3}{2(5)} = 5 + \frac{0.3}{10} = 5 + 0.03$$

$$\therefore \quad \sqrt{25.3} \sim 5.03.$$

(*ii*) To find approximate value of $\sqrt{49.5}$

Let $y = \sqrt{x}$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow dy = \frac{dx}{2\sqrt{x}} \qquad ...(ii)$$

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (*i*),

$$y + \Delta y = \sqrt{x + \Delta x} = \sqrt{49.5} = \sqrt{49 + 0.5} \qquad ...(iii)$$

Comparing $\sqrt{x + \Delta x}$ with $\sqrt{49 + 0.5}$,

$$x = 49 \text{ and } \Delta x = 0.5 \qquad ...(iv)$$

From eqn. (*iii*), $\sqrt{49.5} = y + \Delta y \sim y + dy$

$$\sim \sqrt{x} + \frac{dx}{2\sqrt{x}} \quad \text{(From } (i) \text{ and } (ii))$$

$$\sim \sqrt{x} + \frac{dx}{2\sqrt{x}} \sim \sqrt{x} + \frac{\Delta x}{2\sqrt{x}}$$

$$\therefore \quad \sqrt{49.5} \sim \sqrt{49} + \frac{0.5}{2\sqrt{49}} \qquad \text{[By } (iv)]$$

$$= 7 + \frac{0.5}{2(7)} = 7 + \frac{0.5}{14} = 7 + 0.0357 = 7.0357$$

(*iii*) To find approximate value of $\sqrt{0.6}$

Let $y = \sqrt{x}$ ...(*i*)

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}} \quad \Rightarrow \quad dy = \dfrac{dx}{2\sqrt{x}}$ ...(*ii*)

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (*i*),

$y + \Delta y = \sqrt{x + \Delta x} = \sqrt{0.6} = \sqrt{0.60}$

$= \sqrt{0.64 - 0.04}$ ...(*iii*) ($\because$ $0.64 - 0.60 = 0.04$)

Comparing $\sqrt{x + \Delta x}$ with $\sqrt{0.64 - 0.04}$,

we have $x = 0.64$ and $\Delta x = -0.04$ ...(*iv*)

From eqn. (*iii*), $\sqrt{0.6} = y + \Delta y \sim y + dy$

$\sim \sqrt{x} + \dfrac{dx}{2\sqrt{x}}$ (From (*i*) and (*ii*))

$\sim \sqrt{x} + \dfrac{\Delta x}{2\sqrt{x}} \sim \sqrt{0.64} - \dfrac{0.04}{2\sqrt{0.64}}$

$\therefore \quad \sqrt{0.6} \sim 0.8 - \dfrac{0.04}{2(0.8)} = 0.8 - \dfrac{0.04}{1.6} = 0.8 - \dfrac{4}{100} \times \dfrac{10}{16}$

$= 0.8 - \dfrac{1}{40} = 0.8 - 0.025 = 0.775.$

(*iv*) To find approximate value of $(0.009)^{1/3}$

Let $y = x^{1/3}$ ...(*i*) by looking at power (index) $\dfrac{1}{3}$ of 0.009.

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{3} x^{-2/3} = \dfrac{1}{3x^{2/3}} \Rightarrow dy = \dfrac{dx}{3x^{2/3}} \sim \dfrac{\Delta x}{3(x^{1/3})^2}$ ...(*ii*)

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (*i*),

$y + \Delta y = (x + \Delta x)^{1/3} = (0.009)^{1/3} = (0.008 + 0.001)^{1/3}$ ...(*iii*)

0.009 has been written as 0.008 + 0.001 because we know that the cube root of 0.008 *i.e.,* $(0.008)^{1/3} = 0.2$

Comparing $(x + \Delta x)^{1/3}$ with $(0.008 + 0.001)^{1/3}$, we have

$x = 0.008$ and $\Delta x = 0.001$ ...(*iv*)

From eqn. (*iii*), $(0.009)^{1/3} = y + \Delta y$

$\sim y + dy = x^{1/3} + \dfrac{dx}{3x^{2/3}} \sim x^{1/3} + \dfrac{\Delta x}{3(x^{1/3})^2}$

(From (*i*) and (*ii*)) $\sim (0.008)^{1/3} + \dfrac{(0.001)}{3((0.008)^{1/3})^2}$

$\therefore \quad (0.009)^{1/3} \sim 0.2 + \dfrac{0.001}{3(0.2)^2} = 0.2 + \dfrac{0.001}{3(0.04)}$

$= 0.2 + \dfrac{0.001}{0.12}$ $[(0.008)^{1/3} = ((0.2)^3)^{1/3} = 0.2]$

$\sim 0.2 + 0.0083 = 0.2083.$

(*v*) To find approximate value of $(0.999)^{1/10}$

Let $y = x^{1/10}$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = \frac{1}{10}x^{-9/10} = \frac{1}{10x^{9/10}}$$

$$\Rightarrow \quad dy = \frac{dx}{10(x^{1/10})^9} \sim \frac{\Delta x}{10(x^{1/10})^9} \quad ...(ii)$$

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (*i*),

$y + \Delta y = (x + \Delta x)^{1/10} = (0.999)^{1/10} = (1 - 0.001)^{1/10}$ ...(*iii*)

Comparing $x = 1$ and $\Delta x = -0.001$ ...(*iv*)

From eqn. (*iii*), $(0.999)^{1/10} = y + \Delta y \sim y + dy$

$$\sim x^{1/10} + \frac{\Delta x}{10(x^{1/10})^9} \quad \text{[From } (i)$$

and (*ii*)]

$$\sim (1)^{1/10} - \frac{0.001}{10(1^{1/10})^9} = 1 - \frac{0.001}{10}$$

$$= 1 - 0.0001 = 0.9999.$$

(*vi*) To find approximate value of $(15)^{1/4}$

Let $y = x^{1/4}$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = \frac{1}{4}x^{1/4 - 1} = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}}$$

$$\therefore \quad dy = \frac{dx}{4(x^{1/4})^3} \sim \frac{\Delta x}{4(x^{1/4})^3} \quad ...(ii)$$

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (*i*),

$y + \Delta y = (x + \Delta x)^{1/4} = (15)^{1/4} = (16 - 1)^{1/4}$ ...(*iii*)

Comparing, $x = 16$ and $\Delta x = -1$ ...(*iv*)

From eqn. (*iii*), $(15)^{1/4} = y + \Delta y \sim y + dy$

$$= x^{1/4} + \frac{\Delta x}{4(x^{1/4})^3} \quad \text{(From } (i) \text{ and } (ii))$$

$$\sim (16)^{1/4} - \frac{1}{4((16)^{1/4})^3} \quad \text{(From } (iv))$$

$$= 2 - \frac{1}{4 \times 2^3} \quad (\because \quad (16)^{1/4} = (2^4)^{1/4} = 2)$$

$$\therefore \quad (15)^{1/4} \sim 2 - \frac{1}{32} = \frac{64 - 1}{32} = \frac{63}{32} = 1.96875.$$

(*vii*) To find approximate value of $(26)^{1/3}$

Let $y = x^{1/3}$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

$$\therefore \quad dy = \frac{dx}{3x^{2/3}} \sim \frac{\Delta x}{3(x^{1/3})^2} \quad ...(ii)$$

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (*i*),

$y + \Delta y = (x + \Delta x)^{1/3} = (26)^{1/3} = (27 - 1)^{1/3}$ ...(*iii*)

Comparing, $x = 27$ and $\Delta x = -1$ ...(*iv*)

From (iii), $(26)^{1/3} = y + \Delta y \sim y + dy$

$$\sim x^{1/3} + \frac{\Delta x}{3(x^{1/3})^2} \quad \text{[From } (iii) \text{ and } (ii)\text{]}$$

$$\sim (27)^{1/3} - \frac{1}{3((27)^{1/3})^2} \quad \text{[From } (iii)\text{]}$$

$$\therefore \quad (26)^{1/3} \sim 3 - \frac{1}{3(3)^2} \quad [\because \ (27)^{1/3} = (3^3)^{1/3} = 3]$$

$$= 3 - \frac{1}{27} = \frac{81-1}{27} = \frac{80}{27} = 2.9629.$$

$(viii)$ To find approximate value of $(255)^{1/4}$

Let $\quad y = x^{1/4}$ ...(i)

$$\therefore \quad \frac{dy}{dx} = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}}$$

$$\therefore \quad dy = \frac{dx}{4x^{3/4}} \sim \frac{\Delta x}{4(x^{1/4})^3} \quad ...(ii)$$

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (i),

$y + \Delta y = (x + \Delta x)^{1/4} = (255)^{1/4} = (256 - 1)^{1/4}$ ...(iii)

Comparing, $x = 256$ and $\Delta x = -1$ ...(iv)

From (iii), $(255)^{1/4} = y + \Delta y$

$$\sim x^{1/4} + \frac{\Delta x}{4(x^{1/4})^3} \quad \text{(From } (i) \text{ and } (ii)\text{)}$$

$$\sim (256)^{1/4} - \frac{1}{4((256)^{1/4})^3} \sim 4 - \frac{1}{4(4)^3}$$

$[\because \ (256)^{1/4} = (4^4)^{1/4} = 4]$

$$\sim 4 - \frac{1}{256} = \frac{1024-1}{256} = \frac{1023}{256} \sim 3.9961.$$

(ix) To find approximate value of $(82)^{1/4}$

Let $\quad y = x^{1/4}$ ...(i)

$$\therefore \quad \frac{dy}{dx} = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}}$$

$$\therefore \quad dy = \frac{dx}{4(x^{1/4})^3} \sim \frac{\Delta x}{4(x^{1/4})^3} \quad ...(ii)$$

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (i),

$y + \Delta y = (x + \Delta x)^{1/4} = (82)^{1/4} = (81 + 1)^{1/4}$ ...(iii)

Comparing, $x = 81$ and $\Delta x = 1$ ...(iv)

From (iii), $(81)^{1/4} = y + \Delta y \sim y + dy$

$$\sim x^{1/4} + \frac{\Delta x}{4(x^{1/4})^3} \quad \text{[From } (i) \text{ and } (ii)\text{]}$$

$$\sim (81)^{1/4} + \frac{1}{4((81)^{1/4})^3} = 3 + \frac{1}{4(3)^3}$$

$$\sim 3 + \frac{1}{108} = \frac{324+1}{108} = \frac{325}{108} = 3.0092.$$

(x) To find approximate value of $(401)^{1/2} = \sqrt{401}$

Let $\quad y = x^{1/2} = \sqrt{x}$ ...(i)

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\therefore \quad dy = \frac{dx}{2\sqrt{x}} \sim \frac{\Delta x}{2\sqrt{x}} \qquad ...(ii)$$

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (i),

$$y + \Delta y = \sqrt{x + \Delta x} = \sqrt{401} = \sqrt{400 + 1} \qquad ...(iii)$$

Comparing, $x = 400$ and $\Delta x = 1$ ...(iv)

From (iii), $\sqrt{401} = y + \Delta y \sim y + dy$

$$\sim \sqrt{x} + \frac{\Delta x}{2\sqrt{x}} \qquad \text{[From (i) and (ii)]}$$

$$\sim \sqrt{400} + \frac{1}{2\sqrt{400}} = 20 + \frac{1}{40} = \frac{800 + 1}{40} = \frac{801}{40} \sim 20.025.$$

(xi) To find approximate value of $(0.0037)^{1/2} = \sqrt{0.0037}$

Let $\quad y = \sqrt{x}$ ...(i)

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow dy = \frac{dx}{2\sqrt{x}} \sim \frac{\Delta x}{2\sqrt{x}} \qquad ...(ii)$$

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (i),

$$y + \Delta y = \sqrt{x + \Delta x} = \sqrt{0.0037} = \sqrt{0.0036 + 0.0001} \qquad ...(iii)$$

$(\because \; 0.0037 - 0.0036 = 0.0001)$

Comparing with $x + \Delta x$, $x = 0.0036$ and $\Delta x = 0.0001$ ...(iv)

From (iii), $\sqrt{0.0037} = y + \Delta y \sim y + dy$

$$= \sqrt{x} + \frac{\Delta x}{2\sqrt{x}} \qquad \text{(From (i) and (ii))}$$

$$\sim \sqrt{0.0036} + \frac{0.0001}{2\sqrt{0.0036}}$$

$$= 0.06 + \frac{0.0001}{2(0.06)} \qquad [(0.06)^2 = 0.0036]$$

$$\sim 0.06 + \frac{0.0001}{0.12} \sim 0.06 + 0.000833 \sim 0.060833.$$

(xii) To find approximate value of $(26.57)^{1/3}$

Let $\quad y = x^{1/3}$ ...(i)

$$\therefore \quad \frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3.x^{2/3}}$$

$$\Rightarrow \quad dy = \frac{dx}{3.x^{(2/3)}} \sim \frac{\Delta x}{3(x^{1/3})^2} \qquad ...(ii)$$

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (i),

$$y + \Delta y = (x + \Delta x)^{1/3} = (26.57)^{1/3}$$

$$= (27 - 0.43)^{1/3} \qquad ...(iii) \quad [\because \; 27 - 26.57 = 0.43]$$

Comparing with $x + \Delta x$, $x = 27$ and $\Delta x = -0.43$ ...(iv)

From (iii), $(26.57)^{1/3} = y + \Delta y \sim x^{1/3} + \dfrac{\Delta x}{3(x^{1/3})^2}$

(From (i) and (ii))

$$\sim (27)^{1/3} - \frac{0.43}{3((27)^{1/3})^2} \sim 3 - \frac{0.43}{3(3)^2}$$

$$\sim 3 - \frac{0.43}{27} \sim 3 - 0.0159 \sim 2.9841.$$

$(xiii)$ To find approximate value of $(81.5)^{1/4}$

Let $\quad y = x^{1/4}$ ...(i)

$$\therefore \quad \frac{dy}{dx} = \frac{1}{4} \cdot x^{-3/4} = \frac{1}{4x^{(3/4)}}$$

$$\therefore \quad dy = \frac{dx}{4(x^{3/4})} \sim \frac{\Delta x}{4(x^{1/4})^3} \quad ...(ii)$$

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (i),

$$y + \Delta y = (x + \Delta x)^{1/4} = (81.5)^{1/4} = (81 + 0.5)^{1/4} \quad ...(iii)$$

Comparing with $x + \Delta x$ we have $x = 81$

and $\Delta x = 0.5$ ...(iv)

From (iii), $(81.5)^{1/4} = y + \Delta y \sim y + dy$

$$\sim x^{1/4} + \frac{\Delta x}{4(x^{1/4})^3} \quad \text{(From } (i) \text{ and } (ii))$$

$$\sim (81)^{1/4} + \frac{0.5}{4((81)^{1/4})^3} \sim 3 + \frac{0.5}{4(3)^3}$$

$$\sim 3 + \frac{0.5}{108} \sim 3 + 0.00462 \sim 3.00462.$$

(xiv) To find approximate value of $(3.968)^{3/2}$

Let $y = x^{3/2} = x^{2/2 + 1/2} = x^{1 + 1/2}$

$$= x^1 \, x^{1/2} = x\sqrt{x} \quad ...(i)$$

On looking at power (index) $\dfrac{3}{2}$ of 3.968

$$\therefore \quad \frac{dy}{dx} = \frac{3}{2}x^{1/2} \quad \therefore \quad dy = \frac{3}{2}x^{1/2}\, dx \sim \frac{3}{2}\sqrt{x}\ \Delta x \quad ...(ii)$$

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (i),

$$y + \Delta y = (x + \Delta x)^{3/2} = (3.968)^{3/2} = (4 - 0.032)^{3/2} \quad ...(iii)$$

Comparing with $x + \Delta x$, we have $x = 4$ and

$$\Delta x = -\,0.032 \quad ...(iv)$$

From (iii), $(3.968)^{3/2} = y + \Delta y \sim y + dy$

$$\sim x\sqrt{x} + \frac{3}{2}\sqrt{x}\ \Delta x \quad \text{(From } (i) \text{ and } (ii))$$

$$\sim 4\sqrt{4} + \frac{3}{2}\sqrt{4}\,(-\,0.032) \quad [\text{By } (iv)]$$

$$\sim 4(2) - \frac{3}{2}(2)(0.032) \sim 8 - 3(0.032)$$
$$\sim 8 - 0.096 \sim 7.904.$$

(*xv*) To find approximate value of $(32.15)^{1/5}$

Let $y = x^{1/5}$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = \frac{1}{5}x^{-4/5} = \frac{1}{5x^{4/5}} \quad \therefore \quad dy = \frac{dx}{5x^{4/5}} \sim \frac{\Delta x}{5(x^{1/5})^4}$ ...(*ii*)

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (*i*),

$y + \Delta y = (x + \Delta x)^{1/5} = (32.15)^{1/5} = (32 + 0.15)^{1/5}$ ...(*iii*)

Comparing with $x + \Delta x$, we have $x = 32$ and $\Delta x = 0.15$ ...(*iv*)

From (*iii*), $(32.15)^{1/5} = y + \Delta y \sim y + dy$

$\sim x^{1/5} + \frac{\Delta x}{5(x^{1/5})^4}$ (From (*i*) and (*ii*))

$\sim (32)^{1/5} + \frac{0.15}{5((32)^{1/5})^4} \sim 2 + \frac{0.15}{5(2)^4} \quad (\because \ (32)^{1/5} = (2^5)^{1/5} = 2)$

$\sim 2 + \frac{0.15}{80} \sim 2 + 0.001875 \sim 2.001875.$

2. Find the approximate value of $f(2.01)$ where $f(x) = 4x^2 + 5x + 2$.

Sol. Let $y = f(x) = 4x^2 + 5x + 2$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = f'(x) = 8x + 5$

$\therefore \quad dy = (8x + 5)dx \sim (8x + 5)\Delta x$ ...(*ii*)

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (*i*),

$y + \Delta y = f(x + \Delta x) = f(2.01) = f(2 + 0.01)$ ...(*iii*)

Comparing $f(x + \Delta x)$ with $f(2 + 0.01)$, we have

$x = 2$ and $\Delta x = 0.01$ ...(*iv*)

From (*iii*), $f(2.01) = y + \Delta y \sim y + dy$

$\sim (4x^2 + 5x + 2) + (8x + 5)\ \Delta x$ (From (*i*) and (*ii*))

Putting $x = 2$ and $\Delta x = 0.01$ from (*iv*),

$\sim (4(4) + 5(2) + 2) + (8(2) + 5)\ (0.01)$

$\sim 28 + 21(0.01) \sim 28 + 0.21 \sim 28.21.$

3. Find the approximate value of $f(5.001)$ where $f(x) = x^3 - 7x^2 + 15$.

Sol. Let $y = f(x) = x^3 - 7x^2 + 15$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = f'(x) = 3x^2 - 14x$

$\therefore \quad dy = (3x^2 - 14x)dx \sim (3x^2 - 14x)\ \Delta x$ ...(*ii*)

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (*i*),

$y + \Delta y = f(x + \Delta x) = f(5.001) = f(5 + 0.001)$ ...(*iii*)

Comparing $f(x + \Delta x)$ with $f(5 + 0.001)$, we have

$x = 5$ and $\Delta x = 0.001$ ...(*iv*)

From (*iii*), $f(5.001) = y + \Delta y \sim y + dy$

$$\sim (x^3 - 7x^2 + 15) + (3x^2 - 14x)\Delta x \text{ (From } (i) \text{ and } (ii))$$

Putting $x = 5$ and $\Delta x = 0.001$ from (*iv*), we have

$$\sim (125 - 175 + 15) + (75 - 70)(0.001)$$
$$\sim -35 + 5(0.001) \sim -35 + 0.005$$
$$\sim -34.995.$$

4. Find the approximate change in the volume of a cube of side x metres caused by increasing the side by 1%.

Sol. We know that volume of a cube of side x metres is given by

$$V = x^3 \qquad ...(i)$$

$$\therefore \qquad \frac{dV}{dx} = 3x^2 \qquad ...(ii)$$

Given: Increase in side = 1% of $x = \frac{1}{100}x$

(Positive sign is being taken because it is given that side of cube is increasing)

i.e., $\qquad \Delta x = \frac{x}{100} \qquad ...(iii)$

We know that approximate change in volume V of cube

$$= \Delta V \sim dV = \frac{dV}{dx}dx$$

$$\sim \frac{dV}{dx}\Delta x \sim 3x^2\left(\frac{x}{100}\right) \qquad | \text{ From } (ii) \text{ and } (iii)$$

$$\sim \frac{3}{100}x^3$$

$$\sim 0.03x^3 \text{ m}^3.$$

5. Find the approximate change in the surface area of a cube of side x metres caused by decreasing the side by 1%.

Sol. We know that surface area of a cube of side x is given by $S = 6x^2$

$$\therefore \qquad \frac{dS}{dx} = 12x$$

Decrease in side = – 1% of x = – $0.01x$ [Negative sign is being taken because it is given that side of the cube is decreasing]

$\Rightarrow \qquad \Delta x = -0.01x$

Approximate change in S = Approximate value of ΔS

$$= dS = \left(\frac{dS}{dx}\right) dx$$

$$= (12x)(-0.01x) \qquad [\because \; dx = \Delta x]$$

$$= -0.12x^2 \text{ m}^2$$

Hence, the approximate change in surface is $-0.12x^2$ m^2, *i.e.,* the surface **decreases** by approximately $0.12x^2$ m^2.

6. If the radius of a sphere is measured as 7 m with an error of 0.02 m, then find the approximate error in calculating its volume.

Sol. Let r be the radius of the sphere and Δr be the error in measuring the radius.

Then $r = 7$ m and $\Delta r = 0.02$ m.

Volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$

$\therefore \qquad \frac{dV}{dr} = \frac{4}{3}\pi.3r^2$

Approximate error in calculating the volume

$= $ Approximate value of ΔV

$= dV = \left(\frac{dV}{dr}\right) dr = \left(\frac{4}{3}\pi 3r^2\right) dr$

$= 4\pi(7)^2 (0.02) \qquad [\because \ dr \sim \Delta r]$

$= 3.92\pi \text{ m}^3 = 3.92 \times \frac{22}{7} \text{ m}^3$

$= 12.32 \text{ m}^3$

Hence, the approximate error in calculating volume is 12.32 m^3.

7. If the radius of a sphere is measured as 9 m with an error of 0.03 m, then find the approximate error in calculating its surface area.

Sol. Let x m be the radius of the sphere.

$\therefore$ S, surface area of sphere = $\mathbf{4\pi x^2}$

$\therefore \qquad \frac{dS}{dx} = 4\pi(2x) = 8\pi x$

$\therefore \qquad dS = 8\pi x \, dx \sim 8\pi x \, \Delta x \qquad ...(i)$

Given: $x = 9$ m and error $\Delta x = 0.03$ m $\qquad ...(ii)$

Putting $x = 9$ and $\Delta x = 0.03$ from (ii) in (i),

Error ΔS in surface area of sphere

$\sim dS = 8\pi(9)(0.3) = 72(0.03)\pi = 2.16\pi \text{ m}^2$.

(**Note.** Error can be positive or negative)

8. If $f(x) = 3x^2 + 15x + 5$, then the approximate value of $f(3.02)$ is

(A) 47.66 (B) 57.66 (C) 67.66 (D) 77.66.

Sol. Let $\qquad y = f(x) = 3x^2 + 15x + 5 \qquad ...(i)$

$\therefore \qquad \frac{dy}{dx} = f'(x) = 6x + 15$

$\therefore \qquad dy = (6x + 15)dx \sim (6x + 15)\Delta x \qquad ...(ii)$

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (i),

$y + \Delta y = f(x + \Delta x) = f(3.02) = f(3 + 0.02) \qquad ...(iii)$

Comparing $f(x + \Delta x)$ with $f(3 + 0.02)$, we have

$x = 3$ and $\Delta x = 0.02 \qquad ...(iv)$

From (iii), $f(3.02) = y + \Delta y \sim y + dy$

$\sim (3x^2 + 15x + 5) + (6x + 15)\Delta x \qquad$ (From (i) and (ii))

$\sim (3(9) + 15(3) + 5) + (6(3) + 15)(0.02)$

$= (72 + 5) + (33)(0.02)$

$\sim 77 + 0.66 \sim 77.66$

$\therefore$ Option (D) is the correct answer.

9. The approximate change in the volume of a cube of side x metres caused by increasing the side by 3% is

(A) 0.06 x^3 m^3 (B) 0.6 x^3 m^3 (C) 0.09 x^3 m^3 (D) 0.9 x^3 m^3.

Sol. We know that volume of a cube of side x metres is given by

$$V = x^3 \quad \text{...}(i)$$

$$\therefore \quad \frac{dV}{dx} = 3x^2 \quad \text{...}(ii)$$

Given: Increase in side of cube = 3% = $\frac{3}{100}x$

(Positive sign is being taken because it is given that side of cube is increasing)

i.e., $$\Delta x = \frac{3x}{100} \quad \text{...}(iii)$$

We know that approximate change in volume of cube

$$= \Delta V \sim dV = \frac{dV}{dx} dx$$

$$\sim \frac{dV}{dx} \Delta x \sim 3x^2 \left(\frac{3x}{100}\right) \quad | \text{ From } (i) \text{ and } (iii)$$

$$\sim \frac{9}{100} x^3 \sim 0.09x^3 \text{ m}^3.$$

$\therefore$ Option (C) is the correct answer.

Exercise 6.5

1. Find the maximum and minimum values, if any, of the following functions given by

(*i*) $f(x) = (2x - 1)^2 + 3$ (*ii*) $f(x) = 9x^2 + 12x + 2$

(*iii*) $f(x) = -(x - 1)^2 + 10$ (*iv*) $g(x) = x^3 + 1$.

Sol. (*i*) **Given:** $f(x) = (2x - 1)^2 + 3$

We know that for all $x \in R$, $(2x - 1)^2 \geq 0$

$\Rightarrow$ Adding 3 to both sides, $(2x - 1)^2 + 3 \geq 0 + 3$

$\Rightarrow$ $f(x) \geq 3$

The minimum value of $f(x)$ is 3 and is obtained when $2x - 1 = 0$, [$\because$ Minimum value of $(2x - 1)^2$ is 0] *i.e.*, when $x = \frac{1}{2}$. There is no maximum value of $f(x)$.

[$\because$ Maximum value of $f(x) = (2x - 1)^2 + 3 \to \infty$ as $x \to \infty$ and hence does not exist].

(*ii*) **Given:** $f(x) = 9x^2 + 12x + 2$

Making coefficient of x^2 unity,

$$= 9\left[x^2 + \frac{12x}{9} + \frac{2}{9}\right] = 9\left[x^2 + \frac{4x}{3} + \frac{2}{9}\right]$$

Add and subtract $\left(\frac{1}{2}\text{ coeff. of } x\right)^2$

$$= \left(\frac{1}{2} \times \frac{4}{3}\right)^2 = \left(\frac{2}{3}\right)^2,$$

$$= 9\left[x^2 + \frac{4x}{3} + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2 + \frac{2}{9}\right] = 9\left[\left(x + \frac{2}{3}\right)^2 - \frac{4}{9} + \frac{2}{9}\right]$$

or $f(x) = 9\left(x + \frac{2}{3}\right)^2 - 4 + 2 = 9\left(x + \frac{2}{3}\right)^2 - 2$ ...(*i*)

We know that for all $x \in$ R, $9\left(x + \frac{2}{3}\right)^2 \geq 0$

Adding – 2 to both sides, $9\left(x + \frac{2}{3}\right)^2 - 2 \geq -2$

$\Rightarrow$ Using (*i*), $f(x) \geq -2$

$\therefore$ Minimum value of $f(x)$ is – 2 and is obtained when

$$x + \frac{2}{3} = 0 \text{ i.e., when } x = -\frac{2}{3}.$$

From (*i*), maximum value of $f(x) \to \infty$ as $x \to \infty$ (or as $x \to -\infty$) and hence does not exist.

(*iii*) **Given:** $f(x) = -(x-1)^2 + 10$ ...(*i*)

We know that for all $x \in$ R, $(x-1)^2 \geq 0$

Multiplying by – 1, $-(x-1)^2 \leq 0$

Adding 10 to both sides, $-(x-1)^2 + 10 \leq 10$

$\Rightarrow$ Using (*i*), $f(x) \leq 10$

$\therefore$ Maximum value of $f(x)$ is 10 and is obtained when $x - 1 = 0$ *i.e.,* when $x = 1$.

From (*i*), minimum value of $f(x) \to -\infty$ as $x \to \infty$ (or as $x \to -\infty$) and hence does not exist.

(*iv*) **Given:** $g(x) = x^3 + 1$ ...(*i*)

As $x \to \infty$, $g(x) \to \infty$

As $x \to -\infty$, $g(x) \to -\infty$

$\therefore$ Maximum value of $g(x)$ does not exist and also minimum value of $g(x)$ does not exist.

2. Find the maximum and minimum values, if any, of the following functions given by

(*i*) $f(x) = |x + 2| - 1$ (*ii*) $g(x) = -|x + 1| + 3$

(*iii*) $h(x) = \sin(2x) + 5$ (*iv*) $f(x) = |\sin 4x + 3|$

(*v*) $h(x) = x + 1, x \in (-1, 1)$

Sol. (*i*) **Given:** $f(x) = |x + 2| - 1$ ...(*i*)

We know that for all $x \in$ R, $|x + 2| \geq 0$

Adding – 1 to both sides, $|x + 2| - 1 \geq -1$

$\Rightarrow$ Using (*i*), $f(x) \geq -1$

$\therefore$ Minimum value of $f(x)$ is -1 and is obtained when $x + 2 = 0$ *i.e.,* when $x = -2$.

From (i), maximum value of $f(x) \to \infty$ as $x \to \infty$ (or as $x \to -\infty$) and hence does not exist.

(ii) **Given:** $g(x) = -|x + 1| + 3$

We know that for all $x \in R$, $|x + 1| \geq 0$

Multiplying by -1 to both sides

$\Rightarrow -|x + 1| \leq 0$

Adding 3 to both sides,

$\Rightarrow -|x + 1| + 3 \leq 3 \Rightarrow g(x) \leq 3$

$\therefore$ The maximum value of $g(x)$ is 3 and is obtained when $|x + 1| = 0$, *i.e.,* when $x + 1 = 0$ *i.e.,* when $x = -1$. There is no minimum value of $g(x)$. [Because minimum value of $g(x) = -|x + 1| + 3 \to -\infty$ as $x \to \pm\infty$ and hence does not exist].

(iii) **Given:** $h(x) = \sin(2x) + 5$ ...(i)

We know that for all $x \in R$, $-1 \leq \sin 2x \leq 1$

Adding 5 to all sides, $-1 + 5 \leq \sin 2x + 5 \leq 1 + 5$

$\Rightarrow 4 \leq h(x) \leq 6$ (By (i))

$\therefore$ Minimum value of $h(x)$ is 4 and maximum value is 6.

(iv) **Given:** $f(x) = |\sin 4x + 3|$

We know that for all $x \in R$, $-1 \leq \sin 4x \leq 1$

Adding 3 throughout,

$2 \leq \sin 4x + 3 \leq 4 \Rightarrow 2 \leq |\sin 4x + 3| \leq 4$

[$\because$ $\sin 4x + 3 \geq 2$ and hence > 0, $\therefore |\sin 4x + 3| = \sin 4x + 3$]

$\therefore$ The minimum value of $f(x)$ is 2 and the maximum value of $f(x)$ is 4.

(v) **Given:** $h(x) = x + 1$, $x \in (-1, 1)$ ...(i)

Given: $x \in (-1, 1) \Rightarrow -1 < x < 1$

Adding 1 to all sides, $1 - 1 < x + 1 < 1 + 1$

i.e., $0 < h(x)$ [By (i)] < 2 ...(ii)

$\therefore$ Neither minimum value nor maximum value of $h(x)$ exists.

($\because$ Equality sign is absent at both ends of inequality (ii). We know from (ii) that minimum value of $h(x)$ is > 0 and maximum value is < 2 but what exactly they are can't be said).

3. Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

(i) $f(x) = x^2$ **(ii) $g(x) = x^3 - 3x$**

(iii) $h(x) = \sin x + \cos x$, $0 < x < \dfrac{\pi}{2}$

(iv) $f(x) = \sin x - \cos x$, $0 < x < 2\pi$

(v) $f(x) = x^3 - 6x^2 + 9x + 15$ **(vi) $g(x) = \dfrac{x}{2} + \dfrac{2}{x}$, $x > 0$.**

(*vii*) $g(x) = \dfrac{1}{x^2 + 2}$ **(*viii*) $f(x) = x\sqrt{1 - x}$, $x > 0$.**

Sol. (*i*) **Given:** $f(x) = x^2$ ...(*i*)

$\therefore$ $f'(x) = 2x$ and $f''(x) = 2$

Putting $f'(x) = 0$ to get turning points, we have $2x = 0$

or $x = \dfrac{0}{2} = 0$

(Turning point)

Let us apply second derivative test.

When $x = 0$, $f''(x) = 2$ (positive)

$\therefore$ $x = 0$ is a point of local minima and local minimum value $= f(0) = 0^2$ [From (*i*)]

$= 0$

Therefore, local minima at $x = 0$ and local minimum value = 0.

(*ii*) **Given:** $g(x) = x^3 - 3x$ ...(*i*)

$\therefore$ $g'(x) = 3x^2 - 3$ and $g''(x) = 6x$

Putting $g'(x) = 0$ to get turning points, we have

$3x^2 - 3 = 0$ or $3(x^2 - 1) = 0$ or $3(x + 1)(x - 1) = 0$

But $3 \neq 0$. Therefore, either $x + 1 = 0$ or $x - 1 = 0$

i.e., $x = -1$ or $x = 1$. (Turning points)

Let us apply second derivative test.

At $x = -1$, $g''(x) = 6x = 6(-1) = -6$ (Negative)

$\therefore$ $x = -1$ is a point of local maxima and local maximum value $= g(-1) = (-1)^3 - 3(-1)$ [From (*i*)] $= -1 + 3 = 2$.

At $x = 1$, $g''(x) = 6x = 6(1) = 6$ (positive)

$\therefore$ $x = 1$ is a point of local minima and local minimum value

$= g(1) = 1^3 - 3(1) = 1 - 3 = -2$

Therefore, Local maximum at $x = -1$ and local maximum value = 2. Local minimum at $x = 1$ and local minimum value = -2.

(*iii*) **Given:** $h(x) = \sin x + \cos x$ $\left(0 < x < \dfrac{\pi}{2}\right)$ ...(*i*)

$\therefore$ $h'(x) = \cos x - \sin x$ and $h''(x) = -\sin x - \cos x$

Putting $h'(x) = 0$ to get turning points,

we have $\cos x - \sin x = 0$ $\Rightarrow$ $-\sin x = -\cos x$

Dividing by $-\cos x$, $\dfrac{\sin x}{\cos x} = 1$ or $\tan x = 1$

which is positive. Therefore, x can have values in both Ist and IIIrd quadrants.

Here, x is only in Ist quadrant

$\left[\because\ 0 < x < \dfrac{\pi}{2} \text{ (given)}\right]$

$\therefore \quad \tan x = 1 = \tan \dfrac{\pi}{4}$

$\therefore \qquad x = \dfrac{\pi}{4}$ (only turning point)

At $\qquad x = \dfrac{\pi}{4},\ h''(x) = -\sin x - \cos x$

$$= -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}} = -\sqrt{2} \text{ is negative.}$$

$\therefore\ x = \dfrac{\pi}{4}$ is a point of local maxima and local maximum value

$$= h\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} \qquad \text{(From } (i))$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$\therefore$ Local maximum at $x = \dfrac{\pi}{4}$, and local maximum value $= \sqrt{2}$.

(*iv*) **Given:** $f(x) = \sin x - \cos x \quad ...(i) \quad (0 < x < 2\pi)$

$\therefore \qquad f'(x) = \cos x + \sin x$

and $\qquad f''(x) = -\sin x + \cos x$

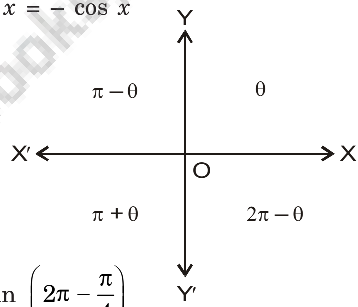
Putting $f'(x) = 0$ to get turning points, we have

$$\cos x + \sin x = 0 \quad \Rightarrow \quad \sin x = -\cos x$$

Dividing by $\cos x$, $\dfrac{\sin x}{\cos x} = -1$

$\Rightarrow \quad \tan x = -1$ is negative.

Therefore, x is in both second and fourth quadrants.



$\therefore \quad \tan x = -1 = -\tan\dfrac{\pi}{4}$

$$= \tan\left(\pi - \frac{\pi}{4}\right) \quad \text{or} \quad \tan\left(2\pi - \frac{\pi}{4}\right)$$

$\Rightarrow \quad \tan x = \tan\dfrac{3\pi}{4} \quad$ or $\quad \tan\dfrac{7\pi}{4}$

$\therefore \qquad x = \dfrac{3\pi}{4}$ and $x = \dfrac{7\pi}{4}$ (Turning points)

Let us apply second derivative test.

At $x = \dfrac{3\pi}{4},\ f''(x) = -\sin x + \cos x = -\sin\dfrac{3\pi}{4} + \cos\dfrac{3\pi}{4}$

$$= -\sin\frac{4\pi - \pi}{4} + \cos\frac{4\pi - \pi}{4} = -\sin\left(\pi - \frac{\pi}{4}\right) + \cos\left(\pi - \frac{\pi}{4}\right)$$

$$= -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}} = -\sqrt{2} \text{ (negative)}$$

$\therefore\ x = \dfrac{3\pi}{4}$ is a point of local maxima and local maximum value

$$= f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} \qquad \text{(From } (i))$$

$$= \sin\left(\pi - \frac{\pi}{4}\right) - \cos\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

At $x = \dfrac{7\pi}{4}$,

$$f''(x) = -\sin x + \cos x = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$$

$$= -\sin\left(\frac{8\pi - \pi}{4}\right) + \cos\left(\frac{8\pi - \pi}{4}\right)$$

$$= -\sin\left(2\pi - \frac{\pi}{4}\right) + \cos\left(2\pi - \frac{\pi}{4}\right)$$

$$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \quad \text{(Positive)}$$

$\therefore\ x = \dfrac{7\pi}{4}$ is a point of local minima and local minimum value

$$= f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} \quad \text{(From } (i))$$

$$= \sin\left(2\pi - \frac{\pi}{4}\right) - \cos\left(2\pi - \frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$$

$$= \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}.$$

Therefore, Local maxima at $x = \dfrac{3\pi}{4}$ and local maximum value $= \sqrt{2}$. Local minima at $x = \dfrac{7\pi}{4}$ and local minimum value $= -\sqrt{2}$.

(*v*) **Given:** $f(x) = x^3 - 6x^2 + 9x + 15$ ...(*i*)

$\therefore\quad f'(x) = 3x^2 - 12x + 9$ and $f''(x) = 6x - 12$

Putting $f'(x) = 0$ to get turning points, we have

$$3x^2 - 12x + 9 = 0$$

Dividing by 3, $x^2 - 4x + 3 = 0$

or $x^2 - x - 3x + 3 = 0$ or $(x - 1)(x - 3) = 0$

$\therefore$ Either $x - 1 = 0$ or $x - 3 = 0$

i.e., $x = 1$ or $x = 3$. (Turning points)

Let us apply second derivative test.

When $x = 1$, $f''(x) = 6x - 12$

$= 6 - 12 = -6$ (negative)

$\therefore$ $x = 1$ is a point of local maxima and local maximum value

$= f(1) = (1)^3 - 6(1)^2 + 9(1) + 15 = 1 - 6 + 9 + 15 = 19$

When $x = 3$ $f''(x) = 6x - 12 = 6(3) - 12 = 6$ (positive)

$\therefore$ $x = 3$ is a point of local minima and local minimum value

$= f(3) = (3)^3 - 6(3)^2 + 9(3) + 15$

$= 27 - 54 + 27 + 15 = 15$

Therefore, Local maxima at $x = 1$ and local maximum value = 19.
Local minima at $x = 3$ and local minimum value = 15.

(*vi*) **Given:** $g(x) = \dfrac{x}{2} + \dfrac{2}{x}, x > 0$

$$\therefore \quad g'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2} = \frac{(x+2)(x-2)}{2x^2} \quad ...(i)$$

For turning points, putting $g'(x) = 0$

$$\Rightarrow \quad \frac{(x+2)(x-2)}{2x^2} = 0$$

$\Rightarrow (x + 2)(x - 2) = 0 \quad \Rightarrow x = -2, 2$

But $x > 0$ (given) $\therefore$ $x = -2$ is rejected.

Hence $x = 2$ is the only turning point.

Let us apply first derivative test

When x is slightly < 2, let $x = 1.9$

From (*i*), $$g'(1.9) = \frac{(1.9+2)(1.9-2)}{2(1.9)^2} = \frac{(+\text{ ve})(-\text{ ve})}{(+\text{ ve})} = -\text{ ve}$$

When x is slightly > 2, let $x = 2.1$

$$g'(2.1) = \frac{(2.1+2)(2.1-2)}{2(2.1)^2} = \frac{(+\text{ ve})(+\text{ ve})}{(+\text{ ve})} = +\text{ ve}$$

Thus, $g'(x)$ changes sign from negative to positive as x increases through 2.

$\therefore$ $x = 2$ is a point of local minima and local minimum value

$$= g(2) = \frac{2}{2} + \frac{2}{2} = 1 + 1 = 2.$$

Note. Second derivative test, $g''(x)$

$$= \frac{d}{dx}\left(\frac{1}{2} - \frac{2}{x^2}\right) = \frac{4}{x^3} \qquad \therefore \quad g''(2) = \frac{4}{8} = \frac{1}{2} > 0$$

$\Rightarrow g(x)$ has local minimum value at $x = 2$ and local minimum value $= g(2) = \dfrac{2}{2} + \dfrac{2}{2} = 1 + 1 = 2.$

(*vii*) **Given:** $h(x) = \dfrac{1}{x^2 + 2} = (x^2 + 2)^{-1}$ ...(*i*)

$\therefore \quad h'(x) = (-1)(x^2 + 2)^{-2}\,(2x) = -\dfrac{2x}{(x^2+2)^2}$

and $\quad h''(x) = -\left[\dfrac{(x^2+2)^2 \,.\, 2 - 2x \,.\, 2(x^2+2)\,2x}{(x^2+2)^4}\right]$

$$= \frac{-2(x^2+2)\,[x^2+2-4x^2)}{(x^2+2)^4} = \frac{-2(2-3x^2)}{(x^2+2)^3}$$

Putting $h'(x) = 0$ to get turning points, we have

$$\frac{-2x}{(x^2+2)^2} = 0 \Rightarrow -2x = 0 \Rightarrow x = \frac{0}{-2} = 0$$

Let us apply second derivative test.

At $x = 0$, $\quad h''(x) = \dfrac{-2(2-3x^2)}{(x^2+2)^3} = \dfrac{-2(2-0)}{(0+2)^3} = \dfrac{-4}{8}$

$$= \frac{-1}{2} \quad \text{(Negative)}$$

$\therefore \quad x = 0$ is a point of local maxima and local maximum value

$$= h(0) = \frac{1}{0+2} = \frac{1}{2} \quad \text{(From } (i))$$

Therefore, Local maxima at $x = 0$ and local maximum value $= \dfrac{1}{2}$.

(*viii*) **Given:** $f(x) = x\sqrt{1-x}$, $x \le 1$

$\therefore \quad f'(x) = x\,\dfrac{1}{2}\,(1-x)^{-1/2}\,\dfrac{d}{dx}(1-x) + \sqrt{1-x}\;.\,1$

$\therefore \quad f'(x) = x\,.\,\dfrac{1}{2\sqrt{1-x}}(-1) + \sqrt{1-x}\;.\,1$

$$= \frac{-x}{2\sqrt{1-x}} + \sqrt{1-x} = \frac{-x+2(1-x)}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}} \quad ...(i)$$

For turning points, putting $f'(x) = 0$

$$\Rightarrow \frac{2-3x}{2\sqrt{1-x}} = 0 \Rightarrow 2-3x = 0 \quad \therefore x = \frac{2}{3}$$

Let us apply first derivative test.

When x is slightly $< \dfrac{2}{3}$, let $x = 0.6$

From (*i*), $\quad f'(0.6) = \dfrac{2-3(0.6)}{2\sqrt{1-0.6}} = \dfrac{2-1.8}{2\sqrt{0.4}} = \dfrac{0.2}{2\sqrt{0.4}} > 0$

When x is slightly $> \dfrac{2}{3}$, let $x = 0.7$

From (*i*), $\quad f'(0.7) = \dfrac{2-3\,.\,(0.7)}{2\sqrt{1-0.7}} = \dfrac{2-2.1}{2\sqrt{0.3}} = \dfrac{-0.1}{2\sqrt{0.3}} < 0$

Thus, $f'(x)$ changes sign from positive to negative as x increases through $\frac{2}{3}$.

$\therefore$ $x = \frac{2}{3}$ is a point of local maxima and local maximum value

$$= f\left(\frac{2}{3}\right) = x\sqrt{1-x} = \frac{2}{3}\sqrt{1-\frac{2}{3}} = \frac{2}{3} \times \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{9}.$$

Note. Apply second derivative test.

Differentiating both sides of (*i*) w.r.t. x,

$$f''(x) = \frac{1}{2} \cdot \frac{\sqrt{1-x}\,.(-3) - (2-3x)\,.\,\frac{1}{2\sqrt{1-x}}(-1)}{1-x}$$

$$f''\left(\frac{2}{3}\right) = \frac{1}{2} \cdot \frac{\left(\frac{1}{\sqrt{3}}\right)(-3) - 0}{\frac{1}{3}} = \frac{-9}{2\sqrt{3}} = -\frac{3\sqrt{3}}{2} < 0$$

$\therefore$ $f(x)$ has local maximum value at $x = \frac{2}{3}$.

4. Prove that the following functions do not have maxima or minima:

(*i*) $f(x) = e^x$ **(*ii*) $g(x) = \log x$**

(*iii*) $h(x) = x^3 + x^2 + x + 1$.

Sol. (*i*) **Given:** $f(x) = e^x$

$\therefore$ $f'(x) = e^x$

Putting $f'(x) = 0$ to get turning points, we have $e^x = 0$. But this gives no real value of x. [$\because$ $e^x > 0$ for all real x]

$\therefore$ No turning point.

Hence $f(x)$ does not have maxima or minima.

(*ii*) **Given:** $g(x) = \log x$

$\therefore$ $g'(x) = \frac{1}{x}$

Putting $g'(x) = 0$ to get turning points, we have

$$\frac{1}{x} = 0 \quad \Rightarrow \quad 1 = 0.$$

But this is impossible.

$\therefore$ No turning point.

Hence $f(x)$ does not have maxima or minima.

(*iii*) $h(x) = x^3 + x^2 + x + 1$

$h'(x) = 3x^2 + 2x + 1$

Putting $h'(x) = 0$, we have $3x^2 + 2x + 1 = 0$

$$\therefore \quad x = \frac{-2 \pm \sqrt{4-12}}{6} = \frac{-2 \pm \sqrt{-8}}{6}$$

$$= \frac{-2 \pm 2\sqrt{2}\, i}{6} = \frac{-1 \pm \sqrt{2}\, i}{3}$$

These values of x are imaginary.

$\therefore$ $h(x)$ does not have maxima or minima.

5. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(i) $f(x) = x^3$, $x \in [-2, 2]$

(ii) $f(x) = \sin x + \cos x$, $x \in [0, \pi]$

(iii) $f(x) = 4x - \frac{1}{2}x^2$, $x \in \left[-2, \frac{9}{2}\right]$

(iv) $f(x) = (x - 1)^2 + 3$, $x \in [-3, 1]$

Sol. (*i*) **Given:** $f(x) = x^3$, $x \in [-2, 2]$ ...(*i*)

$\therefore$ $f'(x) = 3x^2$

Putting $f'(x) = 0$ to get turning points, we have

$3x^2 = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0 \in [-2, 2]$

To find absolute maximum and absolute minimum value of the function, we are to find values of $f(x)$ at (*a*) turning point(s) and (*b*) at end points of the given closed interval $[-2, 2]$.

Putting $x = 0$ in (*i*), $f(0) = 0$

Putting $x = -2$ in (*i*), $f(-2) = (-2)^3 = -8$

Putting $x = 2$ in (*i*), $f(2) = (2)^3 = 8$

Out of these three values of $f(x)$; absolute minimum value $= -8$ and absolute maximum value is 8.

Remark. Absolute maximum and absolute minimum values of $f(x)$ are also called maximum and minimum values of $f(x)$.

(*ii*) **Given:** $f(x) = \sin x + \cos x$, $x \in [0, \pi]$ ...(*i*)

$\therefore$ $f'(x) = \cos x - \sin x$

Putting $f'(x) = 0$ to get turning points, we have

$\cos x - \sin x = 0 \Rightarrow -\sin x = -\cos x$.

Dividing by $-\cos x$, $\tan x = 1$ is positive.

$\therefore$ x is in I and III quadrants.

But $x \in [0, \pi]$ (given) can't be in third quadrant.

$\therefore$ x is in Ist quadrant.

Therefore, $\tan x = 1 = \tan \frac{\pi}{4}$

$\therefore$ $x = \frac{\pi}{4}$

Now let us find values of $f(x)$ at turning point $x = \frac{\pi}{4}$ and at end points $x = 0$ and $x = \pi$ of given closed interval $[0, \pi]$.

Putting $x = \frac{\pi}{4}$ in (*i*), $f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} + \cos \frac{\pi}{4}$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Putting $x = 0$ in (i), $f(0) = \sin 0 + \cos 0$

$= 0 + 1 = 1$

Putting $x = \pi$ in (i), $f(\pi) = \sin \pi + \cos \pi$

$= 0 - 1 = -1$

[$\because$ $\sin \pi = \sin 180° = \sin (180° - 0°) = \sin 0° = 0$

and $\cos \pi = \cos 180° = \cos (180° - 0°) = -\cos 0° = -1$]

$\therefore$ Absolute minimum value $= -1$ and absolute maximum value $= \sqrt{2}$.

(iii) **Given:** $f(x) = 4x - \frac{1}{2}x^2$, $x \in \left[-2, \frac{9}{2}\right]$ ...(i)

$\therefore \quad f'(x) = 4 - \frac{1}{2}(2x) = 4 - x.$

Putting $f'(x) = 0$ to find turning points,

we have $4 - x = 0$

i.e., $-x = -4$ *i.e.,* $x = 4 \in \left[-2, \frac{9}{2}\right]$

Now let us find values of $f(x)$ at turning point $x = 4$ and at end points $x = -2$ and $x = \frac{9}{2}$ of the given closed interval $\left[-2, \frac{9}{2}\right]$.

Putting $x = 4$ in (i), $f(4) = 16 - \frac{1}{2}(16) = 16 - 8 = 8$

Putting $x = -2$ in (i), $f(-2) = 4(-2) - \frac{1}{2}(4) = -8 - 2 = -10$

Putting $x = \frac{9}{2}$ in (i), $f\left(\frac{9}{2}\right) = 4\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right)^2$

$$= 18 - \frac{81}{8} = \frac{144 - 81}{8} = \frac{63}{8}$$

$\therefore$ Absolute minimum value is -10 and absolute maximum value is 8.

(iv) **Given:** $f(x) = (x - 1)^2 + 3$, $x \in [-3, 1]$ ...(i)

$\therefore \quad f'(x) = 2(x - 1)\frac{d}{dx}(x - 1) + 0 = 2(x - 1)$

Putting $f'(x) = 0$ to find turning points,

we have $2(x - 1) = 0$

$\Rightarrow \quad x - 1 = \frac{0}{2} = 0 \quad \Rightarrow \quad x = 1 \in [-3, 1]$

Now let us find values of $f(x)$ at turning point $x = 1$ and at the end point $x = -3$ of the given closed

interval [– 3, 1] ($\because$ the other end point $x = 1$ has already come out to be a turning point)

Putting $x = -3$ in (*i*), $f(-3) = (-3-1)^2 + 3$
$= (-4)^2 + 3 = 16 + 3 = 19$

Putting $x = 1$ in (*i*), $f(1) = (1-1)^2 + 3 = 0 + 3 = 3$

$\therefore$ Absolute minimum value is 3 and absolute maximum value is 19.

Note. To find absolute maximum or absolute minimum value of a function $f(x)$ when only one turning point comes out to be there and no closed interval is given, then we get only one value of $f(x)$ at such points and out of one value of $f(x)$ we can't make a decision about maximum value or minimum value. In such problems, we have to depend upon local minimum value and local maximum value.

6. Find the maximum profit that a company can make, if the profit function is given by

$$p(x) = 41 + 24x - 18x^2.$$

Sol. Given: Profit function is $p(x) = 41 + 24x - 18x^2$ ...(*i*)

$\therefore \quad p'(x) = 24 - 36x \quad$ and $\quad p''(x) = -36$

(The logic of finding $p''(x)$ is given in the note above)

Putting $p'(x) = 0$ to get turning points, we have

$$24 - 36x = 0 \Rightarrow -36x = -24$$

$$\Rightarrow \qquad x = \frac{24}{36} = \frac{2}{3}$$

At $x = \frac{2}{3}$, $p''(x) = -36$ (Negative)

$\therefore \quad p(x)$ has a local maximum value and hence maximum value at $x = \frac{2}{3}$.

Putting $x = \frac{2}{3}$ in (*i*),

Maximum profit $= 41 + 24\left(\frac{2}{3}\right) - 18\left(\frac{4}{9}\right) = 41 + 16 - 8 = 49$

Remark. The original statement in N.C.E.R.T. book $p(x) = 41 - 24x - 18x^2$ is wrong, because with this $p(x)$, turning point comes out to be $x = -\frac{2}{3}$ which being the number of units produced or sold can't be negative.

7. Find both the maximum value and the minimum value of $3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval [0, 3].

Sol. Let $\quad f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$ on [0, 3] ...(*i*)

$\therefore \quad f'(x) = 12x^3 - 24x^2 + 24x - 48$

Putting $f'(x) = 0$ to find turning points, we have

$$12x^3 - 24x^2 + 24x - 48 = 0$$

Dividing every term by 12, $x^3 - 2x^2 + 2x - 4 = 0$

or $x^2(x-2)+2(x-2)=0$ or $(x-2)(x^2+2)=0$

$\therefore$ Either $x-2=0$ or $x^2+2=0$

$\Rightarrow$ $x=2$ $\Rightarrow$ $x^2=-2$

$\Rightarrow$ $x=\pm\sqrt{-2}$

These values of x are imaginary and hence rejected.

Turning point $x=2\in[0,3]$.

Now let us find values of $f(x)$ at turning point $x=2$ and end points $x=0$ and $x=3$ of closed interval [0, 3].

Putting $x=2$ in (i), $f(2)=3(16)-8(8)+12(4)-48(2)+25$
$=48-64+48-96+25=-39$

Putting $x=0$ in (i), $f(0)=25$

Putting $x=3$ in (i), $f(3)=3(81)-8(27)+12(9)-48(3)+25$
$=243-216+108-144+25$
$=27-36+25=16.$

$\therefore$ Minimum (absolute) value is -39 (at $x=2$) and maximum (absolute) value is 25 (at $x=0$).

8. At what points on the interval [0, 2π] does the function sin 2*x* attain its maximum value?

Sol. Let $f(x)=\sin 2x$, then $f'(x)=2\cos 2x$

For maxima or minima, $f'(x)=0$

$\Rightarrow \cos 2x=0 \;\therefore\; 2x=(2n+1)\dfrac{\pi}{2}$ or $x=(2n+1)\dfrac{\pi}{4}$

Putting $n=0,1,2,3;\; x=\dfrac{\pi}{4},\dfrac{3\pi}{4},\dfrac{5\pi}{4},\dfrac{7\pi}{4}\in[0,2\pi]$

Now let us find values of $f(x)$ at these turning points.

Now $f(x)=\sin 2x$

$\therefore \left[f(2n+1)\dfrac{\pi}{4}\right]=\sin(2n+1)\dfrac{\pi}{2}=\sin\left(n\pi+\dfrac{\pi}{2}\right)=(-1)^n\sin\dfrac{\pi}{2}$

$[\because \sin(n\pi+\alpha)=(-1)^n\sin\alpha;\; n\in I]$

$=(-1)^n$

Putting $n=0,1,2,3$,

$f\left(\dfrac{\pi}{4}\right)=(-1)^0=1,$ $f\left(\dfrac{3\pi}{4}\right)=(-1)^1=-1$

$f\left(\dfrac{5\pi}{4}\right)=(-1)^2=1,$ $f\left(\dfrac{7\pi}{4}\right)=(-1)^3=-1$

Also let us find $f(x)$ at the end-points $x=0$ and $x=2\pi$ of $[0, 2\pi]$.

$f(0)=\sin 0=0,\; f(2\pi)=\sin 4\pi=0$ $[\because \sin n\pi=0$ where $n\in I]$

$\therefore f(x)$ attains its maximum value 1 at $x=\dfrac{\pi}{4}$ and $x=\dfrac{5\pi}{4}$.

Hence, the required points are $\left(\dfrac{\pi}{4},1\right)$ and $\left(\dfrac{5\pi}{4},1\right)$.

9. What is the maximum value of the function sin x + cos x?

Sol. Let $f(x) = \sin x + \cos x$ ...(*i*)

$\therefore \quad f'(x) = \cos x - \sin x$

Putting $f'(x) = 0$ to find turning points, we have

$\cos x - \sin x = 0 \quad \Rightarrow \quad -\sin x = -\cos x$

Dividing by $-\cos x$, $\dfrac{\sin x}{\cos x} = 1 \quad \Rightarrow \quad \tan x = 1 = \tan \dfrac{\pi}{4}$

$\therefore \quad x = n\pi + \dfrac{\pi}{4}$ where $n \in Z$ (turning points)

($\because$ If $\tan \theta = \tan \alpha$, then $\theta = n\pi + \alpha$ where $n \in Z$)

Putting $x = n\pi + \dfrac{\pi}{4}$ in (*i*),

$$f\left(n\pi + \frac{\pi}{4}\right) = \sin\left(n\pi + \frac{\pi}{4}\right) + \cos\left(n\pi + \frac{\pi}{4}\right)$$

$$= (-1)^n \sin \frac{\pi}{4} + (-1)^n \cos \frac{\pi}{4}$$

[$\because \sin(n\pi + \alpha) = (-1)^n \sin \alpha$ and $\cos(n\pi + \alpha) = (-1)^n \cos \alpha$]

$= (-1)^n \dfrac{1}{\sqrt{2}} + (-1)^n \dfrac{1}{\sqrt{2}} = 2(-1)^n \dfrac{1}{\sqrt{2}}$ ($\because \; t + t = 2t$)

$= (-1)^n \sqrt{2}$

If n is even; then $(-1)^n = 1$ and then $f\left(n\pi + \dfrac{\pi}{4}\right) = \sqrt{2}$

If n is odd, then $(-1)^n = -1$; and then $f\left(n\pi + \dfrac{\pi}{4}\right) = -\sqrt{2}$

$\therefore$ Maximum value of $f(x)$ is $\sqrt{2}$.

Note. Minimum value of $f(x)$ is $-\sqrt{2}$.

10. Find the maximum value of $2x^3 - 24x + 107$ in the interval [1, 3]. Find the maximum value of the same function in [– 3, – 1].

Sol. Let $f(x) = 2x^3 - 24x + 107$ ...(*i*)

$\therefore \quad f'(x) = 6x^2 - 24$

Let us put $f'(x) = 0$ to find turning points.

i.e., $6x^2 - 24 = 0 \quad \Rightarrow \quad 6x^2 = 24$

Dividing by 6, $x^2 = 24 \quad \Rightarrow \quad x = \pm 2$

$\therefore \quad x = -2$ and $x = 2$ are two turning points. ...(*ii*)

(*a*) Let us find maximum value of $f(x)$ given by (*i*) in the interval [1, 3].

From (*ii*), turning point $x = -2 \notin [1, 3]$.

So let us find values of $f(x)$ at turning point $x = 2$ and at end points $x = 1$ and $x = 3$ of closed interval [1, 3].

Putting $x = 2$ in (*i*), $f(2) = 2(8) - 24(2) + 107$

$= 16 - 48 + 107 = 123 - 48 = 75$

Putting $x = 1$ in (i), $f(1) = 2 - 24 + 107 = 109 - 24 = 85$

Putting $x = 3$ in (i), $f(3) = 2(27) - 24(3) + 107$

$= 54 - 72 + 107 = 161 - 72 = 89$

$\therefore$ Maximum value of $f(x)$ given by (i) in $[1, 3]$ is 89 (at $x = 3$).

(b) Let us find maximum value of $f(x)$ given by (i) in the interval $[-3, -1]$.

From (ii), turning point $x = 2 \notin [-3, -1]$

So let us find values of $f(x)$ at turning point $x = -2$ and at end points $x = -3$ and $x = -1$ of closed interval $[-3, -1]$

Putting $x = -2$ in (i), $f(-2) = 2(-8) - 24(-2) + 107$

$= -16 + 48 + 107 = 139$

Putting $x = -3$ in (i), $f(-3) = 2(-27) - 24(-3) + 107$

$= -54 + 72 + 107 = 125$

Putting $x = -1$ in (i), $f(-1) = 2(-1) - 24(-1) + 107$

$= -2 + 24 + 107 = 129$

$\therefore$ Maximum value of $f(x)$ is 139 (at $x = -2$).

11. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval [0, 2]. Find the value of a.

Sol. Let $f(x) = x^4 - 62x^2 + ax + 9$

$\therefore$ $f'(x) = 4x^3 - 124x + a$ ...(i)

Because $f(x)$ attains its maximum value at $x = 1$ in the interval $[0, 2]$, therefore, $f'(1) = 0$.

Putting $x = 1$ in (i), $f'(1) = 4 - 124 + a = 0$

or $a - 120 = 0$ or $a = 120$.

12. Find the maximum and minimum value of $x + \sin 2x$ on $[0, 2\pi]$.

Sol. Let $f(x) = x + \sin 2x$ ...(i)

$\therefore$ $f'(x) = 1 + 2 \cos 2x$

Putting $f'(x) = 0$ to find turning points, we have

$1 + 2 \cos 2x = 0 \Rightarrow 2 \cos 2x = -1$

$$\Rightarrow \quad \cos 2x = \frac{-1}{2} = -\cos \frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos \frac{2\pi}{3}$$

$$\therefore \quad 2x = 2n\pi \pm \frac{2\pi}{3} \text{ where } n \in Z$$

[$\because$ If $\cos \theta = \cos \alpha$, then $\theta = 2n\pi \pm \alpha$ where $n \in Z$]

Dividing by 2, $x = n\pi \pm \dfrac{\pi}{3}$ where $n \in Z$

For $n = 0$, $x = \pm \dfrac{\pi}{3}$. But $x = -\dfrac{\pi}{3} \notin [0, 2\pi]$ $\therefore$ $x = \dfrac{\pi}{3}$

For $n = 1$, $x = \pi \pm \dfrac{\pi}{3} = \pi + \dfrac{\pi}{3}$ and $\pi - \dfrac{\pi}{3}$

i.e., $\dfrac{4\pi}{3}$ and $\dfrac{2\pi}{3}$ and both belong to $[0, 2\pi]$

For $n = 2$, $x = 2\pi \pm \frac{\pi}{3}$. But $x = 2\pi + \frac{\pi}{3} > 2\pi$ and hence $\notin [0, 2\pi]$

$$\therefore \quad x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

It can be easily observed that for all other $n \in Z$,

$$x = n\pi \pm \frac{\pi}{3} \notin [0, 2\pi].$$

$\therefore$ The only turning points of $f(x)$ given by (i) which belong to given closed interval $[0, 2\pi]$ are

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

Now let us find values of $f(x)$ at these four turning points and at the end points $x = 0$ and $x = 2\pi$ of $[0, 2\pi]$.

Putting $x = \frac{\pi}{3}$ in (i), $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3}$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2} = 1.05 + 0.87$$

$$= 1.92 \text{ nearly}$$

$$\left(\because \frac{\pi}{3} = \frac{\frac{22}{7}}{3} = \frac{22}{21} = 1.05 \text{ and } \frac{\sqrt{3}}{2} = \frac{1.732}{2} = 0.866 = 0.87\right)$$

$$\text{and} \left(\because \sin\frac{2\pi}{3} = \sin\frac{3\pi - \pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\right)$$

Putting $x = \frac{2\pi}{3}$ in (i), $f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3}$

$$= 2\pi - \frac{\sqrt{3}}{2} = 2.10 - 0.87 = 1.23 \text{ nearly}$$

$$\left(\because \sin\frac{4\pi}{3} = \sin\frac{3\pi + \pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = \frac{-\sqrt{3}}{2}\right)$$

Putting $x = \frac{4\pi}{3}$ in (i), $f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin\frac{8\pi}{3}$

$$= \frac{4\pi}{3} + \frac{\sqrt{3}}{2} = 4(1.05) + 0.87 = 4.20 + 0.87 = 5.07$$

$$\left(\because \sin\frac{8\pi}{3} = \sin\left(\frac{6\pi + 2\pi}{3}\right) = \sin\left(2\pi + \frac{2\pi}{3}\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}\right)$$

Putting $x = \frac{5\pi}{3}$ in (i), $f\left(\frac{5\pi}{3}\right)$

$$= \frac{5\pi}{3} + \sin\frac{10\pi}{3} = \frac{5\pi}{3} - \frac{\sqrt{3}}{2}$$

$= 5(1.05) - 0.87 = 5.25 - 0.87 = 4.38$ nearly

$$\left[\because \sin\frac{10\pi}{3} = \sin\frac{6\pi + 4\pi}{3} = \sin\left(2\pi + \frac{4\pi}{3}\right) = \sin\frac{4\pi}{3} = \frac{-\sqrt{3}}{2}\right]$$

Putting $x = 0$ in (*i*), $f(0) = 0 + \sin 0 = 0$

Putting $x = 2\pi$ in (*i*), $f(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi$

$= 2(3.14) = 6.28$ nearly ($\because$ $\sin n\pi = 0$ for every integer n)

$\therefore$ Maximum value $= 2\pi$ (at $x = 2\pi$)

and minimum value $= 0$ (at $x = 0$).

13. Find two numbers whose sum is 24 and whose product is as large as possible.

Sol. Let the two numbers be x and y.

Their sum $= 24$ (given) $\Rightarrow$ $x + y = 24$

$\therefore$ $y = 24 - x$ ...(*i*)

Let z denote their product *i.e.,* product of x and y

i.e., $z = xy$

Putting $y = 24 - x$ from (*i*),

$z = x(24 - x) = 24x - x^2$

Now z is a function of x only.

$\therefore$ $\dfrac{dz}{dx} = 24 - 2x$ and $\dfrac{d^2z}{dx^2} = -2$

Putting $\dfrac{dz}{dx} = 0$ to find turning points, we have

$24 - 2x = 0$ *i.e.,* $-2x = -24$. Therefore, $x = 12$.

At $x = 12$, $\dfrac{d^2z}{dx^2} = -2$ (negative)

$\therefore$ $x = 12$ is a point of (local) maxima.

(See Note at the end of solution of Q. No. 5)

$\therefore$ z is maximum at $x = 12$.

Putting $x = 12$ in (*i*), $y = 24 - 12 = 12$

$\therefore$ The two required numbers are 12 and 12.

14. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

Sol. Here $x + y = 60$, $x > 0$, $y > 0$ ...(*i*) (Given condition)

Let $P = xy^3$ ...(*ii*) (To be maximised)

To express P in terms of one independent variable, (here better y, because power of y is larger in the value of P), we have from (*i*)

$x = 60 - y$,

Putting $x = 60 - y$ in (*ii*), $P = (60 - y)y^3 = 60y^3 - y^4$

$\therefore$ $\dfrac{dP}{dy} = 180y^2 - 4y^3 = 4y^2(45 - y)$ ...(*iii*)

For max. or min., Putting $\dfrac{dP}{dy} = 0$

$\Rightarrow$ $4y^2(45 - y) = 0$ $\Rightarrow y = 0, 45$

Rejecting $y = 0$ because $y > 0 \quad \therefore y = 45$

When y is slightly < 45, from (*iii*), $\dfrac{dP}{dy} = (+\text{ ve})(+\text{ ve}) = +\text{ ve}$

When y is slightly > 45, from (*iii*), $\dfrac{dP}{dy} = (+\text{ ve})(-\text{ ve}) = -\text{ ve}$

Thus, $\dfrac{dP}{dy}$ changes sign from + ve to – ve as y increases through 45.

$\therefore$ P is maximum when $y = 45$.

Hence, xy^3 is maximum when $x = 60 - 45 = 15$

and $\qquad y = 45.$

15. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.

Sol. Given: $x + y = 35 \quad \Rightarrow \quad y = 35 - x \qquad ...(i)$

Let $\qquad z = x^2\, y^5$

Putting $y = 35 - x$ from (*i*), $\quad z = x^2\,(35 - x)^5$

Now z is a function of x alone.

$$\therefore \quad \frac{dz}{dx} = x^2 \,.\, 5(35 - x)^4\,(-1) + (35 - x)^5\, 2x$$

$$\text{or} \quad \frac{dz}{dx} = x(35 - x)^4\,[-5x + (35 - x)\,2]$$

$$\text{or} \quad \frac{dz}{dx} = x(35 - x)^4\,(-5x + 70 - 2x) \quad = x(35 - x)^4\,(70 - 7x)$$

$$\text{or} \quad \frac{dz}{dx} = 7x\,(35 - x)^4\,(10 - x) \qquad ...(ii)$$

Putting $\dfrac{dz}{dx} = 0$ to find turning points, we have

$$7x(35 - x)^4\,(10 - x) = 0$$

But $7 \neq 0 \quad \therefore$ Either $x = 0$ or $35 - x = 0$ or $10 - x = 0$

i.e., $x = 0$ or $x = 35$ or $x = 10$.

Now $x = 0$ is rejected because x is positive number (given).

Also, $x = 35$ is rejected because for $x = 35$, from (*i*)

$\quad y = 35 - 35 = 0$; but y is given to be positive.

$\therefore \quad x = 10$ is the only admissible turning point.

Let us apply first derivative test because finding $\dfrac{d^2z}{dx^2}$ is tedious as you and we think so.

We know that $(35 - x)^4$ is never negative because index 4 is even.

When x is slightly < 10 (say $x = 9.8$); from (*ii*),

$$\frac{dz}{dx} = (+)\,(+)\,(+) = (+)$$

When x is slightly > 10 (say $x = 10.1$); from (*ii*),

$$\frac{dz}{dx} = (+)\,(+)\,(-) = (-)$$

$\therefore \quad \dfrac{dz}{dx}$ changes sign from (+) to (–) as x increases while passing through 10.

$\therefore \quad x = 10$ gives a point of (local) maxima.

(See Note at the end of solution of Q. No. 5)

i.e., z is maximum when $x = 10$.

Putting $x = 10$ in (*i*), $y = 35 - 10 = 25$.

$\therefore$ The two required numbers are 10 and 25.

16. Find two positive numbers whose sum is 16 and sum of whose cubes is minimum.

Sol. Let the two positive numbers be x and y.

Given: $x + y = 16 \quad \Rightarrow \quad y = 16 - x$ ...(*i*)

Let z denote the sum of their cubes *i.e.*, $z = x^3 + y^3$.

Putting $y = 16 - x$ from (*i*), $z = x^3 + (16 - x)^3$

$\Rightarrow \quad z = x^3 + (16)^3 - x^3 - 48x\,(16 - x)$

$[\because \quad (a - b)^3 = a^3 - b^3 - 3ab(a - b)]$

$\Rightarrow \quad z = (16)^3 - 768x + 48x^2$

Now z is a function of x alone.

$$\therefore \quad \frac{dz}{dx} = -\,768 + 96x \text{ and } \frac{d^2z}{dx^2} = 96$$

Putting $\dfrac{dz}{dx} = 0$ to find turning points, we have

$$-\,768 + 96x = 0 \quad \Rightarrow \quad 96x = 768 \quad \Rightarrow \quad x = \frac{768}{96} = 8$$

At $x = 8$, $\dfrac{d^2z}{dx^2} = 96 \quad (+)$

$\therefore \quad x = 8$ is a point of (local minima.

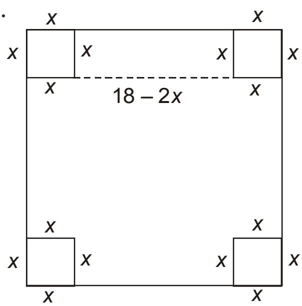
(See **Note** at the end of solution of Q. No. 5)

$\therefore \quad z$ is minimum when $x = 8$.

Putting $x = 8$ in (*i*), $y = 16 - 8 = 8$

$\therefore$ The required numbers are 8 and 8.

17. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?



Sol. Given: Each side of square piece of tin is 18 cm.

Let x cm be the side of each of the four squares cut off from each corner.

Then dimensions of the open box formed by folding the flaps after cutting off squares are $(18 - 2x)$, $(18 - 2x)$, x cm.

Let z denote the volume of the open box.

$\therefore \quad z$ = length × breadth × height $= (18 - 2x)(18 - 2x)x$

or $\quad z = (18 - 2x)^2\, x = (324 + 4x^2 - 72x)x$

or $\quad z = 4x^3 - 72x^2 + 324x$

$$\therefore \quad \frac{dz}{dx} = 12x^2 - 144x + 324 \quad \text{and} \quad \frac{d^2z}{dx^2} = 24x - 144$$

Putting $\frac{dz}{dx} = 0$ to find turning points, we have

$$12x^2 - 144x + 324 = 0$$

Dividing by 12, $x^2 - 12x + 27 = 0$

$\Rightarrow \quad x^2 - 9x - 3x + 27 = 0 \quad \Rightarrow \quad x(x - 9) - 3(x - 9) = 0$

$\Rightarrow \quad (x - 9)(x - 3) = 0$

$\therefore$ Either $\quad x - 9 = 0$ or $x - 3 = 0$

i.e., $\quad x = 9$ or $x = 3$

But $x = 9$ is rejected because for $x = 9$,

length of box $= 18 - 2x = 18 - 18 = 0$ which is clearly impossible.

$\therefore$ $x = 3$ is the only turning point.

At $x = 3$, $\frac{d^2z}{dx^2} = 24x - 144 = 72 - 144 = -72$ (Negative)

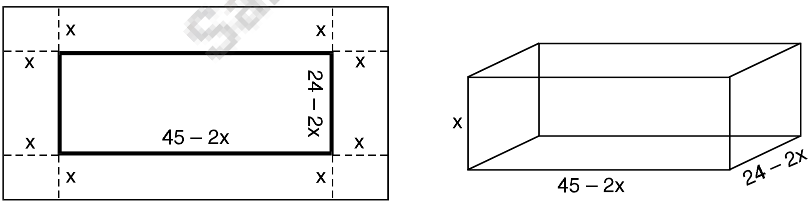
$\therefore$ z is maximum at $x = 3$.

i.e., side of (each) square to be cut off from each corner for maximum volume is 3 cm.

Remark. The reader is suggested to take a paper sheet in square shape and cut off four equal squares from four corners and fold the flaps to form a box for himself or herself.

18. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off a square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum?

Sol. Dimensions of the rectangular sheet of tin are 45 cm and 24 cm. Let the side of the square cut off from each corner be x cm. Therefore, dimensions of the box are $45 - 2x$, $24 - 2x$ and x cm.



The volume V of the box in cubic cm is given by

$$V = (45 - 2x)(24 - 2x)(x) \qquad \text{[product of three dimensions]}$$

$$= x\,(1080 - 138x + 4x^2) = 1080x - 138x^2 + 4x^3$$

$\therefore \quad \dfrac{dV}{dx} = 1080 - 276x + 12x^2$ and $\dfrac{d^2V}{dx^2} = -276 + 24x$

For max. or min. put $\dfrac{dV}{dx} = 0$

$\Rightarrow \quad 1080 - 276x + 12x^2 = 0,$

Dividing by 12, $x^2 - 23x + 90 = 0$

$\Rightarrow \quad (x - 5)(x - 18) = 0 \quad \therefore x = 5, 18$

But $x = 18$ is impossible because otherwise the dimension

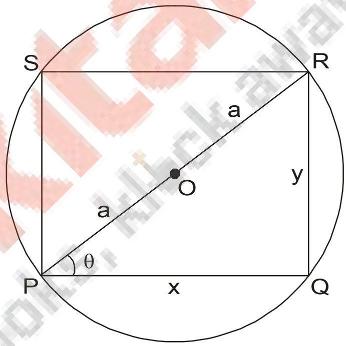
$24 - 2x = 24 - 36 = -12$ is negative. $\therefore x = 5$

$$\left[\frac{d^2V}{dx^2}\right]_{x=5} = -276 + 120 = -156 < 0$$

$\Rightarrow$ V is maximum when $x = 5$

Hence, the box with maximum volume is obtained by cutting off equal squares of side 5 cm.

19. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.



Sol. Let PQRS be the rectangle inscribed in a given circle with centre O and radius a.

Let x and y be the length and breadth of the rectangle ($\therefore x > 0$ and $y > 0$)

In right angled triangle PQR, By Pythagoras Theorem,

$$PQ^2 + QR^2 = PR^2$$

or $\quad x^2 + y^2 = (2a)^2$ (given condition)

or $\quad y^2 = 4a^2 - x^2 \quad \therefore y = \sqrt{4a^2 - x^2}$ ...(*i*)

Let A denote the area of the rectangle.

$\therefore \quad A = xy$ ...(*ii*) (A is to be maximised)

To express A in terms of one independent variable, putting the value of y from (*i*) in (*ii*), we have

$$A = x\sqrt{4a^2 - x^2}$$

Let $\quad z = A^2 = x^2(4a^2 - x^2) = 4a^2x^2 - x^4$ ...(*iii*)

Let us maximise z (= A^2)

From (*iii*), $\quad \dfrac{dz}{dx} = 8a^2x - 4x^3$

and $\quad \dfrac{d^2z}{dx^2} = 8a^2 - 12x^2$

For max. or min.; put $\dfrac{dz}{dx} = 0$

$\therefore\ 8a^2x - 4x^3 = 0 \quad$ or $\quad 4x(2a^2 - x^2) = 0$

But x being side of rectangle cannot be zero.

$\therefore\ 2a^2 - x^2 = 0 \quad$ or $\quad x^2 = 2a^2$

$\therefore \qquad x = \sqrt{2}\ .\ a \qquad\qquad (\because\ x > 0)$

At $\qquad x = \sqrt{2}\ a,\ \dfrac{d^2z}{dx^2} = 8a^2 - 12(2a^2) = 8a^2 - 24a^2$

$= -\ 16a^2$ is negative.

$\therefore \qquad z\ (= A^2)$ is maximum when $x = \sqrt{2}\,a$.

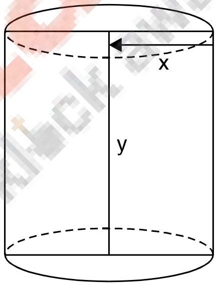
Putting $x = \sqrt{2}\,a$ in (i), $y = \sqrt{4a^2 - 2a^2} = \sqrt{2a^2} = \sqrt{2}\ .\ a$

$\therefore\ x = y = \sqrt{2}\,a\ \therefore$ A is maximum when $x = y = \sqrt{2}\,a$.

Hence, the area of the inscribed rectangle is maximum when it is a square.

20. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

Sol. Let x be the radius of the circular base and y be the height of closed right circular cylinder.



Total surface area of cylinder is given $(x > 0,\ y > 0)$

$\Rightarrow$ It is constant = S (say)

$\therefore$ Curved surface area + area of two ends = S

$\Rightarrow 2\pi xy + 2\pi x^2 = S$ (Given condition)

Dividing every term by 2π

[To get a simpler relation in independent variables x and y]

$$xy + x^2 = \frac{S}{2\pi} = k \text{ (say)}$$

$$\therefore \qquad xy = k - x^2 \text{ or } y = \frac{k - x^2}{x} \qquad ...(i)$$

Let z denote the volume of cylinder

$$\therefore \qquad z = \pi x^2 y \qquad ...(ii)$$

[Here z is to be maximised]. Putting the value of y from (i) in (ii) [to express z in terms of one independent variable x]

$$z = \pi x^2 \left(\frac{k - x^2}{x}\right) \quad \text{or} \quad z = \pi x\,(k - x^2) = \pi\,(kx - x^3)$$

$$\therefore\ \frac{dz}{dx} = \pi\,(k - 3x^2) \text{ and } \frac{d^2z}{dx^2} = \pi\,(-\,6x) = -\,6\pi x$$

For max. or min. put $\dfrac{dz}{dx} = 0\ \therefore\ \ \pi\,(k - 3x^2) = 0$

But $\pi \neq 0$ $\therefore$ $k - 3x^2 = 0$ or $3x^2 = k$ or $x^2 = \frac{k}{3}$ $\therefore$ $x = \sqrt{\frac{k}{3}}$

At $\quad x = \sqrt{\frac{k}{3}}, \quad \frac{d^2z}{dx^2} = -6\pi x = -6\pi\sqrt{\frac{k}{3}}$ is negative

$\therefore \quad z$ is max. at $\quad x = \sqrt{\frac{k}{3}}$ ...(*iii*)

Putting $\quad x = \sqrt{\frac{k}{3}}$ in (*i*), $y = \dfrac{k - \frac{k}{3}}{\sqrt{\frac{k}{3}}} = \dfrac{2\frac{k}{3}}{\sqrt{\frac{k}{3}}}$

$= 2\sqrt{\frac{k}{3}}$ $\quad \left[\because \frac{t}{\sqrt{t}} = \sqrt{t}\right]$

or $\quad y = 2\sqrt{\frac{k}{3}} = 2x$ [By (*iii*)]

i.e., Height = Diameter

Hence, the volume of cylinder is maximum when its height is equal to the diameter of its base.

Remark 1. Right circular cylinder $\Rightarrow$ Closed right circular cylinder.

Remark 2. Total surface area of open cylinder $= 2\pi xy + \pi x^2$.

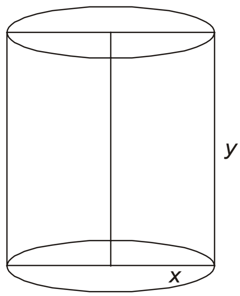
21. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?

Sol. Let x cm be the radius and y cm be the height of closed cylinder.

Given: Volume of closed cylinder = 100 cu cm

$\Rightarrow \quad \pi x^2 y = 100$

$\Rightarrow \quad y = \frac{100}{\pi x^2}$ ...(*i*)



Let z denote the surface area of cylinder.

$$\therefore \quad z = \underbrace{2\pi xy}_{\text{(Curved surface area)}} + \underbrace{2\pi x^2}_{\text{(Area of two ends)}}$$

or $\quad z = 2\pi(xy + x^2)$

Putting $y = \frac{100}{\pi x^2}$ from (*i*),

$$z = 2\pi\left(x \cdot \frac{100}{\pi x^2} + x^2\right) = 2\pi\left(\frac{100}{\pi x} + x^2\right) = 2\pi\left(\frac{100}{\pi}x^{-1} + x^2\right)$$

Now z is a function of x alone.

$$\therefore \quad \frac{dz}{dx} = 2\pi\left(-\frac{100}{\pi}x^{-2} + 2x\right)$$

$$\text{and } \frac{d^2z}{dx^2} = 2\pi\left(\frac{200}{\pi}x^{-3} + 2\right)$$

Putting $\frac{dz}{dx} = 0$ to find turning points, we have

$$2\pi\left(\frac{-100}{\pi x^2} + 2x\right) = 0. \text{ But } 2\pi \neq 0$$

$$\therefore \quad \frac{-100}{\pi x^2} + 2x = 0 \Rightarrow \frac{-100}{\pi x^2} = -2x$$

Cross-multiplying, $2\pi x^3 = 100$

$$\Rightarrow \quad x^3 = \frac{100}{2\pi} = \frac{50}{\pi}$$

$$\therefore \quad x = \left(\frac{50}{\pi}\right)^{1/3}$$

At $\quad x = \left(\frac{50}{\pi}\right)^{1/3}, \; \frac{d^2z}{dx^2} = 2\pi\left(\frac{200}{\pi x^3} + 2\right)$

$$= 2\pi\left(\frac{200}{\pi\left(\frac{50}{\pi}\right)} + 2\right) = 2\pi(4 + 2) = 12\pi \text{ (positive)}$$

$\therefore$ z is minimum (local) when radius $x = \left(\frac{50}{\pi}\right)^{1/3}$ cm ...(*ii*)

(See Note at the end of solution of Q. No. 5)

Putting $x = \left(\frac{50}{\pi}\right)^{1/3}$ in (*i*), $y = \frac{100}{\pi\left(\frac{50}{\pi}\right)^{2/3}}$

$$\Rightarrow \quad \text{Height } y = 2 \,.\, \frac{\frac{50}{\pi}}{\left(\frac{50}{\pi}\right)^{2/3}} = 2\left(\frac{50}{\pi}\right)^{1-2/3} = 2\left(\frac{50}{\pi}\right)^{1/3} \text{ cm.}$$

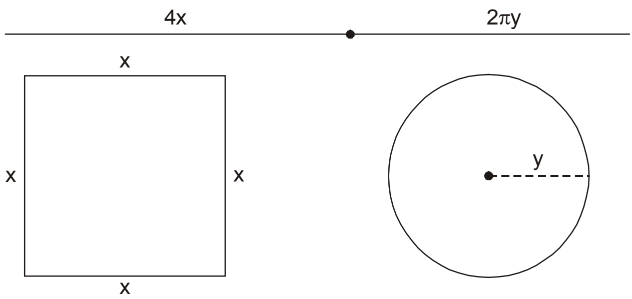
Remark. $y = 2x$ (By (*ii*))

22. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

Sol. Let x metres be the side of the square and y metres, the radius of the circle.

Length of wire = Perimeter of square + Circumference of circle

$= 4x + 2\pi y$



According to the question, $4x + 2\pi y = 28$ (Given condition)

Dividing by 2, $\quad 2x + \pi y = 14 \quad \therefore\ y = \dfrac{14 - 2x}{\pi}$ ...(*i*)

Area of square $= x^2$ sq. m. Area of circle $= \pi y^2$ sq. m

Let A denote their combined area, then

$$A = x^2 + \pi y^2 \qquad \text{[Here A is to be minimised]}$$

Putting the value of y from eqn. (*i*),

[To express A in terms of one independent variable x]

$$A = x^2 + \pi\left(\frac{14 - 2x}{\pi}\right)^2 = x^2 + \pi\left(\frac{2(7 - x)}{\pi}\right)^2 = x^2 + \pi \cdot \frac{4}{\pi^2}(7 - x)^2$$

or $\quad A = x^2 + \dfrac{4}{\pi}(7 - x)^2$

$\therefore \quad \dfrac{dA}{dx} = 2x - \dfrac{8}{\pi}(7 - x)$ and $\dfrac{d^2A}{dx^2} = 2 + \dfrac{8}{\pi}$

For max. or min., $\dfrac{dA}{dx} = 0 \quad \therefore\ 2x - \dfrac{8}{\pi}(7 - x) = 0$

$\therefore \quad 2x = \dfrac{8}{\pi}(7 - x)$

or $\quad 2\pi x = 56 - 8x \quad$ or $\quad (2\pi + 8)x = 56$

$\therefore \quad x = \dfrac{56}{2\pi + 8} = \dfrac{56}{2(\pi + 4)} = \dfrac{28}{\pi + 4}$.

Also $\dfrac{d^2A}{dx^2} = 2 + \dfrac{8}{\pi}$ is +ve.

$\therefore$ A is minimum when $\quad x = \dfrac{28}{\pi + 4}$. ...(*ii*)

Hence, the wire should be cut at a distance $4x = \dfrac{112}{\pi + 4}$ m from one end.

Note 1. Length of circle $= 2\pi y$.

Putting the value of y from (*i*),

$$= 28 - 4x \qquad \text{...(iii)}$$

Putting the value of x from (*ii*),

Length of circle $= 28 - \dfrac{112}{\pi + 4} = \dfrac{28\pi}{\pi + 4}$

$\therefore$ The length of two parts (square and circle) are respectively $\dfrac{112}{\pi + 4}$ m and $\dfrac{28\pi}{\pi + 4}$ m.

Note 2. Side of square $= x = \dfrac{28}{\pi + 4}$

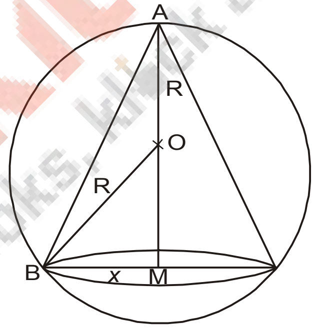
From (*i*), Radius of circle $= y = \dfrac{14 - 2x}{\pi} = \dfrac{14 - 2\left(\dfrac{28}{\pi + 4}\right)}{\pi}$

$$= \frac{14\pi + 56 - 56}{\pi(\pi + 4)} = \frac{14\pi}{\pi(\pi + 4)} = \frac{14}{\pi + 4},$$

Therefore, diameter of circle $= \dfrac{28}{\pi + 4}$

$\therefore$ Side of square = Diameter of circle.

23. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\dfrac{8}{27}$ of the volume of the sphere.



Sol. Let O be the centre and R be the radius of the given sphere.

Let BM $= x$ and AM $= y$ be the radius and height of **any** cone inscribed in the given sphere.

In right angled triangle OMB,

By Pythagoras Theorem,

$$OM^2 + BM^2 = OB^2$$

$\Rightarrow \quad (y - R)^2 + x^2 = R^2 \quad [\because \; OM = AM - OA = y - R]$

$\Rightarrow \quad y^2 + R^2 - 2Ry + x^2 = R^2 \quad \Rightarrow \quad x^2 + y^2 - 2Ry = 0$

$\Rightarrow \quad x^2 = 2Ry - y^2$ ...(*i*)

Let z denote the volume of any cone inscribed in the given sphere.

$\therefore \quad z = \dfrac{1}{3}\pi x^2 y$

Putting the value of x^2 from (*i*),

$$z = \frac{\pi}{3}(2Ry - y^2)y = \frac{\pi}{3}(2Ry^2 - y^3) \quad ...(ii)$$

Now z is a function of y alone.

$\therefore \quad \dfrac{dz}{dy} = \dfrac{\pi}{3}(4Ry - 3y^2)$ and $\dfrac{d^2z}{dy^2} = \dfrac{\pi}{3}(4R - 6y)$

Putting $\dfrac{dz}{dy} = 0$ to find turning points, we have $\dfrac{\pi}{3}(4Ry - 3y^2) = 0$

But $\frac{\pi}{3} \neq 0$. Therefore $4Ry - 3y^2 = 0 \Rightarrow -3y^2 = -4Ry$

Dividing both sides by $-y \neq 0$,

$$3y = 4R \Rightarrow y = \frac{4R}{3}$$

At $\quad y = \frac{4R}{3}, \frac{d^2z}{dy^2} = \frac{\pi}{3}(4R - 6y) = \frac{\pi}{3}(4R - 8R)$

$$= \frac{\pi}{3}(-4R) = -\frac{4R}{3} \quad \text{(Negative)}$$

$\therefore \quad z$ is maximum at $y = \frac{4R}{3}$ ...(*iii*)

Putting $y = \frac{4R}{3}$ from (*iii*) in (*i*), we have

$$x^2 = 2R \cdot \frac{4R}{3} - \left(\frac{4R}{3}\right)^2 = \frac{8R^2}{3} - \frac{16R^2}{9}$$

$$= 8R^2\left(\frac{1}{3} - \frac{2}{9}\right) = 8R^2\left(\frac{3-2}{9}\right) \Rightarrow x^2 = \frac{8R^2}{9}$$

$\therefore \quad$ Maximum volume z of the cone

$$= \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi \cdot \frac{8R^2}{9} \cdot \frac{4R}{3} = \frac{8}{27} \cdot \frac{4\pi}{3} R^3$$

$$= \frac{8}{27}\text{(Volume of the sphere).}$$

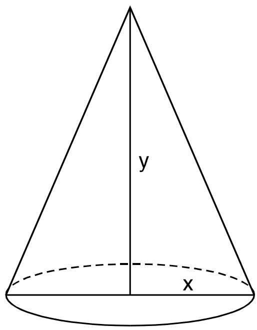
24. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

Sol. Let x be the base radius and y be the height of cone.

Given Volume $\Rightarrow$ Volume of the cone is constant and = V (say)

$\therefore \quad \frac{1}{3}\pi x^2 y = V$ (Given condition)

$\therefore \quad x^2 y = \frac{3V}{\pi} = k \quad$ (say) ...(*i*)



Let S denote the curved surface of the cone

$\therefore \; S = \pi x \sqrt{x^2 + y^2}$ **(formula S = πrl)**

(S is to be minimised here)

Let $\quad z = S^2 = \pi^2 x^2 (x^2 + y^2)$ ...(*ii*)

Putting $x^2 = \frac{k}{y}$ from (*i*) in (*ii*) to get z as a function of single independent variable y.

[Here, we are getting z as a simpler function of y as compared to z as a function of x]

$\therefore \quad z = \pi^2 \dfrac{k}{y}\left(\dfrac{k}{y} + y^2\right) = \pi^2 k \left(\dfrac{k}{y^2} + y\right)$

or $\quad z = \pi^2 k\ (ky^{-2} + y)$

$\therefore \quad \dfrac{dz}{dy} = \pi^2 k\ [-2ky^{-3} + 1]$ and $\dfrac{d^2z}{dy^2} = \pi^2 k\ [6k\ y^{-4}] = \dfrac{6\pi^2 k^2}{y^4}$

For max. or min., put $\dfrac{dz}{dy} = 0$

$\therefore \quad \pi^2 k \left(-\dfrac{2k}{y^3} + 1\right) = 0$ But $\pi^2 k \neq 0$

$\therefore \quad -\dfrac{2k}{y^3} + 1 = 0$ or $\dfrac{2k}{y^3} = 1$

$\therefore \quad y^3 = 2k \quad \therefore \quad y = (2k)^{1/3}$

At $\quad y = (2k)^{1/3}, \dfrac{d^2z}{dy^2} = \dfrac{6\pi^2 k^2}{(2k)^{4/3}}$ which is positive.

$\therefore$ z is least when $y = (2k)^{1/3}$ ...(*iii*)

$\therefore$ From (*i*), $x^2 = \dfrac{k}{y} = \dfrac{k}{(2k)^{1/3}}$ [Using (*iii*)]

or $\quad x^2 = \dfrac{2k}{2(2k)^{1/3}} = \dfrac{(2k)^{2/3}}{2} = \dfrac{y^2}{2}$ [By (*iii*)]

or $\quad y^2 = 2x^2 \quad \therefore \quad y = \sqrt{2}\,x$

$\therefore$ z or S is least when height $= \sqrt{2}$ (radius of base).

25. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

Sol. Let x be the base radius, y the height, l the given slant height and θ, the semi-vertical angle of cone. $(x > 0, y > 0)$

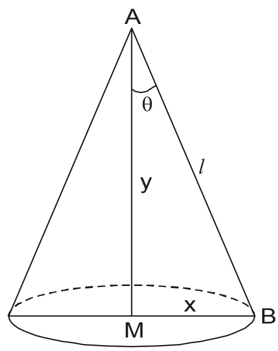
In ΔAMB,

By Pythagoras Theorem, $x^2 + y^2 = l^2$

$\therefore \quad x^2 = l^2 - y^2$ ...(*i*)

Let V denote the volume of cone, then

$$V = \frac{1}{3}\pi x^2 y$$ ...(*ii*) [V is to be maximised here]



Putting $x^2 = l^2 - y^2$ from (*i*) in (*ii*), to express V as a function of single independent variable y.

$$V = \frac{1}{3}\pi(l^2 - y^2)\,y$$

or $$V = \frac{\pi}{3}(l^2 y - y^3)$$

$$\therefore \quad \frac{dV}{dy} = \frac{\pi}{3}(l^2 - 3y^2)$$

and $\dfrac{d^2V}{dy^2} = \dfrac{\pi}{3}(-6y) = -2\pi y$

For max. or min. put $\dfrac{dV}{dy} = 0$

$\therefore \dfrac{\pi}{3}(l^2 - 3y^2) = 0$ But $\dfrac{\pi}{3} \neq 0$

$\therefore \quad l^2 - 3y^2 = 0 \quad$ or $\quad 3y^2 = l^2 \quad$ or $\quad y^2 = \dfrac{l^2}{3}$

$\therefore \quad y = \dfrac{l}{\sqrt{3}} \qquad (\because y > 0)$

At $\quad y = \dfrac{l}{\sqrt{3}}, \dfrac{d^2V}{dy^2} = -2\pi y = -\dfrac{2\pi l}{\sqrt{3}}$ which is negative.

$\therefore$ V is maximum at $y = \dfrac{l}{\sqrt{3}}$.

Putting $y = \dfrac{l}{\sqrt{3}}$ in eqn. (*i*), $\quad x^2 = l^2 - \dfrac{l^2}{3} = \dfrac{2l^2}{3}$

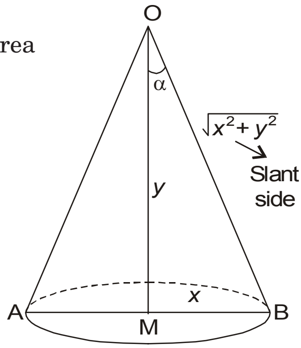
$\therefore \quad x = \sqrt{2}\dfrac{l}{\sqrt{3}}$

In right angled ΔAMB, $\quad \tan\theta = \dfrac{MB}{AM} = \dfrac{x}{y} = \dfrac{\sqrt{2}\dfrac{l}{\sqrt{3}}}{\dfrac{l}{\sqrt{3}}} = \sqrt{2}$

$\therefore$ Semi-vertical angle $\qquad \theta = \tan^{-1}\sqrt{2}$.

26. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\dfrac{1}{3}\right)$.

Sol. Let x be the radius of base of cone and y be its height. (Total) surface area of cone is given.



$\therefore \underset{\substack{\downarrow \\ \text{(Curved Surface area)}}}{\boldsymbol{\pi rl}} + \underset{\substack{\downarrow \\ \text{(Area of base)}}}{\boldsymbol{\pi r^2}} =$ Given Surface area

$\Rightarrow \quad \pi x\sqrt{x^2 + y^2} + \pi x^2 = S$ (say)

Dividing both sides by π,

$x\sqrt{x^2 + y^2} + x^2 = \dfrac{S}{\pi} = k$ (say)

$\Rightarrow \quad x\sqrt{x^2 + y^2} = k - x^2$

Squaring both sides, we have $x^2(x^2 + y^2) = (k - x^2)^2$

or $\quad x^4 + x^2y^2 = k^2 + x^4 - 2kx^2$

$\Rightarrow \quad x^2y^2 + 2kx^2 = k^2 \qquad \Rightarrow \quad x^2(y^2 + 2k) = k^2$

$$\Rightarrow \quad x^2 = \frac{k^2}{y^2 + 2k} \qquad ...(i)$$

Let z denote the volume of the cone.

$$\therefore \quad z = \frac{1}{3}\pi x^2 y$$

Putting the value of x^2 from (i),

$$z = \frac{1}{3}\pi \frac{k^2}{y^2 + 2k} y = \frac{1}{3}\pi k^2 \frac{y}{y^2 + 2k}$$

$$\therefore \quad \frac{dz}{dy} = \frac{1}{3}\pi k^2 \frac{d}{dy} \frac{y}{y^2 + 2k}$$

or $$\frac{dz}{dy} = \frac{1}{3}\pi k^2 \left[\frac{(y^2 + 2k) . 1 - y . 2y}{(y^2 + 2k)^2}\right] \quad \text{(By quotient rule)}$$

$$\therefore \quad \frac{dz}{dy} = \frac{1}{3}\pi k^2 \frac{(2k - y^2)}{(y^2 + 2k)^2} \qquad ...(ii)$$

Putting $\frac{dz}{dy} = 0$ to find turning points, we have

$$\frac{\pi k^2 (2k - y^2)}{3 (y^2 + 2k)^2} = 0 \Rightarrow \pi k^2 (2k - y^2) = 0$$

But $\pi k^2 \neq 0 \quad \therefore \quad 2k - y^2 = 0 \quad \Rightarrow \quad y^2 = 2k$

$\therefore \quad y = \pm \sqrt{2k}$

Rejecting negative sign because height y of cone can't be negative.

$\therefore \quad y = \sqrt{2k}$ is the only turning point.

Let us apply first derivative test.

(because finding $\frac{d^2z}{dx^2}$ looks to be tedious)

Now in R.H.S. of (ii), $\frac{\pi k^2}{3(y^2 + 2k)^2} > 0$ clearly.

When y is slightly $< \sqrt{2k}$; then $y^2 < 2k$

$\Rightarrow \quad 0 < 2k - y^2 \quad \Rightarrow \quad 2k - y^2 > 0,$

therefore from (ii), $\frac{dz}{dy} > 0$ *i.e.*, (positive)

When y is slightly $> \sqrt{2k}$, then $y^2 > 2k \quad \Rightarrow \quad 0 > 2k - y^2$

$\Rightarrow \quad 2k - y^2 < 0$; therefore from (ii) $\frac{dz}{dy} < 0$ *i.e.*, (negative)

$\therefore \quad \frac{dz}{dy}$ changes sign from (+) to (–) as y increases through $\sqrt{2k}$

$\therefore \quad$ Volume z is maximum at $y = \sqrt{2k}$

Putting $y = \sqrt{2k}$ in (*i*), $x^2 = \dfrac{k^2}{2k + 2k} = \dfrac{k^2}{4k} = \dfrac{k}{4}$

$\therefore \qquad x = \sqrt{\dfrac{k}{4}} = \dfrac{\sqrt{k}}{2}$

Let α be the semi-vertical angle of the cone.
In right angled ΔOMB,

$$\sin \alpha = \frac{\text{MB}}{\text{OB}} = \frac{x}{\sqrt{x^2 + y^2}}$$

Putting values of x and y, $\sin \alpha = \dfrac{\dfrac{\sqrt{k}}{2}}{\sqrt{\dfrac{k}{4} + 2k}} = \dfrac{\dfrac{\sqrt{k}}{2}}{\sqrt{\dfrac{9k}{4}}} = \dfrac{\dfrac{\sqrt{k}}{2}}{3\dfrac{\sqrt{k}}{2}} = \dfrac{1}{3}$

$\therefore \qquad \alpha = \sin^{-1} \dfrac{1}{3}$.

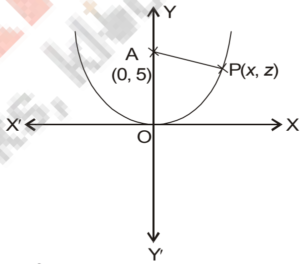
Choose the correct answer in the Exercises 27 to 29.

27. The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is

(A) $(2\sqrt{2}, 4)$ (B) $(2\sqrt{2}, 0)$ (C) (0, 0) (D) (2, 2).

Sol. Equation of the curve (upward parabola here) is

$x^2 = 2y \qquad ...(i)$



The given point is A(0, 5).
Let P(x, y) be any point on curve (*i*).

$\therefore$ Distance z = AP

$= \sqrt{(x-0)^2 + (y-5)^2}$

| Distance formula

Let $\qquad Z = z^2 = x^2 + (y - 5)^2$

Putting $\qquad x^2 = 2y$ from (*i*),

$Z = 2y + (y - 5)^2 = 2y + y^2 + 25 - 10y$

or $\qquad Z = y^2 - 8y + 25$

$\therefore \qquad \dfrac{dZ}{dy} = 2y - 8$ and $\dfrac{d^2Z}{dy^2} = 2$

Putting $\dfrac{dZ}{dy} = 0$ to get turning point(s), we have

$\dfrac{dZ}{dy} = 0$ *i.e.*, $2y - 8 = 0 \Rightarrow 2y = 8 \Rightarrow y = 4$

At $\qquad y = 4$, $\dfrac{d^2Z}{dy^2} = 2$ is (+ ve)

$\therefore$ Z$(= z^2)$ is minimum and hence z is minimum at $y = 4$.

Putting $y = 4$ in (*i*), $x^2 = 8 \quad \therefore \quad x = \pm\sqrt{8} = \pm 2\sqrt{2}$.

$\therefore \quad (2\sqrt{2}, 4)$ and $(-2\sqrt{2}, 4)$ are two points on curve (*i*) which are nearest to the given point (0, 5).

$\therefore$ Option (A) is correct answer.

28. For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is

(A) 0 **(B) 1** **(C) 3** **(D) $\frac{1}{3}$.**

Sol. Given: Let $f(x) = \frac{1-x+x^2}{1+x+x^2}$ ...(*i*)

$$\therefore \quad f'(x) = \frac{(1+x+x^2)\frac{d}{dx}(1-x+x^2) - (1-x+x^2)\frac{d}{dx}(1+x+x^2)}{(1+x+x^2)^2}$$

$$= \frac{(1+x+x^2)(-1+2x) - (1-x+x^2)(1+2x)}{(1+x+x^2)^2}$$

or $$f'(x) = \frac{-1+2x-x+2x^2-x^2+2x^3-1-2x+x+2x^2-x^2-2x^3}{(1+x+x^2)^2}$$

or $$f'(x) = \frac{-2+2x^2}{(1+x+x^2)^2} = \frac{-2(1-x^2)}{(1+x+x^2)^2}$$

Let us put $f'(x) = 0$ to get turning points.

Therefore $\frac{-2(1-x^2)}{(1+x+x^2)^2} = 0 \Rightarrow -2(1-x^2) = 0$

But $-2 \neq 0$. Therefore, $1 - x^2 = 0$ or $-x^2 = -1$

$\therefore \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$

$\therefore \quad x = -1$ and $x = 1$ are two turning points.

Let us find values of $f(x)$ at these two turning points only because no closed interval is given to be domain of $f(x)$.

Putting $x = -1$ in (*i*), $f(-1) = \frac{1+1+1}{1-1+1} = 3$

Putting $x = 1$ in (*i*), $f(1) = \frac{1-1+1}{1+1+1} = \frac{1}{3}$

Therefore, minimum value of $f(x)$ is $\frac{1}{3}$.

$\therefore$ Option (D) is the correct answer.

Note. Maximum value of $f(x)$ for the above question is 3.

29. The maximum value of $[x(x-1)+1]^{1/3}$, $0 \le x \le 1$ is

(A) $\left(\frac{1}{3}\right)^{1/3}$ **(B) $\frac{1}{2}$** **(C) 1** **(D) $\frac{1}{3}$.**

Sol. Let $f(x) = (x(x-1)+1)^{1/3} = (x^2 - x + 1)^{1/3}$, $0 \le x \le 1$ ...(*i*)

$$\therefore \quad f'(x) = \frac{1}{3}(x^2 - x + 1)^{-2/3}\frac{d}{dx}(x^2 - x + 1)$$

or $$f'(x) = \frac{(2x-1)}{3(x^2-x+1)^{2/3}}$$

Putting $f'(x) = 0$ to get turning points, we have

$$\frac{2x-1}{3(x^2-x+1)^{2/3}} = 0$$

Cross-multiplying $2x - 1 = 0 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$

This turning point $x = \frac{1}{2}$ belongs to the given closed interval $0 \le x \le 1$ *i.e.,* $[0, 1]$.

Now let us find values of $f(x)$ at the turning point $x = \frac{1}{2}$ and end points $x = 0$ and $x = 1$ of given closed interval $[0, 1]$.

Putting $x = \frac{1}{2}$ in (*i*),

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{4} - \frac{1}{2} + 1\right)^{1/3} = \left(\frac{1-2+4}{4}\right)^{1/3} = \left(\frac{3}{4}\right)^{1/3} < 1.$$

Putting $x = 0$ in (*i*), $f(0) = (1)^{1/3} = 1$

Putting $x = 1$ in (*i*), $f(1) = (1 - 1 + 1)^{1/3} = (1)^{1/3} = 1$

$\therefore$ Maximum value of $f(x)$ is 1.

$\therefore$ Option (C) is the correct answer.

Note. Minimum value of $f(x)$ for the above question is $\left(\frac{3}{4}\right)^{1/3}$.

MISCELLANEOUS EXERCISE

1. Using differentials, find the approximate value of each of the following:

(*a*) $\left(\frac{17}{81}\right)^{1/4}$ **(*b*)** $(33)^{-1/5}$.

Sol. (*a*) To find approximate value of $\left(\frac{17}{81}\right)^{1/4}$.

Let $y = x^{1/4}$ ...(*i*)

$\therefore \frac{dy}{dx} = \frac{1}{4}x^{1/4-1} = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}}$

$\therefore dy = \frac{dx}{4(x^{1/4})^3} \sim \frac{\Delta x}{4(x^{1/4})^3}$ ...(*ii*)

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (*i*),

$y + \Delta y = (x + \Delta x)^{1/4} = \left(\frac{17}{81}\right)^{1/4} = \left(\frac{16}{81}+\frac{1}{81}\right)^{1/4}$ ...(*iii*)

Comparing $\left(\frac{16}{81}+\frac{1}{81}\right)^{1/4}$ with $(x + \Delta x)^{1/4}$ we have

$x = \frac{16}{81}$ and $\Delta x = \frac{1}{81}$ ...(*iv*)

$$\therefore \quad x^{1/4} = \left(\frac{16}{81}\right)^{1/4} = \left(\left(\frac{2}{3}\right)^4\right)^{1/4} = \frac{2}{3} \quad ...(v)$$

From (*iii*), $\left(\frac{17}{81}\right)^{1/4} = y + \Delta y \sim y + dy \sim x^{1/4} + \frac{\Delta x}{4(x^{1/4})^3}$

[From (*i*) and (*ii*)]

Now putting values from (*iv*) and (*v*), $\left(\frac{17}{81}\right)^{1/4} \sim \frac{2}{3} + \frac{\frac{1}{81}}{4\left(\frac{2}{3}\right)^3}$

$$\sim \frac{2}{3} + \frac{\frac{1}{81}}{4 \times \frac{8}{27}} = \frac{2}{3} + \frac{1}{81} \times \frac{27}{32}$$

$$\sim \frac{2}{3} + \frac{1}{96} = \frac{64+1}{96} = \frac{65}{96} = 0.677$$

(*b*) To find approximate value of $(33)^{-\frac{1}{5}}$

Let $\quad y = x^{-\frac{1}{5}} \quad ...(i)$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{5}x^{-\frac{1}{5}-1} = -\frac{1}{5}x^{-\frac{6}{5}} = -\frac{1}{5x^{\frac{6}{5}}}$$

$$\therefore \quad dy = \frac{-dx}{5x^{\frac{6}{5}}} \sim \frac{-\Delta x}{5(x^{\frac{1}{5}})^6} \quad ...(ii)$$

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in (*i*),

$$y + \Delta y = (x + \Delta x)^{-\frac{1}{5}} = (33)^{-\frac{1}{5}} = (32+1)^{-\frac{1}{5}} \quad ...(iii)$$

Comparing $x = 32$ and $\Delta x = 1$ $\quad ...(iv)$

$$\therefore \quad x^{-\frac{1}{5}} = (32)^{-\frac{1}{5}} = (2^5)^{-\frac{1}{5}} = 2^{-1} = \frac{1}{2} \quad ...(v)$$

From (*iii*), $(33)^{-\frac{1}{5}} = y + \Delta y \sim y + dy$

$$\sim x^{-\frac{1}{5}} - \frac{\Delta x}{5(x^{\frac{1}{5}})^6}$$ [From (*i*) and (*ii*)]

Now putting values from (*iv*) and (*v*) $(33)^{-\frac{1}{5}} \sim \frac{1}{2} - \frac{1}{5(2)^6}$

$$\sim \frac{1}{2} - \frac{1}{5(64)} = \frac{1}{2} - \frac{1}{320} = \frac{160-1}{320} = \frac{159}{320}$$

$\sim 0.4968 \sim 0.497$

2. Show that the function given by $f(x) = \dfrac{\log x}{x}$ has maximum at $x = e$.

Sol. Here $f(x) = \dfrac{\log x}{x}$, $x > 0$ ...(*i*)

$$\therefore\ f'(x) = \frac{x \cdot \frac{1}{x} - \log x \cdot 1}{x^2} = \frac{1 - \log x}{x^2} \quad ...(ii)$$

$$\text{and } f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \log x) \cdot 2x}{x^4} = \frac{-x - 2x + 2x \log x}{x^4}$$

$$= \frac{2x \log x - 3x}{x^4} = \frac{x(2\log x - 3)}{x^4} = \frac{2\log x - 3}{x^3} \quad ...(iii)$$

For local max. or local min., put $f'(x) = 0$

$\Rightarrow \dfrac{1 - \log x}{x^2} = 0$

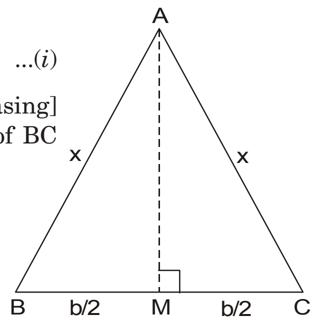
$\Rightarrow 1 - \log x = 0 \Rightarrow \log x = 1 = \log e \Rightarrow x = e$

Now from (*iii*), $f''(e) = \dfrac{2\log e - 3}{e^3} = \dfrac{2-3}{e^3} = -\dfrac{1}{e^3} < 0.$

$\Rightarrow x = e$ is a point of local maxima and hence $f(x)$ has a maximum (value) at $x = e$.

3. The two equal sides of an isosceles triangle with fixed base *b* are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

Sol. Let BC = b be the fixed base and AB = AC = x be the two equal sides of the isosceles triangle ABC.



Given: $\dfrac{dx}{dt} = -3$ cm/s ...(*i*)

[Negative sign because of decreasing]

Draw AM ⊥ BC, then M is mid-point of BC ($\because$ AB = AC (given))

$\therefore$ BM = CM = $\dfrac{1}{2} b$

Area of ΔABC is $(\Delta) = \dfrac{1}{2}$ BC × AM $\quad\Big|\ \dfrac{1}{2}$ Base × Height

$$= \frac{b}{2}\sqrt{AC^2 - MC^2} = \frac{b}{2}\sqrt{x^2 - \frac{b^2}{4}}$$

$$= \frac{b}{2}\sqrt{\frac{4x^2 - b^2}{4}} = \frac{b}{4}\sqrt{4x^2 - b^2}$$

$$\therefore \qquad \frac{d\Delta}{dt} = \frac{d}{dt}\left(\frac{b}{4}\sqrt{4x^2 - b^2}\right)$$

$$= \frac{b}{4} \times \frac{d}{dx}\left(\sqrt{4x^2 - b^2}\right) \times \frac{dx}{dt} \text{ [By Chain Rule]}$$

$$= \frac{b}{4} \times \frac{8x}{2\sqrt{4x^2 - b^2}} \times (-3) \quad \text{[Using } (i)\text{]}$$

$$\left[\because \frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}\right]$$

or $\qquad \dfrac{d\Delta}{dt} = -\dfrac{3bx}{\sqrt{4x^2 - b^2}}$ cm^2/s

When the two equal sides are equal to the base (given) *i.e.*, when $x = b$,

We have on putting $x = b$, $\dfrac{d\Delta}{dt} = -\dfrac{3b \times b}{\sqrt{4b^2 - b^2}} = -\dfrac{3b^2}{\sqrt{3}\,b}$

$$= -\sqrt{3}\,b \text{ cm}^2\text{/s}$$

Hence, the area is decreasing $\left(\because \dfrac{d\Delta}{dt} \text{ is negative}\right)$ at the rate of $\sqrt{3}\,b$ cm^2/s.

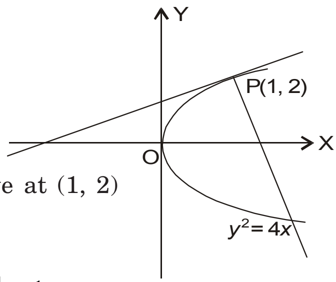
4. Find the equation of the normal to the curve $y^2 = 4x$ at the point (1, 2).

Sol. Equation of the curve (parabola) is

$$y^2 = 4x \qquad \text{...}(i)$$

Differentiating both sides of (i) w.r.t. x, $2y\dfrac{dy}{dx} = 4$

$$\therefore \qquad \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$



$\therefore$ Slope of the tangent to the curve at the point P(1, 2) to curve (i) is $(x = 1, y = 2)$ is $\dfrac{2}{2} = 1$.

$\therefore$ Slope of the normal to the curve at (1, 2) is $\dfrac{-1}{m}$

$$= \frac{-1}{1} = -1 \qquad \left|\ \frac{-1}{m}\right.$$

$\therefore$ Equation of the normal to the curve (i) at (1, 2) is

$$y - 2 = -1\,(x - 1) \quad \text{or} \quad y - 2 = -x + 1$$

or $\qquad x + y = 3.$

5. Show that the normal at any point θ to the curve

$$\boldsymbol{x = a\cos\theta + a\theta\sin\theta,\ y = a\sin\theta - a\theta\cos\theta}$$

is at a constant distance from the origin.

Sol. The parametric equations of the curve are

$$x = a \cos \theta + a \theta \sin \theta,\ y = a \sin \theta - a \theta \cos \theta$$

or $\quad x = a (\cos \theta + \theta \sin \theta),\ y = a (\sin \theta - \theta \cos \theta)$

$\therefore \quad \dfrac{dx}{d\theta} = a [- \sin \theta + \theta \cos \theta + \sin \theta] = a \theta \cos \theta$

and $\quad \dfrac{dy}{d\theta} = a [\cos \theta - (- \theta \sin \theta + \cos \theta)]$

$$= a [\cos \theta + \theta \sin \theta - \cos \theta] = a \theta \sin \theta$$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{a\theta \sin\theta}{a\theta\cos\theta} = \tan \theta$

= slope of tangent at any point (x, y) *i.e.,* point θ.

$\Rightarrow$ Slope of normal at any point $\theta = -\dfrac{1}{\tan\theta} = -\cot\theta = -\dfrac{\cos\theta}{\sin\theta}$

$\therefore$ Equation of normal at any point θ *i.e.,* at (x, y)

$= (a (\cos \theta + \theta \sin \theta),\ a (\sin \theta - \theta \cos \theta))$ is

$$y - a (\sin \theta - \theta \cos \theta) = -\frac{\cos\theta}{\sin\theta} [x - a (\cos \theta + \theta \sin \theta)]$$

or $\quad y \sin \theta - a \sin^2 \theta + a \theta \cos \theta \sin \theta$

$$= - x \cos \theta + a \cos^2 \theta + a \theta \sin \theta \cos \theta$$

or $\quad x \cos \theta + y \sin \theta = a (\sin^2 \theta + \cos^2 \theta)$

or $\quad x \cos \theta + y \sin \theta = a$ or $x \cos \theta + y \sin \theta - a = 0$

$\therefore$ Distance of normal from origin (0, 0)

$$= \frac{|0 + 0 - a|}{\sqrt{\cos^2\theta + \sin^2\theta}} \qquad \left| \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \right.$$

$= a$ which is a constant.

6. Find the intervals in which the function *f* given by

$$\boldsymbol{f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x}}$$ **is (*i*) increasing (*ii*) decreasing.**

Sol. Given: $f(x) = \dfrac{4 \sin x - 2x - x \cos x}{2 + \cos x}$

$$= \frac{4 \sin x - x (2 + \cos x)}{2 + \cos x} = \frac{4 \sin x}{2 + \cos x} - \frac{x(2 + \cos x)}{2 + \cos x}$$

$\Rightarrow \quad f(x) = \dfrac{4 \sin x}{2 + \cos x} - x$

$\therefore \quad f'(x) = \dfrac{(2 + \cos x)\dfrac{d}{dx}(4 \sin x) - 4 \sin x \dfrac{d}{dx}(2 + \cos x)}{(2 + \cos x)^2} - 1$

or $\quad f'(x) = \dfrac{(2 + \cos x)(4 \cos x) - 4 \sin x (- \sin x)}{(2 + \cos x)^2} - 1$

$$= \frac{8 \cos x + 4 \cos^2 x + 4 \sin^2 x}{(2 + \cos x)^2} - 1 = \frac{8 \cos x + 4}{(2 + \cos x)^2} - 1$$

$$[\because \quad 4\cos^2 x + 4\sin^2 x = 4(\cos^2 x + \sin^2 x) = 4]$$

$$\Rightarrow \quad f'(x) = \frac{8\cos x + 4 - (2 + \cos x)^2}{(2 + \cos x)^2}$$

$$= \frac{8\cos x + 4 - 4 - \cos^2 x - 4\cos x}{(2 + \cos x)^2}$$

$$\Rightarrow \quad f'(x) = \frac{4\cos x - \cos^2 x}{(2 + \cos x)^2} = \cos x \, \frac{(4 - \cos x)}{(2 + \cos x)^2} \quad ...(i)$$

Now $4 - \cos x > 0$ for all real x because we know that $-1 \le \cos x \le 1$ always. Also $(2 + \cos x)^2$ being a square of a real number is > 0.

$\therefore$ (*i*) $f(x)$ is increasing if $f'(x) \ge 0$

i.e., if $\cos x \ge 0$ (From (*i*))

i.e., if x lies in Ist and 4th quadrants.

i.e., $f(x)$ is increasing for $0 \le x \le \frac{\pi}{2}$ and $\frac{3\pi}{2} \le x \le 2\pi$.

(*ii*) $f(x)$ is decreasing if $f'(x) \le 0$

i.e., if $\cos x \le 0$ (From (*i*))

i.e., if x lies in IInd and III quadrants

$\therefore$ $f(x)$ is decreasing for $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.

7. Find the intervals in which the function *f* given by

$$f(x) = x^3 + \frac{1}{x^3}, \; x \ne 0 \text{ is}$$

(*i*) increasing **(*ii*) decreasing.**

Sol. (*i*) **Given:** $f(x) = x^3 + \frac{1}{x^3}$, $x \ne 0$

$$\Rightarrow \quad f(x) = x^3 + x^{-3}$$

$$\therefore \quad f'(x) = 3x^2 - 3x^{-4} = 3\left(x^2 - \frac{1}{x^4}\right)$$

Step I. Forming factors $= 3\left(\frac{x^6 - 1}{x^4}\right) = \frac{3}{x^4}[(x^2)^3 - 1^3]$

$$f'(x) = \frac{3}{x^4}(x^2 - 1)\,((x^2)^2 + x^2 \,.\, 1 + 1^2)$$

$$[\because \quad a^3 - b^3 = (a - b)\,(a^2 + ab + b^2)]$$

or $$f'(x) = \frac{3}{x^4}\,(x^4 + x^2 + 1)\,(x + 1)(x - 1) \quad ...(i)$$

(**Note.** For forming factors, we could also use $a^2 - b^2 = (a - b(a + b)$ but $a^3 - b^3$ is much better here)

Step II. Let us find turning points by putting $f'(x) = 0$

$\therefore$ From (*i*), $\dfrac{3(x^4 + x^2 + 1)(x + 1)(x - 1)}{x^4} = 0$

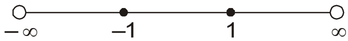
Cross-multiplying, $3(x^4 + x^2 + 1)(x + 1)(x - 1) = 0$

But $3(x^4 + x^2 + 1)$ is positive for all real x

(as both terms of x have even powers and all terms are positive) and hence $\neq 0$.

$\therefore$ Either $x + 1 = 0$ or $x - 1 = 0$

i.e., $x = -1$ or $x = 1$.



These two turning points $x = -1$ and $x = 1$ divide the whole real line into three sub-intervals $(-\infty, -1]$, $[-1, 1]$ and $[1, \infty)$.

In R.H.S. of (1), $\dfrac{3(x^4 + x^2 + 1)}{x^4}$ is positive for all real $x \neq 0$.

Step III.

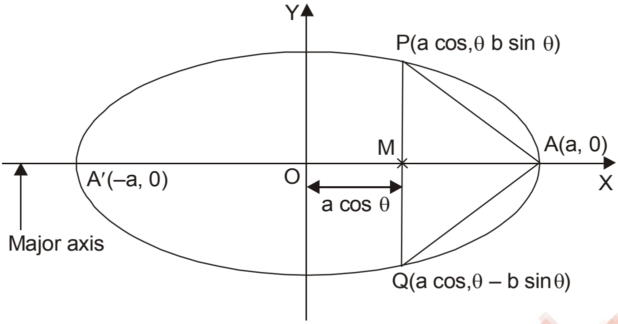
Values of x	**sign of $f'(x)$ $= \dfrac{3(x^4 + x^2 + 1)}{x^4}$ $(x + 1)(x - 1)$**	**Nature of $f(x)$**
$(-\infty, -1]$ *i.e.,* $x \leq -1$	For example, at $x = -2$, $f'(x = (+)\ (-)\ (-) = (+)$ or 0 at $x = -1$	Increasing $\uparrow$
$[-1, 1]$ *i.e.,* $-1 \leq x \leq 1$	For example, at $x = \dfrac{1}{2}$, $f'(x = (+)\ (+)\ (-)$ $= (-)$ or 0 both at $x = -1, 1$	Decreasing $\downarrow$
$[1, \infty)$ *i.e.,* $x \geq 1$	For example, at $x = 2$, $f'(x = (+)\ (+)\ (+)$ $= (+)$ or 0 at $x = 1$	Increasing $\uparrow$

$\therefore$ $f(x)$ is (*i*) an increasing function for $x \leq -1$ and for $x \geq 1$ and (*ii*) decreasing function for $-1 \leq x \leq 1$.

8. Find the maximum area of an isosceles triangle inscribed in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.

Sol. Equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ ...(*i*)

We also know that the extremities of the major axis of the ellipse are A(a, 0) and A′($-a$, 0) ($\Rightarrow$ OA = a, OA′ = a)



Comparing eqn. (i) with

$$\cos^2\theta + \sin^2\theta = 1,$$

we have $\dfrac{x}{a} = \cos\theta$ and $\dfrac{y}{b} = \sin\theta$

$\therefore$ $x = a\cos\theta$ and $y = b\sin\theta$

$\therefore$ Any point on the ellipse is P($a\cos\theta$, $b\sin\theta$)

Draw PM $\perp$ on x-axis and produce it to meet the ellipse in the point Q.

$\therefore$ OM = $a\cos\theta$ and PM = $b\sin\theta$ (By def. of coordinates of a point)

PM = QM [$\because$ The ellipse (i) is symmetrical about major axis (here x-axis)]

$\therefore$ M is the mid-point of PQ.

$\therefore$ ΔAPQ is isosceles.

Let z denote the area of isosceles triangle ΔAPQ inscribed in the ellipse with one vertex A coinciding with an extremity of major axis.

$$\therefore \quad z = \frac{1}{2}\text{ Base} \times \text{Height} = \frac{1}{2}\text{ PQ . AM}$$

$$= \frac{1}{2}\text{ . 2PM . AM} = \text{PM (OA} - \text{OM)}$$

Putting values of OA, OM and PM

$$= b\sin\theta\,(a - a\cos\theta) = ba\sin\theta\,(1 - \cos\theta)$$

$$= ab\,(\sin\theta - \sin\theta\cos\theta).$$

or $z = \dfrac{ab}{2}(2\sin\theta - 2\sin\theta\cos\theta) = \dfrac{ab}{2}(2\sin\theta - \sin 2\theta)$ ...(i)

Differentiating (i) w.r.t. θ,

$$\frac{dz}{d\theta} = \frac{ab}{2}(2\cos\theta - 2\cos 2\theta) = ab\,(\cos\theta - \cos 2\theta)$$

Again differentiating w.r.t. θ,

$$\frac{d^2z}{d\theta^2} = ab\,(-\sin\theta + 2\sin 2\theta)$$

Putting $\frac{dz}{d\theta} = 0$, we have $ab (\cos \theta - \cos 2\theta) = 0$

But $ab \neq 0$ $\quad \therefore \cos \theta - \cos 2\theta = 0$

or $\quad \cos \theta = \cos 2\theta = \cos (360° - 2\theta)$

$\therefore \quad$ Either $\theta = 2\theta$ or $\theta = 360° - 2\theta$

i.e., $\quad \theta = 0$ or $3\theta = 360°$ $\quad \therefore \theta = 120°$

$\theta = 0$ is impossible because otherwise the point P $(a \cos \theta, b \sin \theta)$ $= (a \cos 0, b \sin 0) = (a, 0)$ will coincide with the point A.

$\therefore \quad \theta = 120°$

At $\quad \theta = 120°, \quad \frac{d^2z}{d\theta^2} = ab\,(-\sin 120° + 2 \sin 240°)$

$$= ab\left[-\frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{2}\right] = ab\left(-\frac{3\sqrt{3}}{2}\right) = \text{Negative}$$

$\therefore \quad z$ is maximum at $\theta = 120°$

Putting $\theta = 120°$ in (*i*),

Maximum Area $= \frac{ab}{2}\,[2 \sin 120° - \sin 240°]$

$$= \frac{ab}{2}\left[\frac{2\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right] = \frac{ab}{2}\left(\frac{3\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{4}\,ab.$$

9. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs ₹ 70 per sq. metre for the base and ₹ 45 per square metre for sides. What is the cost of least expensive tank?

Sol. Given: A tank with rectangular base and rectangular sides, open at the top.

(The reader is suggested to visualize this tank as a room with four rectangular walls, floor but no ceiling).

Given: Depth of tank = 2 m

Let x m be the length and y m be the breadth of the base of tank.

Volume of tank $(= lbh) = x.y.2 = 8$ m^3 (given)

$$\therefore \quad y = \frac{8}{2x} = \frac{4}{x} \qquad ...(i)$$

Now cost of building the base of the tank at the given rate of ₹ 70 per square metre is ₹ $70xy$ ...(*ii*)

Again cost of building the four sides (walls) of the tank at the rate of ₹ 45 per square metre.

$$= 45(x.2 + x.2 + y.2 + y.2) = 45(4x + 4y)$$

$$= ₹\ (180x + 180y) \qquad ...(iii)$$

Let z denote the total cost of building the tank.

Adding (*ii*) and (*iii*), $z = 70xy + 180x + 180y$

Putting $y = \dfrac{4}{x}$ from (i), $z = 70x.\dfrac{4}{x} + 180x + 180.\dfrac{4}{x}$

or $\qquad z = 280 + 180x + \dfrac{720}{x}$ $\qquad$ ...(iv)

$\therefore \qquad \dfrac{dz}{dx} = 0 + 180 - \dfrac{720}{x^2}$ and $\dfrac{d^2z}{dx^2} = \dfrac{1440}{x^3}$

$\left[\because \dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{d}{dx}x^{-1} = (-1)x^{-2} = \dfrac{-1}{x^2} \text{ and } \dfrac{d}{dx}\left(\dfrac{1}{x^2}\right) = \dfrac{d}{dx}x^{-2} = -2x^{-3} = \dfrac{-2}{x^3}\right]$

Putting $\dfrac{dz}{dx} = 0$ to find turning points, we have

$$180 - \frac{720}{x^2} = 0 \;\Rightarrow\; 180 = \frac{720}{x^2} \;\Rightarrow\; 180x^2 = 720$$

$\Rightarrow \quad x^2 = \dfrac{720}{180} = 4 \quad \Rightarrow \quad x = 2$ ($\because$ x being length can't be negative)

At $x = 2$, $\dfrac{d^2z}{dx^2} = \dfrac{1440}{x^3} = \dfrac{1440}{8} = 180$ (+ ve)

$\therefore \quad z$ is minimum at $x = 2$

Putting $x = 2$ in (iv), minimum cost

$$z = 280 + 180(2) + \frac{720}{2} = 280 + 360 + 360$$
$$= 280 + 720 = ₹\ 1000.$$

10. The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

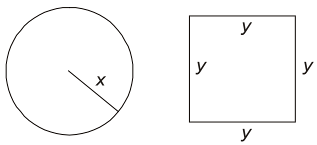
Sol. Let x be the radius of the circle and y be the side of the square.

Given: Perimeter (circumference) of circle + perimeter of square = k

$\Rightarrow \quad 2\pi x + 4y = k$

$\Rightarrow \quad 4y = k - 2\pi x$

$\Rightarrow \quad y = \dfrac{k - 2\pi x}{4}$ $\qquad$ ...(i)



Let z denote the sum of areas of circle and square.

$\therefore \qquad z = \pi x^2 + y^2$ $\qquad$ [Area of square = (side)^2]

Putting the value of y from (i),

$$z = \pi x^2 + \frac{(k - 2\pi x)^2}{16} = \frac{16\pi x^2 + k^2 + 4\pi^2 x^2 - 4k\pi x}{16}$$

or $\qquad z = \dfrac{1}{16}[(16\pi + 4\pi^2)x^2 - 4k\pi x + k^2]$

$\therefore \qquad \dfrac{dz}{dx} = \dfrac{1}{16}[(16\pi + 4\pi^2)2x - 4k\pi]$ and $\dfrac{d^2z}{dx^2} = \dfrac{1}{16}(16\pi + 4\pi^2)2$

Putting $\frac{dz}{dx}$ = 0 to find turning points, we have

$$\frac{1}{16}[(16\pi + 4\pi^2)2x - 4k\pi] = 0$$

$\Rightarrow \quad (16\pi + 4\pi^2)2x - 4k\pi = 0 \times 16 = 0$

$\Rightarrow \quad 4\pi(4 + \pi)2x = 4k\pi$

$\Rightarrow \quad x = \frac{4k\pi}{4\pi(4+\pi)2} = \frac{k}{2(4+\pi)}$

At $x = \frac{k}{2(4+\pi)}$, $\frac{d^2z}{dx^2} = \frac{1}{16}(16\pi + 4\pi^2)2$ is + ve.

$\therefore$ z is minimum when $x = \frac{k}{2(4+\pi)}$ ...(ii)

Putting this value of x in (i),

$$y = \frac{1}{4}\left[k - 2\pi\frac{k}{2(4+\pi)}\right] = \frac{1}{4}\left[k - \frac{\pi k}{4+\pi}\right]$$

$$= \frac{1}{4}\left[\frac{k(4+\pi) - \pi k}{4+\pi}\right] = \frac{4k + \pi k - \pi k}{4(4+\pi)}$$

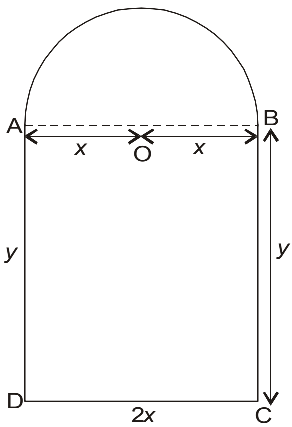
or $\quad y = \frac{4k}{4(4+\pi)} = \frac{k}{4+\pi} = 2\frac{k}{2(4+\pi)}$

$\Rightarrow \quad y = 2x$ (By (ii))

$\therefore$ z (sum of areas) is minimum (least) when side (y) of the square is double the radius (x) of the circle.

11. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Sol. Let x metres be the radius of the semi-circular opening of the window. Therefore one side of rectangle part of window is $2x$. Let y metres be the other side of the rectangle.



$\therefore$ Perimeter of window = Semi-circular arc AB + Length (AD + DC + BC) = 10 m (given)

$\Rightarrow \quad \frac{1}{2}(2\pi x) + y + 2x + y = 10$

$\Rightarrow \quad \pi x + 2x + 2y = 10$

$\Rightarrow \quad 2y = 10 - \pi x - 2x$

$\Rightarrow \quad y = \frac{10 - (\pi + 2)x}{2}$ ...(i)

(It may be noted that length (side) AB can't be a part of the window because if it so, then light after passing through the semi-circle will not enter the rectangle part)

Let z sq. m be the area of the window.
For maximum light to be admitted through the window, area z of window should be maximum.

z = Area of window = Area of semi-circle + Area of rectangle

$$= \frac{1}{2}(\pi x^2) + (2x)y$$

Putting the value of y from (i),

$$z = \frac{1}{2}\pi x^2 + 2x\left[\frac{10-(\pi+2)x}{2}\right] = \frac{1}{2}[\pi x^2 + 20x - 2(\pi + 2)x^2]$$

or $$z = \frac{1}{2}[\pi x^2 + 20x - 2\pi x^2 - 4x^2] = \frac{1}{2}[-\pi x^2 - 4x^2 + 20x]$$

Now z is a function of x alone.

$$\therefore \quad \frac{dz}{dx} = \frac{1}{2}[-2\pi x - 8x + 20]$$

and $$\frac{d^2z}{dx^2} = \frac{1}{2}(-2\pi - 8) = \frac{-2}{2}(\pi + 4) = -(\pi + 4)$$

Putting $\frac{dz}{dx} = 0$ to find turning points, we have

$$\frac{-2\pi x - 8x + 20}{2} = 0 \Rightarrow -2\pi x - 8x + 20 = 0$$

$$\Rightarrow \quad -2x(\pi + 4) = -20 \Rightarrow x = \frac{20}{2(\pi+4)} = \frac{10}{\pi+4}$$

At $x = \frac{10}{\pi+4}$, $\frac{d^2z}{dx^2} = -(\pi + 4)$ is negative.

$\therefore$ z is maximum at $x = \frac{10}{\pi+4}$.

Putting $x = \frac{10}{\pi+4}$ in (i), $y = \dfrac{10-(\pi+2)\dfrac{10}{\pi+4}}{2}$

$$= \frac{10(\pi+4)-10(\pi+2)}{2(\pi+4)} = \frac{10\pi+40-10\pi-20}{2(\pi+4)}$$

$$= \frac{20}{2(\pi+4)} = \frac{10}{\pi+4} \text{ m}$$

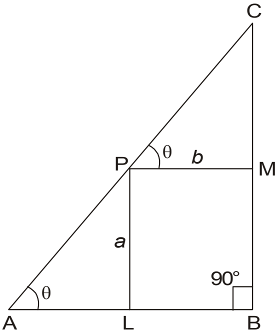
$\therefore$ For maximum light to be admitted through window, dimensions of the window are:

Length of rectangle $= 2x = \frac{20}{\pi+4}$ m

Width of rectangle $= y = \frac{10}{\pi+4}$ m

Radius of semi-circle $= x = \frac{10}{\pi+4}$ m.

12. A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle.
Show that the minimum length of the hypotenuse is $(a^{2/3} + b^{2/3})^{3/2}$.



Sol. Let P be a point on the hypotenuse AC of a right triangle ABC such that PL ($\perp$ AB) = a and PM ($\perp$ BC) = b.
Let $\angle BAC = \angle MPC = \theta$,

then in right-angled ΔALP, $\dfrac{AP}{PL} = \operatorname{cosec}\theta$

$\therefore$ AP = PL cosec θ = a cosec θ

and in right-angled ΔPMC, $\dfrac{PC}{PM} = \sec\theta$

$\therefore$ PC = PM sec θ = b sec 0.
Let AC = z, then

$$z = AP + PC = a \operatorname{cosec}\theta + b \sec\theta,\ 0 < \theta < \frac{\pi}{2} \qquad ...(i)$$

($\because$ θ is an angle of right-angled triangle)

$$\therefore \quad \frac{dz}{d\theta} = -a \operatorname{cosec}\theta \cot\theta + b \sec\theta \tan\theta$$

For maxima or minima, put $\dfrac{dz}{d\theta} = 0$

$$\Rightarrow \quad -a \operatorname{cosec}\theta \cot\theta + b \sec\theta \tan\theta = 0$$

$$\Rightarrow \quad \frac{b \sin\theta}{\cos^2\theta} = \frac{a\cos\theta}{\sin^2\theta}$$

$$\Rightarrow \quad b \sin^3\theta = a\cos^3\theta \quad \Rightarrow \frac{\sin^3\theta}{\cos^3\theta} = \frac{a}{b}$$

$$\Rightarrow \quad \tan^3\theta = \frac{a}{b} \quad \therefore \tan\theta = \left(\frac{a}{b}\right)^{1/3} \qquad ...(ii)$$

Now, $$\frac{d^2z}{d\theta^2} = -a\,[\operatorname{cosec}\theta(-\operatorname{cosec}^2\theta) + \cot\theta(-\operatorname{cosec}\theta\cot\theta)]$$
$$+ b\,[\sec\theta\sec^2\theta + \tan\theta\sec\theta\tan\theta]$$
$= a(\operatorname{cosec}^3\theta + \operatorname{cosec}\theta\cot^2\theta) + b(\sec^3\theta + \sec\theta\tan^2\theta)$
Since $a > 0, b > 0$ and all t-ratios of θ are positive ($\because$ $0 < \theta < \pi/2$)

$$\therefore \quad \frac{d^2z}{d\theta^2} > 0$$

$$\Rightarrow \quad z \text{ is least when } \tan\theta = \left(\frac{a}{b}\right)^{1/3}$$

$$\Rightarrow \quad \sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{a}{b}\right)^{2/3} = \frac{b^{2/3} + a^{2/3}}{b^{2/3}}$$

$$\Rightarrow \qquad \sec\theta = \frac{(a^{2/3}+b^{2/3})^{1/2}}{b^{1/3}}$$

Also $\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \left(\frac{b}{a}\right)^{2/3}$ [By (*ii*)] $= \frac{a^{2/3}+b^{2/3}}{a^{2/3}}$

$$\Rightarrow \qquad \operatorname{cosec}\theta = \frac{(a^{2/3}+b^{2/3})^{1/2}}{a^{1/3}}$$

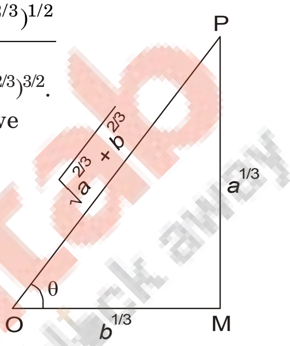
Putting these values of sec θ and cosec θ in (*i*),

$\therefore$ Minimum length of hypotenuse z

$= a \operatorname{cosec}\theta + b \sec\theta$

$= a \cdot \frac{(a^{2/3}+b^{2/3})^{1/2}}{a^{1/3}} + b \cdot \frac{(a^{2/3}+b^{2/3})^{1/2}}{b^{1/3}}$

$= (a^{2/3}+b^{2/3})^{1/2}(a^{2/3}+b^{2/3}) = (a^{2/3}+b^{2/3})^{3/2}.$



Note. Since θ is a positive acute angle, we may draw a right triangle OMP as shown in the figure, then

$$\sec\theta = \frac{OP}{OM} = \frac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}}$$

$$\operatorname{cosec}\theta = \frac{OP}{PM} = \frac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}}.$$

13. Find the points at which the function *f* given by $f(x = (x-2)^4 (x+1)^3$ has

(*i*) local maxima

(*ii*) local minima

(*iii*) point of inflexion.

Sol. Given: $f(x) = (x-2)^4 (x+1)^3$ ...(*i*)

$\therefore \qquad f'(x) = (x-2)^4 \frac{d}{dx}(x+1)^3 + \frac{d}{dx}(x-2)^4 \cdot (x+1)^3$

$= (x-2)^4\, 3(x+1)^2 + 4(x-2)^3 (x+1)^3$

Forming factors $= (x-2)^3 (x+1)^2 [3(x-2) + 4(x+1)]$

$\Rightarrow \qquad f'(x) = (x-2)^3 (x+1)^2 (7x-2)$

[$\because$ $3x - 6 + 4x + 4 = 7x - 2$] ...(*ii*)

Putting $f'(x) = 0$ to get turning points, we have

$$(x-2)^3 (x+1)^2 (7x-2) = 0$$

$\therefore$ Either $x - 2 = 0$ or $x + 1 = 0$ or $7x - 2 = 0$

$\Rightarrow$ $x = 2$, or $x = -1$, or $x = \frac{2}{7}$

Let us apply First derivative Test

(as we and you think that finding $f''(x)$ is tedious)

Clearly the factor $(x+1)^2$ in the value of $f'(x)$ being square of a real number is never negative.

At $x = 2$

When x is slightly < 2, (say $x = 1.9$); from (*ii*)

$$f'(x) = (-)^3 (+) (+) = (-) (+) (+) = (-)$$

When x is slightly > 2, (say $x = 2.1$), from (*ii*)

$$f'(x) = (+)^3 (+) (+) = (+)$$

$\therefore$ $f'(x)$ changes sign from $(-)$ to $(+)$ as x increases through 2.

$\therefore$ $x = 2$ gives a point of local minima.

At $x = -1$

When x is slightly < -1, (say $x = -1 - 0.1 = -1.1$), from (*ii*)

$f'(x) = (-)^3 (+) (-) = (-) (+) (-) = (+)$

When x is slightly > -1 (say $x = -1 + 0.1 = -0.9$),

from (*ii*) $f'(x) = (-)^3 (+) (-) = (+)$

$\therefore$ $f'(x)$ does not change sign as x increases through -1.

$\therefore$ $x = -1$ gives a point of inflexion.

At $x = \frac{2}{7}$

When x is slightly $< \frac{2}{7}$, (say $x = \frac{1}{7}$) from (*ii*)

$$f'(x = (-)^3 (+) (-) = (-) (+) (-) = (+)$$

When x is slightly $> \frac{2}{7}$, (say $x = \frac{3}{7}$) from (*ii*)

$$f'(x) = (-)^3 (+) (+) = (-)$$

$\therefore$ $f'(x)$ changes sign from $(+)$ to $(-)$ as x increases through $\frac{2}{7}$.

$\therefore$ $x = \frac{2}{7}$ gives a point of local maxima.

14. Find the absolute maximum and minimum values of the function f given by

$$f(x) = \cos^2 x + \sin x,\ x \in [0, \pi].$$

Sol. Given: $f(x) = \cos^2 x + \sin x,\ x \in [0, \pi]$ ...(*i*)

$\therefore$ $f'(x) = 2 \cos x \dfrac{d}{dx} (\cos x) + \cos x$

$= -2 \cos x \sin x + \cos x$

$= \cos x (-2 \sin x + 1)$

Let us put $f'(x) = 0$ to get turning points.

$\therefore$ $\cos x (-2 \sin x + 1) = 0$

$\therefore$ Either $\cos x = 0$ or $-2 \sin x + 1 = 0$

i.e., $x = \frac{\pi}{2}$ or $-2 \sin x = -1$ *i.e.,* $\sin x = \frac{1}{2}$.

Now $\sin x = \frac{1}{2}$ is positive and hence x lies in Ist quadrant and second quadrant.

$\therefore \quad \sin x = \frac{1}{2} = \sin \frac{\pi}{6}$ and $\sin \left(\pi - \frac{\pi}{6}\right) = \sin \frac{5\pi}{6}$.

$\therefore \quad x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$

$\therefore$ Turning points are $x = \frac{\pi}{2}$, $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6} \in [0, \pi]$

Let us find values of $f(x)$ at these turning points.

$\therefore$ From (i), $f\left(\frac{\pi}{2}\right) = \cos^2 \frac{\pi}{2} + \sin \frac{\pi}{2} = 0 + 1 = 1$

$$f\left(\frac{\pi}{6}\right) = \cos^2 \frac{\pi}{6} + \sin \frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

$$f\left(\frac{5\pi}{6}\right) = \cos^2 \frac{5\pi}{6} + \sin \frac{\pi}{6} = \left(\frac{-\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

$$\left[\because \cos \frac{5\pi}{6} = \cos \frac{6\pi - \pi}{6} = \cos\left(\pi - \frac{\pi}{6}\right) = -\cos \frac{\pi}{6} = \frac{-\sqrt{3}}{2}\right]$$

Now let us find values of $f(x)$ at the end points $x = 0$ and $x = \pi$ of closed interval $[0, \pi]$

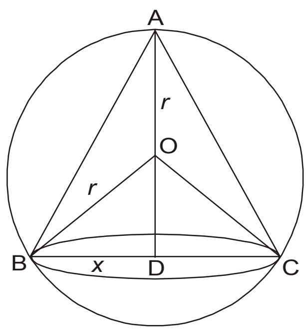
From (i), $f(0) = \cos^2 0 + \sin 0 = 1 + 0 = 1$

and $\quad f(\pi) = \cos^2 \pi + \sin \pi = (-1)^2 + 0 = 1$

[$\because$ $\cos \pi = \cos 180° = \cos (180° - 0) = -\cos 0 = -1$,
and $\sin \pi = \sin 180° = \sin (180° - 0) = \sin 0 = 0$]

Therefore absolute maximum is $\frac{5}{4}$ and absolute minimum is 1.

15. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.



Sol. Let x be the radius of base of cone and y be the height of the cone inscribed in a sphere of radius r.

$\therefore \quad OD = AD - AO = y - r$

In right-angled ΔOBD,

$$OD^2 + BD^2 = OB^2$$

(By Pythagoras Theorem)

$$(y - r)^2 + x^2 = r^2$$

or $\quad y^2 + r^2 - 2ry + x^2 = r^2$

or $\quad x^2 = 2ry - y^2 \quad ...(i)$

Let V denote the volume of the cone.

$\therefore \quad V = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi (2ry - y^2) y \quad$ [By (i)]

or $\quad V = \frac{\pi}{3}(2ry^2 - y^3)$

Diff. w.r.t. y, $\frac{dV}{dy} = \frac{\pi}{3}(4ry - 3y^2)$

and $\quad \frac{d^2V}{dy^2} = \frac{\pi}{3}(4r - 6y)$

Put $\quad \frac{dV}{dy} = 0 \quad \therefore \quad \frac{\pi}{3}(4ry - 3y^2) = 0$

or $\quad \frac{\pi y}{3}(4r - 3y) = 0$

But $\frac{\pi y}{3} \neq 0 \quad \therefore \quad 4r - 3y = 0$ or $y = \frac{4r}{3}$

At $\quad y = \frac{4r}{3}, \quad \frac{d^2V}{dy^2} = \frac{\pi}{3}(4r - 8r) = -\frac{4\pi r}{3} < 0$

$\therefore$ V is maximum at $y = \frac{4r}{3}$.

16. Let f be a function defined on $[a, b]$ such that $f'(x) > 0$, for all $x \in (a, b)$. Then prove that f is an increasing function on (a, b).

Sol. Let I denote the interval (a, b).

Given: $f'(x) > 0$ for all x in an interval I.

Let $x_1, x_2 \in$ I with $x_1 < x_2$.

Since derivability implies continuity, therefore $f(x)$ is continuous in the closed interval $[x_1, x_2]$ and derivable in the open interval (x_1, x_2).

$\therefore$ By Lagrange's Mean Value Theorem, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \text{ where } x_1 < c < x_2$$

$\Rightarrow \quad f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$ where $x_1 < c < x_2$ ...(*i*)

Now $\quad x_1 < x_2 \quad \Rightarrow \quad x_2 - x_1 > 0$

Also $\quad f'(x) > 0$ for all x in **I** (given) $\Rightarrow \quad f'(c) > 0$.

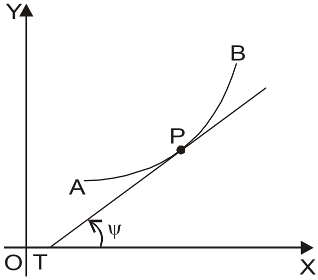
$\therefore$ From (*i*), $\quad f(x_2) - f(x_1) > 0 \quad$ or $\quad f(x_1) < f(x_2)$

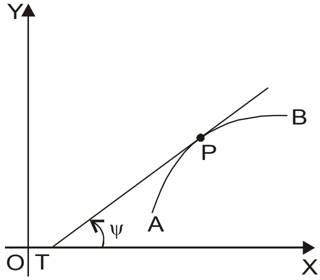
Thus, for every pair of points $x_1, x_2 \in$ **I**, $x_1 < x_2$

$\Rightarrow \quad f(x_1) < f(x_2)$

Hence, $f(x)$ is strictly increasing in **I.**

Remark. Geometrically. Let the graph of a **strictly** increasing function $y = f(x)$ be represented by curve AB. The tangent at *any point* P on the curve makes an **acute** angle ψ with the x-axis, $\left(\text{where } 0 < \psi < \frac{\pi}{2}\right)$.





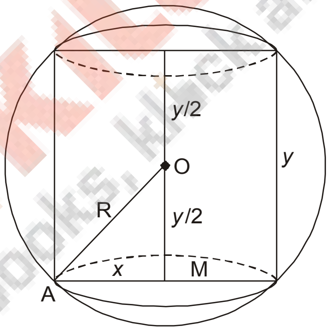
$0 < \psi < \dfrac{\pi}{2} \Rightarrow \tan \psi > 0$ *i.e.,* slope of the tangent is > 0

$\Rightarrow f'(x) > 0.$

Remark. Graph of a strictly **increasing** function is a **rising** graph *i.e.,* graph moves up as x moves to the right.

17. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\dfrac{2R}{\sqrt{3}}$. Also find the maximum volume.

Sol. Let x be the base radius and y be the height of the cylinder inscribed in a sphere having centre O and radius R. $(x > 0, y > 0)$



In right-angled ΔOAM, By Pythagoras Theorem, $AM^2 + OM^2 = OA^2$

i.e., $$x^2 + \left(\frac{y}{2}\right)^2 = R^2$$

$$\therefore \quad x^2 = R^2 - \frac{y^2}{4} \qquad ...(i)$$

Let V denote the volume of right circular cylinder,

$$\therefore \quad V = \pi x^2 y \qquad ...(ii)$$

Putting the value of x^2 from (i) in (ii),

$$V = \pi\left(R^2 - \frac{y^2}{4}\right) y = \pi\left(R^2 y - \frac{1}{4}y^3\right) \qquad ...(iii)$$

$$\therefore \quad \frac{dV}{dy} = \pi\left(R^2 - \frac{3}{4}y^2\right),$$

and $$\frac{d^2V}{dy^2} = \pi\left(-\frac{3}{2}y\right) = -\frac{3\pi y}{2}$$

Put $\dfrac{dV}{dy} = 0 \qquad \therefore \pi\left(R^2 - \dfrac{3}{4}y^2\right) = 0$

But $\pi \neq 0 \qquad \therefore R^2 - \dfrac{3}{4}y^2 = 0$ or $R^2 = \dfrac{3}{4}y^2$

or $\quad y^2 = \dfrac{4R^2}{3} \quad \therefore \; y = \dfrac{2R}{\sqrt{3}} \qquad (\because \; y > 0)$

At $\quad y = \dfrac{2R}{\sqrt{3}}, \; \dfrac{d^2V}{dy^2} = -\dfrac{3\pi}{2}\, y$

$$= -\frac{3\pi}{2}\left(\frac{2R}{\sqrt{3}}\right) = -\pi R\sqrt{3}$$

which is negative.

$\therefore$ V is maximum at $\quad y = \dfrac{2r}{\sqrt{3}}$.

Putting $y = \dfrac{2R}{\sqrt{3}}$ in eqn. (*iii*),

Maximum volume of cylinder $= \pi\left[R^2 \cdot \dfrac{2R}{\sqrt{3}} - \dfrac{1}{4} \cdot \dfrac{4R^2}{3} \cdot \dfrac{2R}{\sqrt{3}}\right]$

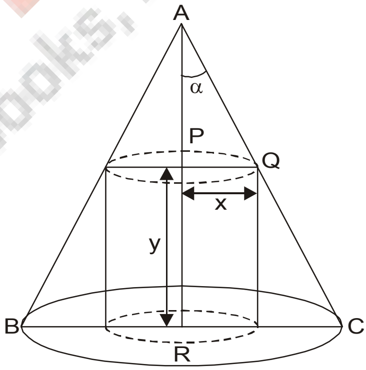
$$= \pi R^2 \frac{2R}{\sqrt{3}}\left(1 - \frac{1}{3}\right) = \frac{2\pi R^3}{\sqrt{3}} \cdot \frac{2}{3} = \frac{4\pi R^3}{3\sqrt{3}}.$$

18. Show that the height of the cylinder of greatest volume which can be inscribed in a right circular cone of height *h* and having semi-vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\dfrac{4}{27}\pi h^3 \tan^2\alpha$.

Sol. Let r be the radius of the given right circular cone of given height h.

Let the radius of the inscribed cylinder be x and its height be y.

In similar triangles APQ and ARC, we have



$$\frac{PQ}{RC} = \frac{AP}{AR}$$

i.e., $\quad \dfrac{x}{r} = \dfrac{h-y}{h}$

$[\because \; AP = AR - PR = h - y]$

Cross-multiplying, $hx = rh - ry$

$\therefore \quad ry = rh - hx = h(r - x)$

$\therefore \quad y = \dfrac{h}{r}(r - x) \qquad$ ...(*i*) (given condition)

Let z denote the volume of the cylinder

$\therefore \quad z = \pi x^2 y \qquad$ ...(*ii*) [Here z is to be maximised]

Putting the value y from (*i*) in (*ii*),

$$z = \pi x^2 \frac{h}{r}(r - x) = \frac{\pi h}{r}(rx^2 - x^3) \qquad \text{...}(iii)$$

$$\therefore \quad \frac{dz}{dx} = \frac{\pi h}{r}(2rx - 3x^2), \quad \frac{d^2z}{dx^2} = \frac{\pi h}{r}(2r - 6x)$$

For max. or min. put $\dfrac{dz}{dx} = 0$

$\therefore \quad \dfrac{\pi h}{r}(2rx - 3x^2) = 0 \quad$ or $\quad \dfrac{\pi h}{r}\, x\,(2r - 3x) = 0$

But $x \neq 0$, $\therefore\ 2r - 3x = 0$ or $3x = 2r$ or $x = \dfrac{2r}{3}$

At $x = \dfrac{2r}{3}$, $\dfrac{d^2z}{dx^2} = \dfrac{\pi h}{r}\left(2r - \dfrac{12r}{3}\right) = \dfrac{\pi h}{r}(-2r) = -2\pi h$

which is negative.

$\therefore\ z$ is maximum at $x = \dfrac{2r}{3}$.

Putting $x = \dfrac{2r}{3}$ in (i), $y = \dfrac{h}{r}\left(r - \dfrac{2r}{3}\right) = \dfrac{h}{r} \cdot \dfrac{r}{3} = \dfrac{h}{3}$

Putting $x = \dfrac{2r}{3}$ in eqn. (iii),

Maximum volume of cylinder $= \dfrac{\pi h}{r}\left[r \cdot \dfrac{4r^2}{9} - \dfrac{8r^2}{27}\right]$

$$= \frac{\pi h}{r}\, r^3\left(\frac{4}{9} - \frac{8}{27}\right) = \pi h r^2\left(\frac{12-8}{27}\right)$$

$$= \frac{4}{27}\pi h r^2 = \frac{4}{27}\pi h\,(h \tan\alpha)^2$$

$$\left[\because \text{In } \Delta\text{ARC}, \frac{\text{RC}}{\text{AR}} = \tan\alpha \text{ i.e., } \frac{r}{h} = \tan\alpha \ \therefore\ r = h\tan\alpha\right]$$

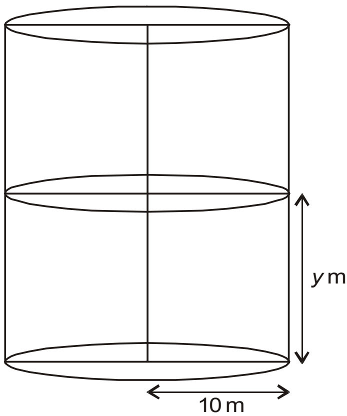
$$= \frac{4}{27}\pi h^3 \tan^2\alpha.$$

Choose the correct answer in the Exercises from 19 to 24:

19. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of

(A) 1 m/h (B) 0.1 m/h

(C) 1.1 m/h (D) 0.5 m/h.



Sol. Let y m be the depth of the wheat in the cylindrical tank of radius 10 m at time t.

$\therefore$ V = Volume of wheat in cylindrical tank at time t

$= \pi(10)^2 y$

$= 100\pi y$ cu. m ...(i)

Given: rate of increase ($\because$ wheat in being filled in the tank) of volume of wheat = $\frac{dV}{dt}$ = 314 cu. m/hr.

$\Rightarrow \quad \frac{d}{dt}(100\pi y)$ (By (*i*)) = 314

$\Rightarrow \quad 100\pi \frac{dy}{dt} = 314$

Using $\pi = \frac{22}{7} = 3.14$ nearly,

$\Rightarrow \quad 100(3.14)\frac{dy}{dt} = 314 \quad \Rightarrow \quad 314y = 314$

$\Rightarrow \quad y = \frac{314}{314} = 1$ m/h

$\therefore$ Option (A) is the correct answer.

20. The slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point (2, – 1) is

(A) $\frac{22}{7}$ (B) $\frac{6}{7}$ (C) $\frac{7}{6}$ (D) $\frac{-6}{7}$.

Sol. Equations of the curve are

$x = t^2 + 3t - 8 \quad ...(i) \quad$ and $\quad y = 2t^2 - 2t - 5 \quad ...(ii)$

$\therefore \quad \frac{dx}{dt} = 2t + 3 \quad$ and $\quad \frac{dy}{dt} = 4t - 2$

$\therefore$ Slope of the tangent to the given curve at point (x, y)

$$= \text{Value of } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t-2}{2t+3} \quad ...(iii)$$

At the given point (2, – 1), $x = 2$ and $y = -1$

Putting $x = 2$ in (*i*), and $y = -1$ in (*ii*),

$2 = t^2 + 3t - 8 \quad$ and $\quad -1 = 2t^2 - 2t - 5$

$\Rightarrow t^2 + 3t - 10 = 0 \quad$ and $\quad 2t^2 - 2t - 4 = 0$

$\Rightarrow t^2 + 5t - 2t - 10 = 0 \quad$ Dividing by 2, $t^2 - t - 2 = 0$

$\Rightarrow t(t + 5) - 2(t + 5) = 0 \quad \Rightarrow \quad t^2 - 2t + t - 2 = 0$

$\Rightarrow (t + 5)(t - 2) = 0 \quad \Rightarrow \quad t(t - 2) + (t - 2) = 0$

$\Rightarrow \quad (t - 2)(t + 1) = 0$

$\Rightarrow t = -5, t = 2 \quad \Rightarrow \quad t = 2, t = -1$

$\therefore$ Common value of t in the two sets of values of t is $t = 2$.

i.e., At the given point (2, – 1), $t = 2$.

Putting $t = 2$ in (*iii*), slope of the tangent to the given curve at the given point (2, – 1) = $\frac{4(2)-2}{2(2)+3} = \frac{6}{7}$

$\therefore$ Option (B) is the correct answer.

21. The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$ if the value of m is

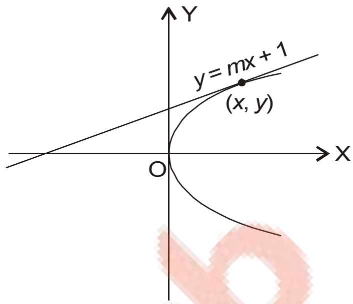
(A) 1 **(B) 2** **(C) 3** **(D) $\frac{1}{2}$.**

Sol. Equation of the curve is $y^2 = 4x$ ...(*i*)

Differentiating both sides of (*i*), w.r.t. x,

$$2y\frac{dy}{dx} = 4 . 1 \therefore \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

$\therefore$ Slope of the tangent to the curve (*i*) at any point $(x, y) = \frac{dy}{dx} = \frac{2}{y}$



$$\therefore \left(\frac{dy}{dx} =\right) \frac{2}{y} = m$$

[$\because$ line $y = mx + 1$ ...(*ii*)
is given to be a tangent to curve (*i*) and its slope is clearly m.]

$\therefore$ $my = 2 \Rightarrow y = \frac{2}{m}$ ...(*iii*)

Let us eliminate x and y from (*i*), (*ii*) and (*iii*).

Putting $y = \frac{2}{m}$ from (*iii*) in (*ii*), $\frac{2}{m} = mx + 1$

$\therefore$ $mx = \frac{2}{m} - 1 = \frac{2-m}{m}$ $\therefore$ $x = \frac{2-m}{m}$ ...(*iv*)

Putting values of x and y from (*iv*) and (*iii*) in (*i*),

$$\frac{4}{m^2} = \frac{4(2-m)}{m^2}$$

Dividing both sides by $\frac{4}{m^2}$, $1 = 2 - m$ $\therefore$ $m = 1$

$\therefore$ Option (A) is the correct answer.

22. The normal at the point (1, 1) on the curve $2y + x^2 = 3$ is

(A) $x + y = 0$ **(B) $x - y = 0$**

(C) $x + y + 1 = 0$ **(D) $x - y = 1$.**

Sol. Equation of the given curve is $2y + x^2 = 3$ ...(*i*)

Differentiating both sides of (*i*) w.r.t. x,

$$2\frac{dy}{dx} + 2x = 0 \Rightarrow 2\frac{dy}{dx} = -2x$$

Dividing by 2, $\frac{dy}{dx} = -x$

$\therefore$ Slope of the tangent at the given point (1, 1)

$= \text{value of } \frac{dy}{dx} \text{ at } (1, 1)$

$= -x = -1 \ (= m)$

$\therefore$ Slope of the normal $= \frac{-1}{m} = \frac{-1}{-1} = 1$

$\therefore$ Equation of the normal at (1, 1) is

$y - 1 = 1(x - 1)$ or $y - 1 = x - 1$

or $\quad -x + y = 0 \qquad$ or $\quad x - y = 0$

$\therefore$ Option (B) is the correct answer.

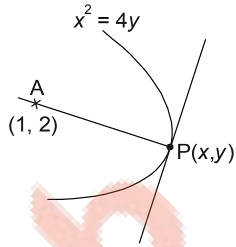
23. The normal to the curve $x^2 = 4y$ passing through (1, 2) is

(A) $x + y = 3$ **(B) $x - y = 3$**

(C) $x + y = 1$ **(D) $x - y = 1$.**

Sol. Equation of curve is

$$x^2 = 4y \qquad ...(i)$$



Let the normal to curve (i) at P (x, y) pass through the point A(1, 2). (Given)

Differentiating (i) w.r.t. x, we have

$$2x = 4\frac{dy}{dx}$$

or
$$\frac{dy}{dx} = \frac{x}{2}$$

$\therefore$ Slope of normal at $(x, y) = -\frac{dx}{dy} = -\frac{2}{x}$ ...(ii)

Again slope of normal (PA) $= \frac{y-2}{x-1}$ ...(iii) $\left|\ \frac{y_2 - y_1}{x_2 - x_1}\right.$

From (ii) and (iii), equating the two values of slope of Normal, we have

$$-\frac{2}{x} = \frac{y-2}{x-1}$$

Cross-multiplying $-2x + 2 = xy - 2x$

or $xy = 2 \qquad \therefore\ y = \frac{2}{x}$

Putting $y = \frac{2}{x}$ in (i), $x^2 = \frac{8}{x}$ or $x^3 = 8 = 2^3$

$\therefore\ x = 2 \qquad \therefore\ y = \frac{2}{x} = \frac{2}{2} = 1 \qquad \therefore$ Point P(2, 1)

At point P(2, 1), from (ii) slope of the normal $= \frac{-2}{x} = \frac{-2}{2} = -1$

$\therefore$ Equation of normal is $y - 1 = -1(x - 2)$

or $\quad y - 1 = -x + 2$

or $\quad x + y = 3.$

$\therefore$ Option (A) is the correct answer.

24. The points on the curve $9y^2 = x^3$, where the normal to the curve make equal intercepts with axes are

(A) $\left(4, \pm\frac{8}{3}\right)$ (B) $\left(4, -\frac{8}{3}\right)$ (C) $\left(4, \pm\frac{3}{8}\right)$ (D) $\left(\pm 4, \frac{3}{8}\right)$.

Sol. Equation of the curve is

$$9y^2 = x^3 \qquad ...(i)$$

Differentiating both sides of (i), w.r.t. x,

$$18y\frac{dy}{dx} = 3x^2 \quad \text{and so} \quad \frac{dy}{dx} = \frac{3x^2}{18y} = \frac{x^2}{6y}$$

$\therefore$ Slope of the tangent to curve (i) at any point (x, y)

$$= \text{value of } \frac{dy}{dx} = \frac{x^2}{6y}$$

$\therefore$ Slope of normal = negative reciprocal = $\frac{-6y}{x^2} = \pm 1$

($\because$ We know that slopes of lines making equal intercepts on the axes are ± 1)

$\Rightarrow \quad -6y = \pm x^2$

Taking positive sign, $\quad -6y = x^2 \quad$ or $\quad y = -\frac{x^2}{6}$ ...(ii)

Let us solve (i) and (ii) for x and y.

Putting $y = -\frac{x^2}{6}$ from (ii) in (i), $9 \cdot \frac{x^4}{36} = x^3$

$\Rightarrow \quad \frac{x^4}{4} = x^3 \quad \Rightarrow \quad x^4 = 4x^3$

Dividing both sides by x^3 ($x \neq 0$ because in none of the four options given, $x = 0$)

$x = 4.$

Putting $x = 4$ in (ii), $y = -\frac{16}{6} = -\frac{8}{3}$.

$\therefore$ One required point is $\left(4, -\frac{8}{3}\right)$

Taking negative sign, $-6y = -x^2 \quad \Rightarrow \quad y = \frac{x^2}{6}$ ...(iii)

Now solving (iii) and (i) for x and y as above (we solved (i) and (ii), the required point is $\left(4, \frac{8}{3}\right)$.

$\therefore$ Required points are $\left(4, \pm\frac{8}{3}\right)$.

$\therefore$ Option (A) is the correct answer.